Part I

Set Theory

Chapter 1

The Iterative Conception

1.1 Extensionality

The very first thing to say is that sets are individuated by their elements. More precisely:

Axiom (Extensionality). If sets A and B have the same elements, then A and B are the same set.

$$\forall A \, \forall B \, (\forall x \, (x \in A \leftrightarrow x \in B) \rightarrow A = B)$$

We assumed this throughout **??**. But it bears repeating. The Axiom of Extensionality expresses the basic idea that a set is determined by its elements. (So sets might be contrasted with *concepts*, where precisely the same objects might fall under many different concepts.)

Why embrace this principle? Well, it is plausible to say that any denial of Extensionality is a decision to abandon anything which might even be called *set theory*. Set theory is no more nor less than the theory of extensional collections.

The real challenge in part I, though, is to lay down principles which tell us *which sets exist*. And it turns out that the only truly "obvious" answer to this question is provably wrong.

1.2 Russell's Paradox (again)

In **??**, we worked with a naïve set theory. But according to a *very* naïve conception, sets are just the extensions of predicates. This naïve thought would mandate the following principle:

> *Naïve Comprehension.* $\{x : \varphi(x)\}$ exists for any formula φ.

Tempting as this principle is, it is provably inconsistent. We saw this in **??**, but the result is so important, and so straightforward, that it's worth repeating. Verbatim.

Theorem 1.1 (Russell's Paradox). *There is no set $R = \{x : x \notin x\}$*

Proof. If $R = \{x : x \notin x\}$ exists, then $R \in R$ iff $R \notin R$, which is a contradiction. □

Russell discovered this result in June 1901. (He did not, though, put the paradox in quite the form we just presented it, since he was considering Frege's set theory, as outlined in *Grundgesetze*. We will return to this in section 1.6.) Russell wrote to Frege on June 16, 1902, explaining the inconsistency in Frege's system. For the correspondence, and a bit of background, see Heijenoort (1967, pp. 124–8).

It is worth emphasising that this two-line proof is a result of *pure logic*. Granted, we implicitly used a (non-logical?) axiom, Extensionality, in our notation $\{x : x \notin x\}$; for $\{x : \varphi(x)\}$ is to be *the unique* (by Extensionality) set of the φs, if one exists. But we can avoid even the hint of Extensionality, just by stating the result as follows: *there is no set whose members are exactly the non-self-membered sets*. And this has nothing much to do with sets. As Russell himself observed, exactly similar reasoning will lead you to conclude: *no man shaves exactly the men who do not shave themselves*. Or: *no pug sniffs exactly the pugs which don't sniff themselves*. And so on. Schematically, the shape of the result is just:

$$\neg \exists x \forall z (Rzx \leftrightarrow \neg Rzz).$$

And that's just a theorem (scheme) of first-order logic. Consequently, we can't avoid Russell's Paradox just by tinkering with our set theory; it arises before we even *get* to set theory. If we're going to use (classical) first-order logic, we simply have to *accept* that there is no set $R = \{x : x \notin x\}$.

The upshot is this. If you want to accept Naïve Comprehension whilst *avoiding* inconsistency, you cannot just tinker with the *set theory*. Instead, you would have to overhaul your *logic*.

Of course, set theories with non-classical logics have been presented. But they are—to say the least—non-standard. The standard approach to Russell's Paradox is to treat it as a straightforward non-existence proof, and then to try to learn how to live with it. That is the approach we will follow.

1.3 Predicative and Impredicative

sth:story:predicative: sec

The Russell set, R, was defined via $\{x : x \notin x\}$. Spelled out more fully, R would be the set which contains all and only those sets which are not non-self-membered. So in defining R, we quantify over the domain which would contain R (if it existed).

This is an *impredicative* definition. More generally, we might say that a definition is impredicative iff it quantifies over a domain which contains the object that is being defined.

In the wake of the paradoxes, Whitehead, Russell, Poincaré and Weyl rejected such impredicative definitions as "viciously circular":

> An analysis of the paradoxes to be avoided shows that they all result from a kind of vicious circle. The vicious circles in question arise from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole[. . . . ¶]
>
> The principle which enables us to avoid illegitimate totalities may be stated as follows: 'Whatever involves *all* of a collection must not be one of the collection'; or, conversely: 'If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.' We shall call this the 'vicious-circle principle,' because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities. (Whitehead and Russell, 1910, p. 37)

If we follow them in rejecting the *vicious-circle principle*, then we might attempt to replace the disastrous Naïve Comprehension Scheme (of section 1.2) with something like this:

Predicative Comprehension. For every formula φ quantifying only over sets: the set$'$ $\{x : \varphi(x)\}$ exists.

So long as sets$'$ are not sets, no contradiction will ensue.

Unfortunately, Predicative Comprehension is not very *comprehensive*. After all, it introduces us to new entities, sets$'$. So we will have to consider formulas which quantify over sets$'$. If they always yield a set$'$, then Russell's paradox will arise again, just by considering the set$'$ of all non-self-membered sets$'$. So, pursuing the same thought, we must say that a formula quantifying over sets$'$ yields a corresponding set$''$. And then we will need sets$'''$, sets$''''$, etc. To prevent a rash of primes, it will be easier to think of these as sets_0, sets_1, sets_2, sets_3, sets_4,.... And this would give us a way into the (simple) theory of types.

There are a few obvious objections against such a theory (though it is not obvious that they are *overwhelming* objections). In brief: the resulting theory is cumbersome to use; it is profligate in postulating different kinds of objects; and it is not clear, in the end, that impredicative definitions are even *all that bad*.

To bring out the last point, consider this remark from Ramsey:

> we may refer to a man as the tallest in a group, thus identifying him by means of a totality of which he is himself a member without there being any vicious circle. (Ramsey, 1925)

Ramsey's point is that "the tallest man in the group" *is* an impredicative definition; but it is obviously perfectly kosher.

One might respond that, in this case, we could pick out the tallest person by *predicative* means. For example, maybe we could just point at the man in question. The objection against impredicative definitions, then, would clearly need

to be limited to entities which can *only* be picked out impredicatively. But even then, we would need to hear more, about why such "essential impredicativity" would be so bad.[1]

Admittedly, impredicative definitions are extremely bad news, if we want our definitions to provide us with something like a recipe for *creating* an object. For, given an impredicative definition, one would genuinely be caught in a vicious circle: to create the impredicatively specified object, one would *first* need to create all the objects (including the impredicatively specified object), since the impredicatively specified object is specified in terms of all the objects; so one would need to create the impredicatively specified object before one had created it itself. But again, this is only a serious objection against "essentially impredicatively" specified sets, if we think of sets as things that we *create*. And we (probably) don't.

As such—for better or worse—the approach which became common does not involve taking a hard line concerning (im)predicativity. Rather, it involves what is now regarded as the cumulative-iterative approach. In the end, this will allow us to stratify our sets into "stages"—a *bit* like the predicative approach stratifies entities into sets$_0$, sets$_1$, sets$_2$, . . . —but we will not postulate any difference in kind between them.

1.4 The Cumulative-Iterative Approach

Here is a slightly fuller statement of how we will stratify sets into stages:

> Sets are formed in *stages*. For each stage S, there are certain stages which are *before* S. At stage S, each collection consisting of sets formed at stages before S is formed into a set. There are no sets other than the sets which are formed at stages. (Shoenfield, 1977, p. 323)

This is a sketch of the *cumulative-iterative conception of set*. It will underpin the formal set theory that we present in part I.

Let's explore this in a little more detail. As Shoenfield describes the process, at every stage, we form new sets from the sets which were available to us from earlier stages. So, on Shoenfield's picture, at the initial stage, stage 0, there are no *earlier* stages, and so *a fortiori* there are no sets available to us from earlier stages.[2] So we form only one set: the set with no elements $\emptyset$. At stage 1, exactly one set is available to us from earlier stages, so only one new set is $\{\emptyset\}$. At stage 2, two sets are available to us from earlier stages, and we form two new sets $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$. At stage 3, four sets are available to us from earlier stages, so we form twelve new sets. . . . As such, the cumulative-iterative picture of the sets will look a bit like this (with numbers indicating stages):

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[1]For more, see Linnebo (2010).

[2]Why should we assume that there *is* a first stage? See the footnote to *Stages-are-ordered* in section 2.1.
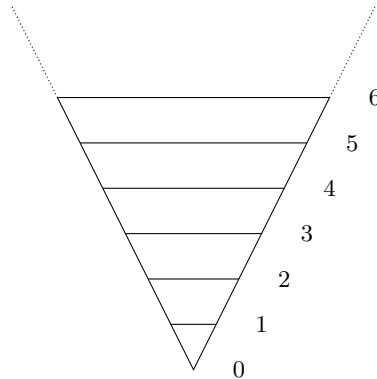
So: why should we embrace this story?

One reason is that it is a nice, tractable story. Given the demise of the most obvious story, i.e., Naïve Comprehension, we are in want of something nice.

But the story is not *just* nice. We have a good reason to believe that any set theory based on this story will be *consistent*. Here is why.

Given the cumulative-iterative conception of set, we form sets at stages; and their elements must be objects which were available *already*. So, for any stage S, we can form the set

$$R_S = \{x : x \notin x \text{ and } x \text{ was available before } S\}$$

The reasoning involved in proving Russell's Paradox will now establish that R_S itself is not available before stage S. And that's not a contradiction. Moreover, if we embrace the cumulative-iterative conception of set, then we shouldn't even have *expected* to be able to form the Russell set itself. For that would be the set of all non-self-membered sets that "will ever be available". In short: the fact that we (provably) can't form the Russell set isn't *surprising*, given the cumulative-iterative story; it's what we would *predict*.

1.5   Urelements or Not?

In the next few chapters, we will try to extract axioms from the cumulative-iterative conception of set. But, before going any further, we need to say something more about *urelements*.

The picture of section 1.4 allowed us only to form new sets from old *sets*. However, we might want to allow that certain *non-sets*—cows, pigs, grains of sand, or whatever—can be elements of sets. In that case, we would start with certain basic elements, *urelements*, and then say that at each stage S we would form "all possible" sets consisting of urelements or sets formed at stages before S (in any combination). The resulting picture would look more like this:
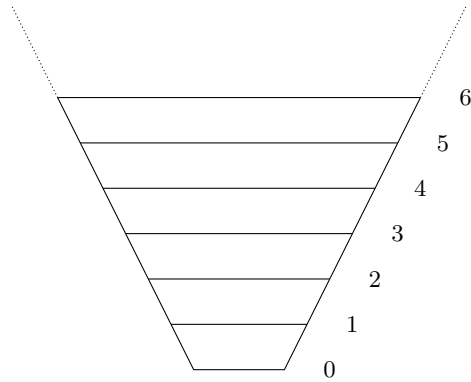
So now we have a decision to take: *Should we allow urelements?*

Philosophically, it makes sense to include urelements in our theorising. The main reason for this is to make our set theory *applicable*. To illustrate the point, recall from **??** that we say that two sets A and B have the same size, i.e., $A \approx B$, iff there is a bijection between them. Now, if the cows in the field and the pigs in the sty both form sets, we can offer a set-theoretical treatment of the claim "there are as many cows as pigs". But if we ban urelements, so that the cows and the pigs do *not* form sets, then that set-theoretical treatment will be unavailable. Indeed, we will have no straightforward ability to apply set theory to anything other than sets themselves. (For more reasons to include urelements, see Potter 2004, pp. vi, 24, 50–1.)

Mathematically, however, it is quite rare to allow urelements. In part, this is because it is *very slightly* easier to formulate set theory without urelements. But, occasionally, one finds more interesting justifications for excluding urelement from set theory:

> In accordance with the belief that set theory is the foundation of mathematics, we should be able to capture all of mathematics by just talking about sets, so our variable should not range over objects like cows and pigs. (Kunen, 1980, p. 8)

So: a focus on applicability would suggest *including* urelements; a focus on a reductive foundational goal (reducing mathematics to pure set theory) might suggest *excluding* them. Mild laziness, too, points in the direction of excluding urelements.

We will follow the laziest path. Partly, though, there is a pedagogical justification. Our aim is to introduce you to the elements of set theory that you would need in order to get started on the philosophy of set theory. And most of that philosophical literature discusses set theories formulated *without* urelements. So this book will, perhaps, be of more use, if it hews fairly closely to that literature.

1.6 Appendix: Frege's Basic Law V

In section 1.2, we explained that Russell's formulated his paradox as a problem for the system Frege outlined in his *Grundgesetze*. Frege's system did not include a direct formulation of Naïve Comprehension. So, in this appendix, we will very briefly explain what Frege's system *did* include, and how it relates to Naïve Comprehension and how it relates to Russell's Paradox.

Frege's system is *second-order*, and was designed to formulate the notion of an *extension of a concept*. Using notation inspired by Frege, we will write $\epsilon x\, F(x)$ for *the extension of the concept F*. This is a device which takes a *predicate*, "F", and turns it into a (first-order) *term*, "$\epsilon x\, F(x)$". Using this device, Frege offered the following *definition* of membership:

$$a \in b =_{\mathrm{df}} \exists G(b = \epsilon x\, G(x) \wedge Ga)$$

roughly: $a \in b$ iff a falls under a concept whose extension is b. (Note that the quantifier "$\exists G$" is second-order.) Frege also maintained the following principle, known as *Basic Law V*:

$$\epsilon x\, F(x) = \epsilon x\, G(x) \leftrightarrow \forall x(Fx \leftrightarrow Gx)$$

roughly: concepts have identical extensions iff they are coextensive. (Again, both "F" and "G" are in predicate position.) Now a simple principle connects membership with property-satisfaction:

Lemma 1.2 (in *Grundgesetze*). $\forall F \forall a(a \in \epsilon x\, F(x) \leftrightarrow Fa)$

Proof. Fix F and a. Now $a \in \epsilon x\, F(x)$ iff $\exists G(\epsilon x\, F(x) = \epsilon x\, G(x) \wedge Ga)$ (by the definition of membership) iff $\exists G(\forall x(Fx \leftrightarrow Gx) \wedge Ga)$ (by Basic Law V) iff Fa (by elementary second-order logic). $\square$

And this yields Naïve Comprehension almost immediately:

Lemma 1.3 (in *Grundgesetze*.). $\forall F \exists s \forall a(a \in s \leftrightarrow Fa)$

Proof. Fix F; now Lemma 1.2 yields $\forall a(a \in \epsilon x\, F(x) \leftrightarrow Fa)$; so $\exists s \forall a(a \in s \leftrightarrow Fa)$ by existential generalisation. The result follows since F was arbitrary. $\square$

Russell's Paradox follows by taking F as given by $\forall x(Fx \leftrightarrow x \notin x)$.

Chapter 2

Steps towards Z

2.1 The Story in More Detail

In section 1.4, we quoted Schoenfield's description of the process of set-formation. We now want to write down a few more principles, to make this story a bit more precise. Here they are:

> *Stages-are-key.* Every set is formed at some stage.

> *Stages-are-ordered.* Stages are ordered: some come *before* others.[1]

> *Stages-accumulate.* For any stage S, and for any sets which were formed *before* stage S: a set is formed at stage S whose members are exactly those sets. Nothing else is formed at stage S.

These are informal principles, but we will be able to use them to vindicate several of the axioms of Zermelo's set theory.

(We should offer a word of caution. Although we will be presenting some completely standard axioms, with completely standard names, the italicized principles we have just presented have no particular names in the literature. We simply monikers which we hope are helpful.)

2.2 Separation

We start with a principle to replace Naïve Comprehension:

Axiom (Scheme of Separation). For every formula $\varphi(x)$, this is an axiom: for any A, the set $\{x \in A : \varphi(x)\}$ exists.

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[1] We will actually assume—tacitly—that the stages are *well-ordered*. What this amounts to is explained in chapter 3. This is a substantial assumption. In fact, using a very clever technique due to Scott (1974), this assumption can be *avoided* and then *derived*. (This will also explain why we should think that there is an initial stage.) We cannot go into that here; for more, see Button (forthcoming).

Note that this is not a single axiom. It is a *scheme* of axioms. There are *infinitely many* Separation axioms; one for every formula $\varphi(x)$. The scheme can equally well be (and normally is) written down as follows:

> For any formula $\varphi(x)$ which does not contain "S", this is an axiom:
>
> $$\forall A \exists S \forall x (x \in S \leftrightarrow (\varphi(x) \wedge x \in A)).$$

In keeping with the convention noted at the start of part I, the formulas φ in the Separation axioms may have parameters.[2]

Separation is immediately justified by our cumulative-iterative conception of sets we have been telling. To see why, let A be a set. So A is formed by some stage S (by *Stages-are-key*). Since A was formed at stage S, all of A's members were formed before stage S (by *Stages-accumulate*). Now in particular, consider all the sets which are members of A and which also satisfy φ; clearly all of these sets, too, were formed before stage S. So they are formed into a set $\{x \in A : \varphi(x)\}$ at stage S too (by *Stages-accumulate*).

Unlike Naïve Comprehension, this avoid Russell's Paradox. For we cannot simply assert the existence of the set $\{x : x \notin x\}$. Rather, *given* some set A, we can assert the existence of the set $R_A = \{x \in A : x \notin x\}$. But all this proves is that $R_A \notin R_A$ and $R_A \notin A$, none of which is very worrying.

However, Separation has an immediate and striking consequence:

Theorem 2.1. *There is no universal set, i.e., $\{x : x = x\}$ does not exist.*

Proof. For reductio, suppose V is a universal set. Then by Separation, $R = \{x \in V : x \notin x\} = \{x : x \notin x\}$ exists, contradicting Russell's Paradox. $\square$

The absence of a universal set—indeed, the open-endedness of the hierarchy of sets—is one of the most fundamental ideas behind the cumulative-iterative conception. So it is worth seeing that, intuitively, we could reach it via a different route. A universal set must be an element of itself. But, on our cumulative-iterative conception, every set appears (for the first time) in the hierarchy at the first stage immediately after all of its elements. But this entails that *no* set is self-membered. For any self-membered set would have to first occur immediately after the stage at which it first occurred, which is absurd. (We will see in Definition 4.15 how to make this explanation more rigorous, by using the notion of the "rank" of a set. However, we will need to have a few more axioms in place to do this.)

Here are a few more consequences of Separation and Extensionality.

Proposition 2.2. *If any set exists, then $\emptyset$ exists.*

Proof. If A is a set, $\emptyset = \{x \in A : x \neq x\}$ exists by Separation. $\square$

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[2]For an explanation of what this means, see the discussion immediately after **??**.

Proposition 2.3. *$A \setminus B$ exists for any sets A and B*

Proof. $A \setminus B = \{x \in A : x \notin B\}$ exists by Separation. □

It also turns out that (almost) arbitrary intersections exist:

Proposition 2.4. *If $A \neq \emptyset$, then $\bigcap A = \{x : (\forall y \in A)x \in y\}$ exists.*

Proof. Let $A \neq \emptyset$, so there is some $c \in A$. Then $\bigcap A = \{x : (\forall y \in A)x \in y\} = \{x \in c : (\forall y \in A)x \in y\}$, which exists by Separation. □

Note the condition that $A \neq \emptyset$, though; for $\bigcap \emptyset$ would be the universal set, vacuously, contradicting Theorem 2.1.

2.3 Union

Proposition 2.4 gave us intersections. But if we want arbitrary unions to exist, we need to lay down another axiom:

Axiom (Union). For any set A, the set $\bigcup A = \{x : (\exists b \in A)x \in b\}$ exists.

$$\forall A \exists U \forall x(x \in U \leftrightarrow (\exists b \in A)x \in b)$$

This axiom is also justified by the cumulative-iterative conception. Let A be a set, so A is formed at some stage S (by *Stages-are-key*). Every member of A was formed *before* S (by *Stages-accumulate*); so, reasoning similarly, every member of every member of A was formed before S. Thus all of *those* sets are available before S, to be formed into a set at S. And that set is just $\bigcup A$.

2.4 Pairs

The next axiom to consider is the following:

Axiom (Pairs). For any sets a, b, the set $\{a, b\}$ exists.

$$\forall a \forall b \exists P \forall x(x \in P \leftrightarrow (x = a \vee x = b))$$

Here is how to justify this axiom, using the iterative conception. Suppose a is available at stage S, and b is available at stage T. Let M be whichever of stages S and T comes later. Then since a and b are both available at stage M, the set $\{a, b\}$ is a possible collection available at any stage after M (whichever is the greater).

But hold on! Why assume that there *are* any stages after M? If there are none, then our justification will fail. So, to justify Pairs, we will have to add another principle to the story we told in section 2.1, namely:

Stages-keep-going. There is no last stage.

Is this principle justified? Nothing in Shoenfield's story stated *explicitly* that there is no last stage. Still, even if it is (strictly speaking) an extra addition to our story, it fits well with the basic idea that sets are formed in stages. We will simply accept it in what follows. And so, we will accept the Axiom of Pairs too.

Armed with this new Axiom, we can prove the existence of plenty more sets. For example:

Proposition 2.5. *For any sets a and b, the following sets exist:*

1. $\{a\}$

2. $a \cup b$

3. $\langle a, b \rangle$

Proof. (1). By Pairs, $\{a, a\}$ exists, which is $\{a\}$ by Extensionality.

(2). By Pairs, $\{a, b\}$ exists. Now $a \cup b = \bigcup\{a, b\}$ exists by Union.

(3). By (1), $\{a\}$ exists. By Pairs, $\{a, b\}$ exists. Now $\{\{a\}, \{a, b\}\} = \langle a, b \rangle$ exists, by Pairs again. □

Problem 2.1. Show that, for any sets a, b, c, the set $\{a, b, c\}$ exists.

Problem 2.2. Show that, for any sets $a_1, \ldots, a_n$, the set $\{a_1, \ldots, a_n\}$ exists.

2.5 Powersets

We will proceed with another axiom:

Axiom (Powersets). For any set A, the set $\wp(A) = \{x : x \subseteq A\}$ exists.

$$\forall A \exists P \forall x (x \in P \leftrightarrow (\forall z \in x) z \in A)$$

Our justification for this is pretty straightforward. Suppose A is formed at stage S. Then all of A's members were available before S (by *Stages-accumulate*). So, reasoning as in our justification for Separation, every subset of A is formed by stage S. So they are all available, to be formed into a single set, at any stage after S. And we know that there is some such stage, since S is not the last stage (by *Stages-keep-going*). So $\wp(A)$ exists.

Here is a nice consequence of Powersets:

Proposition 2.6. *Given any sets A, B, their Cartesian product $A \times B$ exists.*

Proof. The set $\wp(\wp(A \cup B))$ exists by Powersets and Proposition 2.5. So by Separation, this set exists:

$$C = \{z \in \wp(\wp(A \cup B)) : (\exists x \in A)(\exists y \in B) z = \langle x, y \rangle\}.$$

Now, for any $x \in A$ and $y \in B$, the set $\langle x, y \rangle$ exists by Proposition 2.5. Moreover, since $x, y \in A \cup B$, we have that $\{x\}, \{x, y\} \in \wp(A \cup B)$, and $\langle x, y \rangle \in \wp(\wp(A \cup B))$. So $A \times B = C$. □

In this proof, Powerset interacts with Separation. And that is no surprise. Without Separation, Powersets wouldn't be a very *powerful* principle. After all, Separation tells us which subsets of a set exist, and hence determines just how "fat" each Powerset is.

Problem 2.3. Show that, for any sets A, B: (i) the set of all relations with domain A and range B exists; and (ii) the set of all functions from A to B exists.

Problem 2.4. Let A be a set, and let $\sim$ be an equivalence relation on A. Prove that the set of equivalence classes under $\sim$ on A, i.e., $A/_{\sim}$, exists.

2.6 Infinity

We already have enough axioms to ensure that there are infinitely many sets (if there are any). For suppose some set exists, and so $\emptyset$ exists (by Proposition 2.2). Now for any set x, the set $x \cup \{x\}$ exists by Proposition 2.5. So, applying this a few times, we will get sets as follows:

0. $\emptyset$

1. $\{\emptyset\}$

2. $\{\emptyset, \{\emptyset\}\}$

3. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

4. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$

and we can check that each of these sets is distinct.

We have started the numbering from 0, for a few reasons. But one of them is this. It is not that hard to check that the set we have labelled "n" has exactly n members, and (intuitively) is formed at the nth stage.

But. This gives us *infinitely many* sets, but it does not guarantee that there is an *infinite set*, i.e., a set with infinitely many members. And this really matters: unless we can find a (Dedekind) infinite set, we cannot construct a Dedekind algebra. But we want a Dedekind algebra, so that we can treat it as the set of natural numbers. (Compare **??**.)

Importantly, the axioms we have laid down so far do *not* guarantee the existence of any infinite set. So we have to lay down a new axiom:

Axiom (Infinity). There is a set, I, such that $\emptyset \in I$ and $x \cup \{x\} \in I$ whenever $x \in I$.

$$\exists I((\exists o \in I)\forall x \; x \notin o \wedge$$
$$(\forall x \in I)(\exists s \in I)\forall z(z \in s \leftrightarrow (z \in x \vee z = x)))$$

It is easy to see that the set I given to us by the Axiom of Infinity is Dedekind infinite. Its distinguished element is $\emptyset$, and the injection on I is given by $s(x) = x \cup \{x\}$. Now, **??** showed how to extract a Dedekind Algebra from a Dedekind infinite set; and we will treat this as our set of natural numbers. More precisely:

Definition 2.7. Let I be any set given to us by the Axiom of Infinity. Let s be the function $s(x) = x \cup \{x\}$. Let $\omega = \mathrm{clo}_s(\emptyset)$. We call the members of ω the *natural numbers*, and say that n is the result of n-many applications of s to $\emptyset$.

You can now look back and check that the set labelled "n", a few paragraphs earlier, will be treated *as* the number n.

We will discuss this significance of this stipulation in section 2.8. For now, it enables us to prove an intuitive result:

Proposition 2.8. *No natural number is Dedekind infinite.*

Proof. The proof is by induction, i.e., **??**. Clearly $0 = \emptyset$ is not Dedekind infinite. For the induction step, we will establish the contrapositive: if (absurdly) $s(n)$ is Dedekind infinite, then n is Dedekind infinite.

So suppose that $s(n)$ is Dedekind infinite, i.e., there is some injection f with $\mathrm{ran}(f) \subsetneq \mathrm{dom}(f) = s(n) = n \cup \{n\}$. There are two cases to consider.

Case 1: $n \notin \mathrm{ran}(f)$. So $\mathrm{ran}(f) \subseteq n$, and $f(n) \in n$. Let $g = f{\restriction}_n$; now $\mathrm{ran}(g) = \mathrm{ran}(f) \setminus \{f(n)\} \subsetneq n = \mathrm{dom}(g)$. Hence n is Dedekind infinite.

Case 2: $n \in \mathrm{ran}(f)$. Fix $m \in \mathrm{dom}(f) \setminus \mathrm{ran}(f)$, and define a function h with domain $s(n) = n \cup \{n\}$:

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \neq n \\ m & \text{if } f(x) = n \end{cases}$$

So h and f agree everywhere, except that $h(f^{-1}(n)) = m \neq n = f(f^{-1}(n))$. Since f is an injection, $n \notin \mathrm{ran}(h)$; and $\mathrm{ran}(h) \subsetneq \mathrm{dom}(h) = s(n)$. Now n is Dedekind infinite, using the argument of Case 1. $\qquad\square$

The question remains, though, of how we might *justify* the Axiom of Infinity. The short answer is that we will need to add another principle to the story we have been telling. That principle is as follows:

> *Stages-hit-infinity.* There is an infinite stage. That is, there is a stage which (a) is not the first stage, and which (b) has some stages before it, but which (c) has no immediate predecessor.

The Axiom of Infinity follows straightforwardly from this principle. We know that natural number n is formed at stage n. So the set ω is formed at the first infinite stage. And ω itself witnesses the Axiom of Infinity.

This, however, simply pushes us back to the question of how we might justify *Stages-hit-infinity*. As with *Stages-keep-going*, it was not an explicit part of the story we told about the cumulative-iterative hierarchy. But more

than that: nothing in the very idea of an iterative hierarchy, in which sets are formed stage by stage, forces us to think that the process involves an *infinite* stage. It seems perfectly coherent to think that the stages are ordered like the natural numbers.

This, however, gives rise to an obvious problem. In **??**, we considered Dedekind's "proof" that there is a Dedekind infinite set (of thoughts). This may not have struck you as very satisfying. But if *Stages-hit-infinity* is not "forced upon us" by the iterative conception of set (or by "the laws of thought"), then we are still left without an intrinsic justification for the claim that there is a Dedekind infinite set.

There is much more to say here, of course. But hopefully you are now at a point to start thinking about what it might *take* to justify an axiom (or principle). In what follows we will simply take *Stages-hit-infinity* for granted.

2.7 Z⁻: a Milestone

We will revisit *Stages-hit-infinity* in the next section. However, with the Axiom of Infinity, we have reached an important milestone. We now have all the axioms required for the theory **Z⁻**. In detail:

Definition 2.9. The theory **Z⁻** has these axioms: Extensionality, Union, Pairs, Powersets, Infinity, and all instances of the Separation scheme.

The name stands for *Zermelo* set theory (*minus* something which we will come to later). Zermelo deserves the honour, since he essentially formulated this theory in his 1908.[3]

This theory is powerful enough to allow us to do an enormous amount of mathematics. In particular, you *should* look back through **??**, and convince yourself that everything we did, naïvely, could be done more formally within **Z⁻**. (Once you have done that for a bit, you might want to skip ahead and read section 2.9.) So, henceforth, and without any further comment, we will take ourselves to be working in **Z⁻** (at least).

2.8 Selecting our Natural Numbers

In Definition 2.7, we explicitly defined the expression "natural numbers". How should you understand this stipulation? It is not a metaphysical claim, but just a decision to *treat* certain sets as the natural numbers. We touched upon reasons for thinking this in **??**, **??** and **??**. But we can make these reasons even more pointed.

Our Axiom of Infinity follows von Neumann (1925). But here is another axiom, which we could have adopted instead:

---

[3]For interesting comments on the history and technicalities, see Potter (2004, Appendix A).

> *Zermelo's 1908 Axiom of Infinity.* There is a set A such that $\emptyset \in A$ and $(\forall x \in A)\{x\} \in A$.

Had we used Zermelo's axiom, instead of our (von Neumann-inspired) Axiom of Infinity, we would equally well have been given a Dedekind infinite set, and so a Dedekind algebra. On Zermelo's approach, the distinguished element of our algebra would again have been $\emptyset$ (our surrogate for 0), but the injection would have been given by the map $x \mapsto \{x\}$, rather than $x \mapsto x \cup \{x\}$. The simplest upshot of this is that Zermelo treats 2 as $\{\{\emptyset\}\}$, whereas we (with von Neumann) treat 2 as $\{\emptyset, \{\emptyset\}\}$.

Why choose one axiom of Infinity rather than the other? The main practical reason is that von Neumann's approach "scales up" to handle transfinite numbers rather well. We will explore this from chapter 3 onwards. However, from the simple perspective of *doing arithmetic*, both approaches would do equally well. So if someone tells you that the natural numbers *are* sets, the obvious question is: *Which sets are they?*

This precise question was made famous by Benacerraf (1965). But it is worth emphasising that it is just the most famous example of a phenomenon that we have encountered many times already. The basic point is this. Set theory gives us a way to *simulate* a bunch of "intuitive" kinds of entities: the reals, rationals, integers, and naturals, yes; but also ordered pairs, functions, and relations. However, set theory never provides us with a *unique* choice of simulation. There are *always* alternatives which—straightforwardly—would have served us just as well.

2.9 Appendix: Closure, Comprehension, and Intersection

In section 2.7, we suggested that you should look back through the naïve work of **??** and check that it can be carried out in $\mathbf{Z}^-$. If you followed that advice, *one* point might have tripped you up: the use of *intersection* in Dedekind's treatment of *closures*.

sth:z:arbintersections: sec

Recall from **??** that

$$\mathrm{clo}_f(o) = \bigcap\{X : o \in X \text{ and } X \text{ is } f\text{-closed}\}.$$

The general shape of this is a definition of the form:

$$C = \bigcap\{X : \varphi(X)\}.$$

But this should ring alarm bells: since Naïve Comprehension fails, there is no guarantee that $\{X : \varphi(X)\}$ exists. It looks dangerously, then, like such definitions are *cheating*.

Fortunately, they are not cheating; or rather, if they *are* cheating as they stand, then we can engage in some honest toil to render them kosher. That honest toil was foreshadowed in Proposition 2.4, when we explained why $\bigcap A$ exists for any $A \neq \emptyset$. But we will spell it out explicitly.

Given Extensionality, if we attempt to define C as $\bigcap\{X : \varphi(X)\}$, all we are really asking is for an object C which obeys the following:

$$\forall x(x \in C \leftrightarrow \forall X(\varphi(X) \to x \in X)) \qquad (*)$$

Now, suppose there is *some* set, S, such that $\varphi(S)$. Then to deliver eq. (*), we can simply define C using *Separation*, as follows:

$$C = \{x \in S : \forall X(\varphi(X) \to x \in X)\}.$$

We leave it as an exercise to check that this definition yields eq. (*), as desired. And this general strategy will allow us to circumvent any apparent use of Naïve Comprehension in defining intersections. In the particular case which got us started on this line of thought, namely that of $\mathrm{clo}_f(o)$, here is how that would work. We began the proof of **??** by noting that $o \in \mathrm{ran}(f) \cup \{o\}$ and that $\mathrm{ran}(f) \cup \{o\}$ is f-closed. So, we can define what we want thus:

$$\mathrm{clo}_f(o) = \{x \in \mathrm{ran}(f) \cup \{o\} : (\forall X \ni o)(X \text{ is } f\text{-closed} \to x \in X)\}.$$

Chapter 3

Ordinals

3.1 Introduction

In chapter 2, we postulated that there is an infinite-th stage of the hierarchy, in the form of *Stages-hit-infinity* (see also our axiom of Infinity). However, given *Stages-keep-going*, we can't stop at the infinite-th stage; we have to keep going. So: at the next stage after the first infinite stage, we form all possible collections of sets that were available at the first infinite stage; and repeat; and repeat; and repeat; . . .

Implicitly what has happened here is that we have started to invoke an "intuitive" notion of number, according to which there can be numbers *after* all the natural numbers. In particular, the notion involved is that of a *transfinite ordinal*. The aim of this chapter is to make this idea more rigorous. We will explore the general notion of an ordinal, and then explicitly define certain sets to be our ordinals.

3.2 The General Idea of an Ordinal

Consider the natural numbers, in their usual order:

$$0 \ < \ 1 \ < \ 2 \ < \ 3 \ < \ 4 \ < \ 5 \ < \ \cdots$$

We call this, in the jargon, an ω-sequence. And indeed, this general ordering is mirrored in our initial construction of the stages of the set hierarchy. But, now suppose we move 0 to the end of this sequence, so that it comes after all the other numbers:

$$1 \ < \ 2 \ < \ 3 \ < \ 4 \ < \ 5 \ < \ \cdots \ < \ 0$$

We have the same entities here, but ordered in a fundamentally different way: our first ordering had no last element; our new ordering does. Indeed, our new ordering consists of an ω-sequence of entities $(1, 2, 3, 4, 5, \ldots)$, followed by another entity. It will be an $\omega + 1$-sequence.

We can generate even more types of ordering, using just these entities. For example, consider all the even numbers (in their natural order) followed by all the odd numbers (in their natural order):

$$0 \ < \ 2 \ < \ 4 \ < \ \cdots \ < \ 1 \ < \ 3 \ < \ \cdots$$

This is an ω-sequence followed by another ω-sequence; an $\omega + \omega$-sequence.

Well, we can keep going. But what we would like is a general way to understand this talk about *orderings*.

3.3 Well-Orderings

The fundamental notion is as follows:

Definition 3.1. The relation $<$ *well-orders* A iff it meets these two conditions:

1. $<$ is connected, i.e., for all $a, b \in A$, either $a < b$ or $a = b$ or $b < a$;

2. every non-empty subset of A has a $<$-minimal element, i.e., if $\emptyset \neq X \subseteq A$ then $(\exists m \in X)(\forall z \in X)z \not< m$

It is easy to see that three examples we just considered were indeed well-ordering relations.

Problem 3.1. Section 3.2 presented three example orderings on the natural numbers. Check that each is a well-ordering.

Here are some elementary but extremely important observations concerning well-ordering.

Proposition 3.2. *If $<$ well-orders A, then every non-empty subset of A has a unique $<$-least member, and $<$ is irreflexive, asymmetric and transitive.*

Proof. If X is a non-empty subset of A, it has a $<$-minimal element m, i.e., $(\forall z \in X)z \not< m$. Since $<$ is connected, $(\forall z \in X)m \leq z$. So m is the $<$-least element of X.

For irreflexivity, fix $a \in A$; the $<$-least element of $\{a\}$ is a, so $a \not< a$. For transitivity, if $a < b < c$, then since $\{a, b, c\}$ has a $<$-least element, $a < c$. Asymmetry follows from irreflexivity and transitivity $\qquad \square$

Proposition 3.3. *If $<$ well-orders A, then for any formula $\varphi(x)$:*

$$\text{if } (\forall a \in A)((\forall b < a)\varphi(b) \to \varphi(a)), \text{ then } (\forall a \in A)\varphi(a).$$

Proof. We will prove the contrapositive. Suppose $\neg(\forall a \in A)\varphi(a)$, i.e., that $X = \{x \in A : \neg\varphi(x)\} \neq \emptyset$. Then X has an $<$-minimal element, a. So $(\forall b < a)\varphi(b)$ but $\neg\varphi(a)$. $\qquad \square$

This last property should remind you of the principle of strong induction on the naturals, i.e.: if $(\forall n \in \omega)((\forall m < n)\varphi(m) \to \varphi(n))$, then $(\forall n \in \omega)\varphi(n)$. And this property makes well-ordering into a very *robust* notion.[1]

3.4 Order-Isomorphisms

To explain *how* robust well-ordering is, we will start by introducing a method for comparing well-orderings.

Definition 3.4. A *well-ordering* is a pair $\langle A, < \rangle$, such that $<$ well-orders A. The well-orderings $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ are *order-isomorphic* iff there is a bijection $f\colon A \to B$ such that: $x < y$ iff $f(x) \lessdot f(y)$. In this case, we write $\langle A, < \rangle \cong \langle B, \lessdot \rangle$, and say that f is an *order-isomorphism*.

In what follows, for brevity, we will speak of "isomorphisms" rather than "order-isomorphisms". Intuitively, isomorphisms are structure-preserving bijections. Here are some simple facts about isomorphisms.

Lemma 3.5. *Compositions of isomorphisms are isomorphisms, i.e.: if $f\colon A \to B$ and $g\colon B \to C$ are isomorphisms, then $(g \circ f)\colon A \to C$ is an isomorphism.*

Problem 3.2. Prove Lemma 3.5.

Proof. Left as an exercise. $\qquad\square$

Corollary 3.6. $X \cong Y$ *is an equivalence relation.*

Proposition 3.7. *If $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ are isomorphic well-orderings, then the isomorphism between them is unique.*

Proof. Let f and g be isomorphisms $A \to B$. We will prove the result by induction, i.e. using Proposition 3.3. Fix $a \in A$, and suppose (for induction) that $(\forall b < a)f(b) = g(b)$. Fix $x \in B$.

If $x \lessdot f(a)$, then $f^{-1}(x) < a$, so $g(f^{-1}(x)) \lessdot g(a)$, invoking the fact that f and g are isomorphisms. But since $f^{-1}(x) < a$, by our supposition $x = f(f^{-1}(x)) = g(f^{-1}(x))$. So $x \lessdot g(a)$. Similarly, if $x \lessdot g(a)$ then $x \lessdot f(a)$.

Generalising, $(\forall x \in B)(x \lessdot f(a) \leftrightarrow x \lessdot g(a))$. It follows that $f(a) = g(a)$ by **??**. So $(\forall a \in A)f(a) = g(a)$ by Proposition 3.3. $\qquad\square$

This gives some sense that well-orderings are robust. But to continue explaining this, it will help to introduce some more notation.

Definition 3.8. When $\langle A, < \rangle$ is a well-ordering with $a \in A$, let $A_a = \{x \in A : x < a\}$. We say that A_a is a proper *initial segment* of A (and allow that A itself is an improper initial segment of A). Let $<_a$ be the restriction of $<$ to the initial segment, i.e., $< \restriction_{A_a^2}$.

---

[1] A reminder: all formulas can have parameters (unless explicitly stated otherwise).

Using this notation, we can state and prove that no well-ordering is isomorphic to any of its proper initial segments.

Lemma 3.9. *If $\langle A, < \rangle$ is a well-ordering with $a \in A$, then $\langle A, < \rangle \not\cong \langle A_a, <_a \rangle$*

Proof. For reductio, suppose $f \colon A \to A_a$ is an isomorphism. Since f is a bijection and $A_a \subsetneq A$, using Proposition 3.2 let $b \in A$ be the $<$-least element of A such that $b \neq f(b)$. We'll show that $(\forall x \in A)(x < b \leftrightarrow x < f(b))$, from which it will follow by **??** that $b = f(b)$, completing the reductio.

Suppose $x < b$. So $x = f(x)$, by the choice of b. And $f(x) < f(b)$, as f is an isomorphism. So $x < f(b)$.

Suppose $x < f(b)$. So $f^{-1}(x) < b$, since f is an isomorphism, and so $f^{-1}(x) = x$ by the choice of b. So $x < b$. $\qquad\square$

Our next result shows, roughly put, that an "initial segment" of an isomorphism is an isomorphism:

Lemma 3.10. *Let $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ be well-orderings. If $f \colon A \to B$ is an isomorphism and $a \in A$, then $f{\restriction}_{A_a} \colon A_a \to B_{f(a)}$ is an isomorphism.*

Proof. Since f is an isomorphism:

$$
\begin{aligned}
f[A_a] &= f[\{x \in A : x < a\}] \\
&= f[\{f^{-1}(y) \in A : f^{-1}(y) < a\}] \\
&= \{y \in B : y \lessdot f(a)\} \\
&= B_{f(a)}
\end{aligned}
$$

And $f{\restriction}_{A_a}$ preserves order because f does. $\qquad\square$

Our next two results establish that well-orderings are always comparable:

Lemma 3.11. *Let $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ be well-orderings. If $\langle A_{a_1}, <_{a_1} \rangle \cong \langle B_{b_1}, \lessdot_{b_1} \rangle$ and $\langle A_{a_2}, <_{a_2} \rangle \cong \langle B_{b_2}, \lessdot_{b_2} \rangle$, then $a_1 < a_2$ iff $b_1 \lessdot b_2$*

Proof. We will prove *left to right*; the other direction is similar. Suppose both $\langle A_{a_1}, <_{a_1} \rangle \cong \langle B_{b_1}, \lessdot_{b_1} \rangle$ and $\langle A_{a_2}, <_{a_2} \rangle \cong \langle B_{b_2}, \lessdot_{b_2} \rangle$, with $f \colon A_{a_2} \to B_{b_2}$ our isomorphism. Let $a_1 < a_2$; then $\langle A_{a_1}, <_{a_1} \rangle \cong \langle B_{f(a_1)}, \lessdot_{f(a_1)} \rangle$ by Lemma 3.10. So $\langle B_{b_1}, \lessdot_{b_1} \rangle \cong \langle B_{f(a_1)}, \lessdot_{f(a_1)} \rangle$, and so $b_1 = f(a_1)$ by Lemma 3.9. Now $b_1 \lessdot b_2$ as f's domain is B_{b_2}. $\qquad\square$

Theorem 3.12. *Given any two well-orderings, one is isomorphic to an initial segment (not necessarily proper) of the other.*

Proof. Let $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ be well-orderings. Using Separation, let

$$
f = \{\langle a, b \rangle \in A \times B : \langle A_a, <_a \rangle \cong \langle B_b, \lessdot_b \rangle\}.
$$

By Lemma 3.11, $a_1 < a_2$ iff $b_1 \lessdot b_2$ for all $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in f$. So $f \colon \mathrm{dom}(f) \to \mathrm{ran}(f)$ is an isomorphism.

If $a_2 \in \mathrm{dom}(f)$ and $a_1 < a_2$, then $a_1 \in \mathrm{dom}(f)$ by Lemma 3.10; so $\mathrm{dom}(f)$ is an initial segment of A. Similarly, $\mathrm{ran}(f)$ is an initial segment of B. For reductio, suppose both are *proper* initial segments. Then let a be the $<$-least element of $A \setminus \mathrm{dom}(f)$, so that $\mathrm{dom}(f) = A_a$, and let b be the $\prec$-least element of $B \setminus \mathrm{ran}(f)$, so that $\mathrm{ran}(f) = B_b$. So $f\colon A_a \to B_b$ is an isomorphism, and hence $\langle a, b \rangle \in f$, a contradiction. $\qquad\square$

3.5 Von Neumann's Construction of the Ordinals

Theorem 3.12 gives rise to a thought. We could introduce certain objects, called *order types*, to go proxy for the well-orderings. Writing $\mathrm{ord}(A, <)$ for the order type of the well-ordering $\langle A, < \rangle$, we would hope to secure the following two principles:

$$\mathrm{ord}(A, <) = \mathrm{ord}(B, \prec) \text{ iff } \langle A, < \rangle \cong \langle B, \prec \rangle$$
$$\mathrm{ord}(A, <) < \mathrm{ord}(B, \prec) \text{ iff } \langle A, < \rangle \cong \langle B_b, \prec_b \rangle \text{ for some } b \in B$$

Moreover, we might hope to introduce order-types *as certain sets*, just as we can introduce the natural numbers as certain sets.

The most common way to do this—and the approach we will follow—is to define these order-types via certain *canonical* well-ordered sets. These canonical sets were first introduced by von Neumann:

Definition 3.13. The set A is *transitive* iff $(\forall x \in A)x \subseteq A$. Then A is an *ordinal* iff A is transitive and well-ordered by $\in$.

In what follows, we will use Greek letters for ordinals. It follows immediately from the definition that, if α is an ordinal, then $\langle \alpha, \in_\alpha \rangle$ is a well-ordering, where $\in_\alpha = \{\langle x, y \rangle \in \alpha^2 : x \in y\}$. So, abusing notation a little, we can just say that α *itself* is a well-ordering.

Here are our first few ordinals:

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$$

You will note that these are the first few ordinals that we encountered in our Axiom of Infinity, i.e., in von Neumann's definition of ω (see section 2.6). This is no coincidence. Von Neumann's definition of the ordinals treats natural numbers as ordinals, but allows for transfinite ordinals too.

As always, we can now ask: *are* these the ordinals? Or has von Neumann simply given us some sets that we can *treat* as the ordinals? The kinds of discussions one might have about this question are similar to the discussions we had in **??**, **??**, **??**, and section 2.8, so we will not belabour the point. Instead, in what follows, we will simply use "the ordinals" to speak of "the von Neumann ordinals".

3.6 Basic Properties of the Ordinals

We observed that the first few ordinals are the natural numbers. The main reason for developing a theory of ordinals is to extend the principle of induction which holds on the natural numbers. We will build up to this via a sequence of elementary results.

Lemma 3.14. *Every element of an ordinal is an ordinal.*

Proof. Let α be an ordinal with $b \in \alpha$. Since α is transitive, $b \subseteq \alpha$. So $\in$ well-orders b as $\in$ well-orders α.

To see that b is transitive, suppose $x \in c \in b$. So $c \in \alpha$ as $b \subseteq \alpha$. Again, as α is transitive, $c \subseteq \alpha$, so that $x \in \alpha$. So $x, c, b \in \alpha$. But $\in$ well-orders α, so that $\in$ is a transitive relation on α by Proposition 3.2. So since $x \in c \in b$, we have $x \in b$. Generalising, $c \subseteq b$ $\qquad\square$

Corollary 3.15. $\alpha = \{\beta \in \alpha : \beta \text{ is an ordinal}\}$, *for any ordinal* α

Proof. Immediate from Lemma 3.14. $\qquad\square$

The rough gist of the next two main results, Theorem 3.16 and Theorem 3.17, is that the ordinals themselves are well-ordered by membership:

Theorem 3.16 (Transfinite Induction). *For any formula* $\varphi(x)$:

$$\text{if } \exists\alpha\varphi(\alpha), \text{ then } \exists\alpha(\varphi(\alpha) \wedge (\forall\beta \in \alpha)\neg\varphi(\beta))$$

where the displayed quantifiers are implicitly restricted to ordinals.

Proof. Suppose $\varphi(\alpha)$, for some ordinal α. If $(\forall\beta \in \alpha)\neg\varphi(\beta)$, then we are done. Otherwise, as α is an ordinal, it has some $\in$-least element which is φ, and this is an ordinal by Lemma 3.14. $\qquad\square$

Note that we can equally express Theorem 3.16 as the scheme:

$$\text{if } \forall\alpha((\forall\beta \in \alpha)\varphi(\beta) \to \varphi(\alpha)), \text{ then } \forall\alpha\varphi(\alpha)$$

just by taking $\neg\varphi(\alpha)$ in Theorem 3.16, and then performing elementary logical manipulations.

Theorem 3.17 (Trichotomy). $\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha$, *for any ordinals* α *and* β.

Proof. The proof is by double induction, i.e., using Theorem 3.16 twice. Say that x is *comparable* with y iff $x \in y \vee x = y \vee y \in x$.

For induction, suppose that every ordinal in α is comparable with *every* ordinal. For further induction, suppose that α is comparable with every ordinal in β. We will show that α is comparable with β. By induction on β, it will follow that α is comparable with every ordinal; and so by induction on α, *every*

ordinal is comparable with *every* ordinal, as required. It suffices to assume that $\alpha \notin \beta$ and $\beta \notin \alpha$, and show that $\alpha = \beta$.

To show that $\alpha \subseteq \beta$, fix $\gamma \in \alpha$; this is an ordinal by Lemma 3.14. So by the first induction hypothesis, γ is comparable with β. But if either $\gamma = \beta$ or $\beta \in \gamma$ then $\beta \in \alpha$ (invoking the fact that α is transitive if necessary), contrary to our assumption; so $\gamma \in \beta$. Generalising, $\alpha \subseteq \beta$.

Exactly similar reasoning, using the second induction hypothesis, shows that $\beta \subseteq \alpha$. So $\alpha = \beta$. $\qquad\square$

As such, we will sometimes write $\alpha < \beta$ rather than $\alpha \in \beta$, since $\in$ is behaving as an ordering relation. There are no deep reasons for this, beyond familiarity, and because it is easier to write $\alpha \le \beta$ than $\alpha \in \beta \vee \alpha = \beta$.[2]

Here are two quick consequences of our last results, the first of which puts our new notation into action:

Corollary 3.18. *If $\exists\alpha\varphi(\alpha)$, then $\exists\alpha(\varphi(\alpha) \wedge \forall\beta(\varphi(\beta) \to \alpha \le \beta))$. Moreover, for any ordinals α, β, γ, both $\alpha \notin \alpha$ and $\alpha \in \beta \in \gamma \to \alpha \in \gamma$.*

Proof. Just like Proposition 3.2. $\qquad\square$

Problem 3.3. Complete the "exactly similar reasoning" in the proof of Theorem 3.17.

Corollary 3.19. *A is an ordinal iff A is a transitive set of ordinals.*

Proof. Left-to-right. By Lemma 3.14. *Right-to-left.* If A is a transitive set of ordinals, then $\in$ well-orders A by Theorem 3.16 and Theorem 3.17. $\qquad\square$

Now, we glossed Theorem 3.16 and Theorem 3.17 as telling us that $\in$ well-orders the ordinals. However, we have to be *very cautious* about this sort of claim, thanks to the following result:

Theorem 3.20 (Burali-Forti Paradox). *There is no set of all the ordinals*

Proof. For reductio, suppose O is the set of all ordinals. If $\alpha \in \beta \in O$, then α is an ordinal, by Lemma 3.14, so $\alpha \in O$. So O is transitive, and hence O is an ordinal by Corollary 3.19. Hence $O \in O$, contradicting Corollary 3.18. $\qquad\square$

This result is named after Burali-Forti. But, it was Cantor in 1899—in a letter to Dedekind—who first saw clearly the *contradiction* in supposing that there is a set of all the ordinals. As van Heijenoort explains:

> Burali-Forti himself considered the contradiction as establishing, by *reductio ad absurdum*, the result that the natural ordering of the ordinals is just a partial ordering. (Heijenoort, 1967, p. 105)

---

[2]We could write $\alpha \underline{\in} \beta$; but that would be wholly non-standard.

Setting Burali-Forti's mistake to one side, we can summarize the foregoing as follows. Ordinals are sets which are individually well-ordered by membership, and collectively well-ordered by membership (without collectively constituting a set).

Rounding this off, here are some more basic properties about the ordinals which follow from Theorem 3.16 and Theorem 3.17.

Proposition 3.21. *Any strictly descending sequence of ordinals is finite.*

Proof. Any infinite strictly descending sequence of ordinals $\alpha_0 > \alpha_1 > \alpha_2 > \ldots$ has no $<$-minimal member, contradicting Theorem 3.16. $\square$

Proposition 3.22. $\alpha \subseteq \beta \vee \beta \subseteq \alpha$, *for any ordinals* α, β.

Proof. If $\alpha \in \beta$, then $\alpha \subseteq \beta$ as β is transitive. Similarly, if $\beta \in \alpha$, then $\beta \subseteq \alpha$. And if $\alpha = \beta$, then $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$. So by Theorem 3.17 we are done. $\square$

Proposition 3.23. $\alpha = \beta$ *iff* $\alpha \cong \beta$, *for any ordinals* α, β.

Proof. The ordinals are well-orders; so this is immediate from Trichotomy (Theorem 3.17) and Lemma 3.9. $\square$

Problem 3.4. Prove that, if every member of X is an ordinal, then $\bigcup X$ is an ordinal.

3.7 Replacement

In section 3.5, we motivated the introduction of ordinals by suggesting that we could treat them as order-types, i.e., canonical proxies for well-orderings. In order for that to work, we would need to prove that *every well-ordering is isomorphic to some ordinal*. This would allow us to define $\mathrm{ord}(A, <)$ as the ordinal α such that $\langle A, < \rangle \cong \alpha$.

Unfortunately, we *cannot* prove the desired result only the Axioms we provided introduced so far. (We will see why in section 5.2, but for now the point is: we can't.) We need a new thought, and here it is:

Axiom (Scheme of Replacement). For any formula $\varphi(x, y)$, the following is an axiom:

for any A, if $(\forall x \in A)\exists! y\, \varphi(x, y)$, then $\{y : (\exists x \in A)\varphi(x, y)\}$ exists.

As with Separation, this is a scheme: it yields infinitely many axioms, for each of the infinitely many different φ's. And it can equally well be (and normally is) written down thus:

For any formula $\varphi(x, y)$ which does not contain "B", the following is an axiom:

$$\forall A[(\forall x \in A)\exists! y\, \varphi(x, y) \to \exists B \forall y(y \in B \leftrightarrow (\exists x \in A)\varphi(x, y))]$$

On first encounter, however, this is quite a tangled formula. The following quick consequence of Replacement probably gives a *clearer* expression to the intuitive idea we are working with:

Corollary 3.24. *For any term $\tau(x)$, and any set A, this set exists:*

$$\{\tau(x) : x \in A\} = \{y : (\exists x \in A)y = \tau(x)\}.$$

Proof. Since τ is a *term*, $\forall x \exists! y\, \tau(x) = y$. A fortiori, $(\forall x \in A)\exists! y\, \tau(x) = y$. So $\{y : (\exists x \in A)\tau(x) = y\}$ exists by Replacement. $\square$

This suggests that "Replacement" is a good name for the Axiom: given a set A, you can form a new set, $\{\tau(x) : x \in A\}$, by replacing every member of A with its image under τ. Indeed, following the notation for the image of a set under a function, we might write $\tau[A]$ for $\{\tau(x) : x \in A\}$.

Crucially, however, τ is a *term*. It need not be (a name for) a *function*, in the sense of **??**, i.e., a certain set of ordered pairs. After all, if f is a function (in that sense), then the set $f[A] = \{f(x) : x \in A\}$ is just a particular subset of $\operatorname{ran}(f)$, and that is already guaranteed to exist, just using the axioms of $\mathbf{Z}^-$.[3] Replacement, by contrast, is a *powerful* addition to our axioms, as we will see in chapter 5.

3.8 $\mathbf{ZF}^-$: a milestone

The question of how to justify Replacement (if at all) is not straightforward. As such, we will reserve that for chapter 5. However, with the addition of Replacement, we have reached another important milestone. We now have all the axioms required for the theory $\mathbf{ZF}^-$. In detail:

Definition 3.25. The theory $\mathbf{ZF}^-$ has these axioms: Extensionality, Union, Pairs, Powersets, Infinity, and all instances of the Separation and Replacement schemes. Otherwise put, $\mathbf{ZF}^-$ adds Replacement to $\mathbf{Z}^-$.

This stands for *Zermelo–Fraenkel* set theory (*minus* something which we will come to later). Fraenkel gets the honour, since he is credited with the formulation of Replacement in 1922, although the first precise formulation was due to Skolem (1922).

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[3] Just consider $\{y \in \bigcup\bigcup f : (\exists x \in A)y = f(x)\}$.

3.9 Ordinals as Order-Types

Armed with Replacement, and so now working in $\mathbf{ZF}^-$, we can finally prove the result we have been aiming for:

Theorem 3.26. *Every well-ordering is isomorphic to a unique ordinal.*

Proof. Let $\langle A, < \rangle$ be a well-order. By Proposition 3.23, it is isomorphic to at most one ordinal. So, for reductio, suppose $\langle A, < \rangle$ is not isomorphic to *any* ordinal. We will first "make $\langle A, < \rangle$ as small as possible". In detail: if some proper initial segment $\langle A_a, <_a \rangle$ is not isomorphic to any ordinal, there is a least $a \in A$ with that property; then let $B = A_a$ and $\prec\; =\; <_a$. Otherwise, let $B = A$ and $\prec\; =\; <$.

By definition, every proper initial segment of B is isomorphic to some ordinal, which is unique as above. So by Replacement, the following set exists, and is a function:

$$f = \{\langle \beta, b \rangle : b \in B \text{ and } \beta \cong \langle B_b, \prec_b \rangle\}$$

To complete the reductio, we'll show that f is an isomorphism $\alpha \to B$, for some ordinal α.

It is obvious that $\mathrm{ran}(f) = B$. And by Lemma 3.11, f preserves ordering, i.e., $\gamma \in \beta$ iff $f(\gamma) \prec f(\beta)$. To show that $\mathrm{dom}(f)$ is an ordinal, by Corollary 3.19 it suffices to show that $\mathrm{dom}(f)$ is transitive. So fix $\beta \in \mathrm{dom}(f)$, i.e., $\beta \cong \langle B_b, \prec_b \rangle$ for some b. If $\gamma \in \beta$, then $\gamma \in \mathrm{dom}(f)$ by Lemma 3.10; generalising, $\beta \subseteq \mathrm{dom}(f)$. $\qquad\square$

This result licenses the following definition, which we have wanted to offer since section 3.5:

Definition 3.27. If $\langle A, < \rangle$ is a well-ordering, then its order type, $\mathrm{ord}(A, <)$, is the unique ordinal α such that $\langle A, < \rangle \cong \alpha$.

Moreover, this definition licenses two nice principles:

Corollary 3.28. *Where $\langle A, < \rangle$ and $\langle B, \prec \rangle$ are well-orderings:*

$$\mathrm{ord}(A, <) = \mathrm{ord}(B, \prec) \text{ iff } \langle A, < \rangle \cong \langle B, \prec \rangle$$
$$\mathrm{ord}(A, <) \in \mathrm{ord}(B, \prec) \text{ iff } \langle A, < \rangle \cong \langle B_b, \prec_b \rangle \text{ for some } b \in B$$

Proof. The identity holds by Proposition 3.23. To prove the second claim, let $\mathrm{ord}(A, <) = \alpha$ and $\mathrm{ord}(B, \prec) = \beta$, and let $f \colon \beta \to \langle B, \prec \rangle$ be our isomorphism. Then:

$$\alpha \in \beta \text{ iff } f{\restriction}_\alpha \colon \alpha \to B_{f(\alpha)} \text{ is an isomorphism}$$
$$\text{iff } \langle A, < \rangle \cong \langle B_{f(\alpha)}, \prec_{f(\alpha)} \rangle$$
$$\text{iff } \langle A, < \rangle \cong \langle B_b, \prec_b \rangle \text{ for some } b \in B$$

by Proposition 3.7, Lemma 3.10, and Corollary 3.15. $\qquad\square$

3.10 Successor and Limit Ordinals

In the next few chapters, we will use ordinals a great deal. So it will help if we introduce some simple notions.

Definition 3.29. For any ordinal α, its *successor* is $\alpha^+ = \alpha \cup \{\alpha\}$. We say that α is a *successor* ordinal if $\beta^+ = \alpha$ for some ordinal β. We say that α is a *limit* ordinal iff α is neither empty nor a successor ordinal.

The following result shows that this is the right notion of *successor*:

Proposition 3.30. *For any ordinal α:*

1. *$\alpha \in \alpha^+$;*

2. *α^+ is an ordinal;*

3. *there is no ordinal β such that $\alpha \in \beta \in \alpha^+$.*

Proof. Trivially, $\alpha \in \alpha \cup \{\alpha\} = \alpha^+$. Equally, α^+ is a transitive set of ordinals, and hence an ordinal by Corollary 3.19. And it is impossible that $\alpha \in \beta \in \alpha^+$, since then either $\beta \in \alpha$ or $\beta = \alpha$, contradicting Corollary 3.18. $\qquad\square$

This also licenses a variant of proof by transfinite induction:

Theorem 3.31 (Simple Transfinite Induction). *Let $\varphi(x)$ be a formula such that:*

1. *$\varphi(\emptyset)$; and*

2. *for any ordinal α, if $\varphi(\alpha)$ then $\varphi(\alpha^+)$; and*

3. *if α is a limit ordinal and $(\forall\beta \in \alpha)\varphi(\beta)$, then $\varphi(\alpha)$.*

Then $\forall\alpha\varphi(\alpha)$.

Proof. We prove the contrapositive. So, suppose there is some ordinal which is $\neg\varphi$; let γ be the least such ordinal. Then either $\gamma = \emptyset$, or $\gamma = \alpha^+$ for some α such that $\varphi(\alpha)$; or γ is a limit ordinal and $(\forall\beta \in \gamma)\varphi(\beta)$. $\qquad\square$

A final bit of notation will prove helpful later on:

Definition 3.32. If X is a set of ordinals, then $\mathrm{lsub}(X) = \bigcup_{\alpha \in X}\alpha^+$.

Here, "lsub" stands for "least strict upper bound".[4] The following result explains this:

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[4]Some books use "$\sup(X)$" for this. But other books use "$\sup(X)$" for the least *non-strict* upper bound, i.e., simply $\bigcup X$. If X has a greatest element, α, these notions come apart: the least *strict* upper bound is α^+, whereas the least *non-strict* upper bound is just α.

Proposition 3.33. *If X is a set of ordinals, $\mathrm{lsub}(X)$ is the least ordinal greater than every ordinal in X.*

Proof. Let $Y = \{\alpha^+ : \alpha \in X\}$, so that $\mathrm{lsub}(X) = \bigcup Y$. Since ordinals are transitive and every member of an ordinal is an ordinal, $\mathrm{lsub}(X)$ is a transitive set of ordinals, and so is an ordinal by Corollary 3.19.

If $\alpha \in X$, then $\alpha^+ \in Y$, so $\alpha^+ \subseteq \bigcup Y = \mathrm{lsub}(X)$, and hence $\alpha \in \mathrm{lsub}(X)$. So $\mathrm{lsub}(X)$ is strictly greater than every ordinal in X.

Conversely, if $\alpha \in \mathrm{lsub}(X)$, then $\alpha \in \beta^+ \in Y$ for some $\beta \in X$, so that $\alpha \leq \beta \in X$. So $\mathrm{lsub}(X)$ is the *least* strict upper bound on X. $\qquad\square$

Chapter 4

Stages and Ranks

4.1 Defining the Stages as the V_αs

In , we defined well-orderings and the (von Neumann) ordinals. In this chapter, we will use these to characterise the hierarchy of sets *itself*. To do this, recall that in , we defined the idea of successor and limit ordinals. We use these ideas in following definition:

Definition 4.1.

$$V_\emptyset = \emptyset$$
$$V_{\alpha^+} = \wp(V_\alpha) \qquad \text{for any ordinal } \alpha$$
$$V_\alpha = \bigcup_{\gamma < \alpha} V_\gamma \qquad \text{when } \alpha \text{ is a limit ordinal}$$

This will be a definition by *transfinite recursion* on the ordinals. In this regard, we should compare this with recursive definitions of functions on the natural numbers.[1] As when dealing with natural numbers, one defines a base case and successor cases; but when dealing with ordinals, we also need to describe the behaviour of *limit* cases.

This definition of the V_αs will be an important milestone. We have informally motivated our hierarchy of sets as forming sets by *stages*. The V_αs are, in effect, just those stages. Importantly, though, this is an *internal* characterisation of the stages. Rather than suggesting a possible *model* of the theory, we will have defined the stages *within* our set theory.

4.2 The Transfinite Recursion Theorem(s)

The first thing we must do, though, is confirm that is a successful definition. More generally, we need to prove that any attempt to offer

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[1]Cf. the definitions of addition, multiplication, and exponentiation in **??**.

a transfinite by (transfinite) recursion will succeed. That is the aim of this section.

Warning: this is tricky material. The overarching moral, though, is quite simple: Transfinite Induction plus Replacement guarantee the legitimacy of (several versions of) transfinite recursion.[2]

Definition 4.2. Let $\tau(x)$ be a term; let f be a function; let α be an ordinal. We say that f is an α-*approximation* for τ iff both $\mathrm{dom}(f) = \alpha$ and $(\forall \beta \in \alpha) f(\beta) = \tau(f{\restriction}_\beta)$.

Lemma 4.3 (Bounded Recursion). *For any term $\tau(x)$ and any ordinal α, there is a unique α-approximation for τ.*

Proof. We will show that, for any $\gamma \leq \alpha$, there is a unique γ-approximation.

We first establish uniqueness. Let g and h (respectively) be γ- and δ-approximations. A transfinite induction on their arguments shows that $g(\beta) = h(\beta)$ for any $\beta \in \mathrm{dom}(g) \cap \mathrm{dom}(h) = \gamma \cap \delta = \min(\gamma, \delta)$. So our approximations are unique (if they exist), and agree on all values.

To establish existence, we now use a simple transfinite induction (Theorem 3.31) on ordinals $\delta \leq \alpha$.

The empty function is trivially an $\emptyset$-approximation.

If g is a γ-approximation, then $g \cup \{\langle \gamma^+, \tau(g) \rangle\}$ is a γ^+-approximation.

If γ is a limit ordinal and g_δ is a δ-approximation for all $\delta < \gamma$, let $g = \bigcup_{\delta \in \gamma} g_\delta$. This is a function, since our various g_δs agree on all values. And if $\delta \in \gamma$ then $g(\delta) = g_{\delta^+}(\delta) = \tau(g_{\delta^+}{\restriction}_\delta) = \tau(g{\restriction}_\delta)$.

This completes the proof by transfinite induction. $\square$

If we allow ourselves to define a *term* rather than a function, then we can remove the bound α from the previous result. In the statement and proof of the following result, when σ is a term, we let $\sigma{\restriction}_\alpha = \{\langle \beta, \sigma(\beta) \rangle : \beta \in \alpha\}$.

Theorem 4.4 (General Recursion). *For any term $\tau(x)$, we can explicitly define a term $\sigma(x)$, such that $\sigma(\alpha) = \tau(\sigma{\restriction}_\alpha)$ for any ordinal α.*

Proof. For each α, by Lemma 4.3 there is a unique α-approximation, f_α, for τ. Define $\sigma(\alpha)$ as $f_{\alpha^+}(\alpha)$. Now:

$$
\begin{aligned}
\sigma(\alpha) &= f_{\alpha^+}(\alpha) \\
&= \tau(f_{\alpha^+}{\restriction}_\alpha) \\
&= \tau(\{\langle \beta, f_{\alpha^+}(\beta) \rangle : \beta \in \alpha\}) \\
&= \tau(\{\langle \beta, f_\alpha(\beta) \rangle : \beta \in \alpha\}) \\
&= \tau(\sigma{\restriction}_\alpha)
\end{aligned}
$$

noting that $f_\alpha(\beta) = f_{\alpha^+}(\beta)$ for all $\beta < \alpha$, as in Lemma 4.3. $\square$

---

[2]A reminder: all formulas and terms can have parameters (unless explicitly stated otherwise).

Note that Theorem 4.4 is a *schema*. Crucially, we cannot expect σ to define a function, i.e., a certain kind of *set*, since then $\mathrm{dom}(\sigma)$ would be the set of all ordinals, contradicting the Burali-Forti Paradox (Theorem 3.20).

It still remains to show, though, that Theorem 4.4 vindicates our definition of the V_αs. This may not be immediately obvious; but it will become apparent with a last, simple, version of transfinite recursion.

Theorem 4.5 (Simple Recursion). *For any terms $\tau(x)$ and $\theta(x)$ and any set A, we can explicitly define a term $\sigma(x)$ such that:*

$$\sigma(\emptyset) = A$$
$$\sigma(\alpha^+) = \tau(\sigma(\alpha)) \qquad \textit{for any ordinal } \alpha$$
$$\sigma(\alpha) = \theta(\mathrm{ran}(\sigma{\restriction}_\alpha)) \qquad \textit{when } \alpha \textit{ is a limit ordinal}$$

Proof. We start by defining a term, $\xi(x)$, as follows:

$$\xi(x) = \begin{cases} A & \text{if } x \text{ is not a function whose} \\ & \quad \text{domain is an ordinal; otherwise:} \\ \tau(x(\alpha)) & \text{if } \mathrm{dom}(x) = \alpha^+ \\ \theta(\mathrm{ran}(x)) & \text{if } \mathrm{dom}(x) \text{ is a limit ordinal} \end{cases}$$

By Theorem 4.4, there is a term $\sigma(x)$ such that $\sigma(\alpha) = \xi(\sigma{\restriction}_\alpha)$ for every ordinal α; moreover, $\sigma{\restriction}_\alpha$ is a function with domain α. We show that σ has the required properties, by simple transfinite induction (Theorem 3.31).

First, $\sigma(\emptyset) = \xi(\emptyset) = A$.
Next, $\sigma(\alpha^+) = \xi(\sigma{\restriction}_{\alpha^+}) = \tau(\sigma{\restriction}_{\alpha^+}(\alpha)) = \tau(\sigma(\alpha))$.
Last, $\sigma(\alpha) = \xi(\sigma{\restriction}_\alpha) = \theta(\mathrm{ran}(\sigma{\restriction}_\alpha))$, when α is a limit. $\qquad\square$

Now, to vindicate Definition 4.1, just take $A = \emptyset$ and $\tau(x) = \wp(x)$ and $\theta(x) = \bigcup x$. At long last, this vindicates the definition of the V_αs!

4.3 Basic Properties of Stages

To bring out the foundational importance of the definition of the V_αs, we will present a few basic results about them. We start with a definition:[3]

Definition 4.6. The set A is *potent* iff $\forall x((\exists y \in A)x \subseteq y \rightarrow x \in A)$.

Lemma 4.7. *For each ordinal α:*

1. *Each V_α is transitive.*

2. *Each V_α is potent.*

3. *If $\gamma \in \alpha$, then $V_\gamma \in V_\alpha$ (and hence also $V_\gamma \subseteq V_\alpha$ by (1))*

---

[3]There's no standard terminology for "potent"; this is the name used by Button (forthcoming).

Proof. We prove this by a (simultaneous) transfinite induction. For induction, suppose that (1)–(3) holds for each ordinal $\beta < \alpha$.

The case of $\alpha = \emptyset$ is trivial.

Suppose $\alpha = \beta^+$. To show (3), if $\gamma \in \alpha$ then $V_\gamma \subseteq V_\beta$ by hypothesis, so $V_\gamma \in \wp(V_\beta) = V_\alpha$. To show (2), suppose $A \subseteq B \in V_\alpha$ i.e., $A \subseteq B \subseteq V_\beta$; then $A \subseteq V_\beta$ so $A \in V_\alpha$. To show (1), note that if $x \in A \in V_\alpha$ we have $A \subseteq V_\beta$, so $x \in V_\beta$, so $x \subseteq V_\beta$ as V_β is transitive by hypothesis, and so $x \in V_\alpha$.

Suppose α is a limit ordinal. To show (3), if $\gamma \in \alpha$ then $\gamma \in \gamma^+ \in \alpha$, so that $V_\gamma \in V_{\gamma^+}$ by assumption, hence $V_\gamma \in \bigcup_{\beta \in \alpha} V_\beta = V_\alpha$. To show (1) and (2), just observe that a union of transitive (respectively, potent) sets is transitive (respectively, potent). $\qquad\square$

Lemma 4.8. *For each ordinal α, $V_\alpha \notin V_\alpha$.*

Proof. By transfinite induction. Evidently $V_\emptyset \notin V_\emptyset$.

If $V_{\alpha^+} \in V_{\alpha^+} = \wp(V_\alpha)$, then $V_{\alpha^+} \subseteq V_\alpha$; and since $V_\alpha \in V_{\alpha^+}$ by Lemma 4.7, we have $V_\alpha \in V_\alpha$. Conversely: if $V_\alpha \notin V_\alpha$ then $V_{\alpha^+} \notin V_{\alpha^+}$

If α is a limit and $V_\alpha \in V_\alpha = \bigcup_{\beta \in \alpha} V_\beta$, then $V_\alpha \in V_\beta$ for some $\beta \in \alpha$; but then also $V_\beta \in V_\alpha$ so that $V_\beta \in V_\beta$ by Lemma 4.7 (twice). Conversely, if $V_\beta \notin V_\beta$ for all $\beta \in \alpha$, then $V_\alpha \notin V_\alpha$. $\qquad\square$

Corollary 4.9. *For any ordinals α, β: $\alpha \in \beta$ iff $V_\alpha \in V_\beta$*

Proof. Lemma 4.7 gives one direction. Conversely, suppose $V_\alpha \in V_\beta$. Then $\alpha \neq \beta$ by Lemma 4.8; and $\beta \notin \alpha$, for otherwise we would have $V_\beta \in V_\alpha$ and hence $V_\beta \in V_\beta$ by Lemma 4.7 (twice), contradicting Lemma 4.8. So $\alpha \in \beta$ by Trichotomy. $\qquad\square$

All of this allows us to think of each V_α as the αth stage of the hierarchy. Here is why.

Certainly our V_αs can be thought of as being formed in an *iterative* process, for our use of ordinals tracks the notion of iteration. Moreover, if one stage is formed before the other, i.e., $V_\beta \in V_\alpha$, i.e., $\beta \in \alpha$, then our process of formation is *cumulative*, since $V_\beta \subseteq V_\alpha$. Finally, we are indeed forming *all* possible collections of sets that were available at any earlier stage, since any successor stage V_{α^+} is the power-set of its predecessor V_α.

In short: with $\mathbf{ZF}^-$, we are *almost* done, in articulating our vision of the cumulative-iterative hierarchy of sets. (Though, of course, we still need to justify Replacement.)

4.4 Foundation

We are only *almost* done—and not *quite* finished—because nothing in $\mathbf{ZF}^-$ guarantees that *every* set is in some V_α, i.e., that every set is formed at some stage.

Now, there is a fairly straightforward (mathematical) sense in which we don't *care* whether there are sets outside the hierarchy. (If there are any there, we can simply ignore them.) But we have motivated our *concept* of set with the thought that every set is formed at some stage (see *Stages-are-key* in section 2.1). So we will want to preclude the possibility of sets which fall outside of the hierarchy. Accordingly, we must add a new axiom, which ensures that every set occurs somewhere in the hierarchy.

Since the V_αs are our stages, we might simply consider adding the following as an axiom:

Regularity. $\forall A \exists \alpha\, A \subseteq V_\alpha$

This would be a perfectly reasonable approach. However, for reasons that will be explained in the next section, we will instead adopt an alternative axiom:

Axiom (Foundation). $(\forall A \neq \emptyset)(\exists B \in A)A \cap B = \emptyset$.

With some effort, we can show (in $\mathbf{ZF}^-$) that Foundation entails Regularity:

Definition 4.10. For each set A, let:

$$\mathrm{cl}_0(A) = A,$$
$$\mathrm{cl}_{n+1}(A) = \bigcup \mathrm{cl}_n(A),$$
$$\mathrm{trcl}(A) = \bigcup_{n < \omega} \mathrm{cl}_n(A).$$

We call $\mathrm{trcl}(A)$ the *transitive closure* of A.

The name "transitive closure" is apt:

Proposition 4.11. $A \subseteq \mathrm{trcl}(A)$ *and* $\mathrm{trcl}(A)$ *is a transitive set.*   *sth:spine:foundation: subsetoftrcl*

Proof. Evidently $A = \mathrm{cl}_0(A) \subseteq \mathrm{trcl}(A)$. And if $x \in b \in \mathrm{trcl}(A)$, then $b \in \mathrm{cl}_n(A)$ for some n, so $x \in \mathrm{cl}_{n+1}(A) \subseteq \mathrm{trcl}(A)$.   □

Lemma 4.12. *If A is a transitive set, then there is some α such that $A \subseteq V_\alpha$.*   *sth:spine:foundation: lem:TransitiveWellFounded*

Proof. Recalling the definition of "$\mathrm{lsub}(X)$" from Definition 3.32, define two sets:

$$D = \{x \in A : \forall \delta\, x \nsubseteq V_\delta\}$$
$$\alpha = \mathrm{lsub}\{\delta : (\exists x \in A)(x \subseteq V_\delta \wedge (\forall \gamma \in \delta)x \nsubseteq V_\gamma)\}$$

Suppose $D = \emptyset$. So if $x \in A$, then there is some δ such that $x \subseteq V_\delta$ and, by the well-ordering of the ordinals, $(\forall \gamma \in \delta)x \nsubseteq V_\gamma$; hence $\delta \in \alpha$ and so $x \in V_\alpha$ by Lemma 4.7. Hence $A \subseteq V_\alpha$, as required.

So it suffices to show that $D = \emptyset$. For reductio, suppose otherwise. By Foundation, there is some $B \in D \subseteq A$ such that $D \cap B = \emptyset$. If $x \in B$ then $x \in A$, since A is transitive, and since $x \notin D$, it follows that $\exists \delta \; x \subseteq V_\delta$. So now let

$$\beta = \mathrm{lsub}\{\delta : (\exists x \in b)(x \subseteq V_\delta \wedge (\forall \gamma < \delta)x \nsubseteq V_\gamma)\}.$$

As before, $B \subseteq V_\beta$, contradicting the claim that $B \in D$. $\qquad\square$

Theorem 4.13. *Regularity holds.*

Proof. Fix A; now $A \subseteq \mathrm{trcl}(A)$ by Proposition 4.11, which is transitive. So there is some α such that $A \subseteq \mathrm{trcl}(A) \subseteq V_\alpha$ by Lemma 4.12 $\qquad\square$

These results show that $\mathbf{ZF}^-$ proves the conditional *Foundation $\Rightarrow$ Regularity*. In Proposition 4.22, we will show that $\mathbf{ZF}^-$ proves *Regularity $\Rightarrow$ Foundation*. As such, Foundation and Regularity are *equivalent* (modulo $\mathbf{ZF}^-$). But this means that, given $\mathbf{ZF}^-$, we can justify Foundation by noting that it is equivalent to Regularity. And we can justify Regularity immediately on the basis of *Stages-are-key*.

4.5 Z and ZF: A Milestone

With Foundation, we reach another important milestone. We have considered theories $\mathbf{Z}^-$ and $\mathbf{ZF}^-$, which we said were certain theories "minus" a certain something. That certain something is Foundation. So:

Definition 4.14. The theory $\mathbf{Z}$ adds Foundation to $\mathbf{Z}^-$. So its axioms are Extensionality, Union, Pairs, Powersets, Infinity, Foundation, and all instances of the Separation scheme.

The theory $\mathbf{ZF}$ adds Foundation to $\mathbf{ZF}^-$. Otherwise put, $\mathbf{ZF}$ adds all instances of Replacement to $\mathbf{Z}$.

Still, one question might have occurred to you. If Regularity is equivalent over $\mathbf{ZF}^-$ to Foundation, and Regularity's justification is clear, why bother to go around the houses, and take Foundation as our basic axiom, rather than Regularity?

Setting aside historical reasons (to do with who formulated what and when), the basic reason is that Foundation can be presented without employing the definition of the V_αs. That definition relied upon all of the work of section 4.2: we needed to prove Transfinite Recursion, to show that it was justified. But our proof of Transfinite Recursion employed *Replacement*. So, whilst Foundation and Regularity are equivalent modulo $\mathbf{ZF}^-$, they are not equivalent modulo $\mathbf{Z}^-$.

Indeed, the matter is more drastic than this simple remark suggests. Though it goes well beyond this book's remit, it turns out that both $\mathbf{Z}^-$ and $\mathbf{Z}$ are too weak to define the V_αs. So, if you are working only in $\mathbf{Z}$, then Regularity (as we

have formulated it) does not even make *sense*. This is why our official axiom is Foundation, rather than Regularity.

From now on, we will work in **ZF** (unless otherwise stated), without any further comment.

4.6 Rank

Now that we have defined the stages as the V_α's, and we know that every set is a subset of some stage, we can define the *rank* of a set. Intuitively, the rank of A is the first moment at which A is formed. More precisely:

Definition 4.15. For each set A, $\mathrm{rank}(A)$ is the least ordinal α such that $A \subseteq V_\alpha$.

Proposition 4.16. $\mathrm{rank}(A)$ *exists, for any A.*

Proof. Left as an exercise. $\square$

Problem 4.1. Prove Proposition 4.16.

The well-ordering of ranks allows us to prove some important results:

Proposition 4.17. *For any ordinal α, $V_\alpha = \{x : \mathrm{rank}(x) \in \alpha\}$.*

Proof. If $\mathrm{rank}(x) \in \alpha$ then $x \subseteq V_{\mathrm{rank}(x)} \in V_\alpha$, so $x \in V_\alpha$ as V_α is potent (invoking Lemma 4.7 multiple times). Conversely, if $x \in V_\alpha$ then $x \subseteq V_\alpha$, so $\mathrm{rank}(x) \leq \alpha$; now a simple transfinite induction shows that $x \notin V_\alpha$. $\square$

Problem 4.2. Complete the simple transfinite induction mentioned in Proposition 4.17.

Proposition 4.18. *If $B \in A$, then $\mathrm{rank}(B) \in \mathrm{rank}(A)$.*

Proof. $A \subseteq V_{\mathrm{rank}(A)} = \{x : \mathrm{rank}(x) \in \mathrm{rank}(A)\}$ by Proposition 4.17. $\square$

Using this fact, we can establish a result which allows us to prove things about *all sets* by a form of induction:

Theorem 4.19 ($\in$-Induction Scheme). *For any formula φ:*

$$\forall A((\forall x \in A)\varphi(x) \to \varphi(A)) \to \forall A\varphi(A).$$

Proof. We will prove the contrapositive. So, suppose $\neg\forall A\varphi(A)$. By Transfinite Induction (Theorem 3.16), there is some non-φ of least possible rank; i.e. some A such that $\neg\varphi(A)$ and $\forall x(\mathrm{rank}(x) \in \mathrm{rank}(A) \to \varphi(x))$. Now if $x \in A$ then $\mathrm{rank}(x) \in \mathrm{rank}(A)$, by Proposition 4.18, so that $\varphi(x)$; i.e. $(\forall x \in A)\varphi(x) \land \neg\varphi(A)$. $\square$

Here is an informal way to gloss this powerful result. Say that φ is *hereditary* iff whenever every element of a set is φ, the set itself is φ. Then $\in$-Induction tells you the following: if φ is hereditary, every set is φ.

To wrap up the discussion of ranks (for now), we'll prove a few claims which we have foreshadowed a few times.

Proposition 4.20. $\operatorname{rank}(A) = \operatorname{lsub}_{x \in A} \operatorname{rank}(x)$.

Proof. Let $\alpha = \operatorname{lsub}_{x \in A} \operatorname{rank}(x)$. By Proposition 4.18, $\alpha \leq \operatorname{rank}(A)$. But if $x \in A$ then $\operatorname{rank}(x) \in \alpha$, so that $x \in V_\alpha$ by Proposition 4.17, and hence $A \subseteq V_\alpha$, i.e., $\operatorname{rank}(A) \leq \alpha$. Hence $\operatorname{rank}(A) = \alpha$. $\qquad\square$

Corollary 4.21.   *For any ordinal α, $\operatorname{rank}(\alpha) = \alpha$.*

Proof. Suppose for transfinite induction that $\operatorname{rank}(\beta) = \beta$ for all $\beta \in \alpha$. Now $\operatorname{rank}(\alpha) = \operatorname{lsub}_{\beta \in \alpha} \operatorname{rank}(\beta) = \operatorname{lsub}_{\beta \in \alpha} \beta = \alpha$ by Proposition 4.20. $\qquad\square$

Finally, here is a quick proof of the result promised at the end of section 4.4, that $\mathbf{ZF}^-$ proves the conditional *Regularity $\Rightarrow$ Foundation*. (Note that the notion of "rank" and Proposition 4.18 are available for use in this proof since—as mentioned at the start of this section—they can be presented using $\mathbf{ZF}^- +$ Regularity.)

Proposition 4.22 (working in $\mathbf{ZF}^- +$ Regularity).   *Foundation holds.*

Proof. Fix $A \neq \emptyset$, and some $B \in A$ of least possible rank. If $c \in B$ then $\operatorname{rank}(c) \in \operatorname{rank}(B)$ by Proposition 4.18, so that $c \notin A$ by choice of B. $\qquad\square$

Chapter 5

Replacement

5.1 Introduction

Replacement is the axiom scheme which makes the difference between **ZF** and **Z**. We helped ourselves to it throughout chapters 3 to 4. In this chapter, we will finally consider the question: is Replacement justified?

To make the question sharp, it is worth observing that Replacement is really rather *strong*. We will get a sense of just how strong it is, during this chapter (and again in section 8.5). But this will suggest that justification really is required.

We will discuss two kinds of justification. Roughly: an *extrinsic* justification is an attempt to justify an axiom by its fruits; an *intrinsic* justification is an attempt to justify an axiom by suggesting that it is vindicated by the mathematical concepts in question. We will get a greater sense of what this means during this chapter, but it is just the tip of an iceberg. For more, see in particular Maddy (1988a and 1988b).

5.2 The Strength of Replacement

We begin with a simple observation about the strength of Replacement: unless we go beyond **Z**, we cannot prove the existence of any von Neumann ordinal greater than or equal to $\omega + \omega$.

Here is a sketch of why. Working in **ZF**, consider the set $V_{\omega+\omega}$. This set acts as the domain for a *model* for **Z**. To see this, we introduce some notation for the *relativization* of a formula:

Definition 5.1. For any set M, and any formula φ, let φ^M be the formula which results by restricting all of φ's quantifiers to M. That is, replace "$\exists x$" with "$(\exists x \in M)$", and replace "$\forall x$" with "$(\forall x \in M)$".

It can be shown that, for every axiom φ of **Z**, we have that $\mathbf{ZF} \vdash \varphi^{V_{\omega+\omega}}$. But $\omega + \omega$ is not *in* $V_{\omega+\omega}$, by Corollary 4.21. So **Z** is consistent with the non-existence of $\omega + \omega$.

This is why we said, in section 3.7, that Theorem 3.26 cannot be proved without Replacement. For it is easy, within **Z**, to define an explicit well-ordering which intuitively *should* have order-type $\omega + \omega$. Indeed, we gave an informal example of this in section 3.2, when we presented the ordering on the natural numbers given by:

$$n \lessdot m \text{ iff either } n < m \text{ and } m - n \text{ is even,}$$
$$\text{or } n \text{ is even and } m \text{ is odd.}$$

But if $\omega + \omega$ does not exist, this well-ordering is not isomorphic to any ordinal. So **Z** does *not* prove Theorem 3.26.

Flipping things around: Replacement allows us to prove the existence of $\omega + \omega$, and hence must allow us to prove the existence of $V_{\omega + \omega}$. And not just that. For *any* well-ordering we can define, Theorem 3.26 tells us that there is some α isomorphic with that well-ordering, and hence that V_α exists. In a straightforward way, then, Replacement guarantees that the hierarchy of sets must be *very tall*.

Over the next few sections, and then again in section 8.5, we'll get a better sense of better just *how* tall Replacement forces the hierarchy to be. The simple point, for now, is that Replacement really *does* stand in need of justification!

5.3 Extrinsic Considerations about Replacement

We start by considering an *extrinsic* attempt to justify Replacement. Boolos suggests one, as follows.

> [...] the reason for adopting the axioms of replacement is quite simple: they have many desirable consequences and (apparently) no undesirable ones. In addition to theorems about the iterative conception, the consequences include a satisfactory if not ideal theory of infinite numbers, and a highly desirable result that justifies inductive definitions on well-founded relations. (Boolos, 1971, 229)

The gist of Boolos's idea is that we should justify Replacement by its fruits. And the specific fruits he mentions are the things we have discussed in the past few chapters. Replacement allowed us to prove that the von Neumann ordinals were excellent surrogates for the idea of a well-ordering type (this is our "satisfactory if not ideal theory of infinite numbers"). Replacement also allowed us to define the V_αs, establish the notion of rank, and prove $\in$-Induction (this amounts to our "theorems about the iterative conception"). Finally, Replacement allows us to prove the Transfinite Recursion Theorem (this is the "inductive definitions on well-founded relations").

These are, indeed, desirable consequences. But do these desirable consequences suffice to *justify* Replacement? *No.* Or at least, not straightforwardly.

Here is a simple problem. Whilst we have stated some desirable consequences of Replacement, we could have obtained many of them via other means.

This is not as well known as it ought to be, though, so we should pause to explain the situation.

There is a simple theory of sets, Level Theory, or **LT** for short.[1] **LT**'s axioms are just Extensionality, Separation, and the claim that every set is a subset of some *level*, where "level" is cunningly defined so that the levels behave like our friends, the V_αs. So **ZF** proves **LT**; but **LT** is *much* weaker than **ZF**. In fact, **LT** does not give you Pairs, Powersets, Infinity, or Replacement. Let **Zr** be the result of adding Infinity and Powersets to **LT**; this delivers Pairs too, so, **Zr** is at least as strong as **Z**. But, in fact, **Zr** is strictly stronger than **Z**, since it adds the claim that every set has a rank (hence my suggestion that we call it **Zr**). Indeed, **Zr** delivers: a perfectly satisfactory theory of ordinals; results which stratify the hierarchy into well-ordered stages; a proof of ∈-Induction; and a *version* of Transfinite Recursion.

In short: although Boolos didn't know this, all of the desirable consequences which he mentions could have been arrived at *without* Replacement; he simply needed to use **Zr** rather than **Z**.

(Given all of this, why did we follow the conventional route, of teaching you **ZF**, rather than **LT** and **Zr**? There are two reasons. First: for purely historical reasons, starting with **LT** is rather nonstandard; we wanted to equip you to be able to read more standard discussions of set theory. Second: when you are ready to appreciate **LT** and **Zr**, you can simply read Potter 2004 and Button forthcoming.)

Of course, since **Zr** is strictly weaker than **ZF**, there are results which **ZF** proves which **Zr** leaves open. So one could try to justify Replacement on extrinsic grounds by pointing to one of these results. But, once you know how to use **Zr**, it is quite hard to find many examples of things that are (a) settled by Replacement but not otherwise, and (b) are intuitively true. (For more on this, see Potter 2004, §13.2.)

The bottom line is this. To provide a compelling extrinsic justification for Replacement, one would need to find a result which *cannot* be achieved without Replacement. And that's not an easy enterprise.

Let's consider a further problem which arises for any attempt to offer a purely extrinsic justification for Replacement. (This problem is perhaps more fundamental than the first.) Boolos does not just point out that Replacement has many desirable consequences. He also states that Replacement has "(apparently) no undesirable" consequences. But this parenthetical caveat, "apparently," is surely absolutely crucial.

Recall how we ended up here: Naïve Comprehension ran into inconsistency, and we responded to this inconsistency by embracing the cumulative-iterative conception of set. This conception comes equipped with a story which, we hope, assures us of its consistency. But if we cannot justify Replacement from within that story, then we have (as yet) no reason to believe that **ZF** is consistent.

---

[1]The first versions of **LT** are offered by Montague (1965) and Scott (1974); this was simplified, and given a book-length treatment, by Potter (2004); and Button (forthcoming) has recently simplified **LT** further.

Or rather: we have no reason to believe that **ZF** is consistent, apart from the (perhaps merely contingent) fact that no one has discovered a contradiction *yet*. In exactly that sense, Boolos's comment seems to come down to this: "(apparently) **ZF** is consistent". We should demand greater reassurance of consistency than this.

This issue will affect any *purely* extrinsic attempt to justify Replacement, i.e., any justification which is couched solely in terms of the (known) consequences of **ZF**. As such, we will want to look for an *intrinsic* justification of Replacement, i.e., a justification which suggests that the story which we told about sets somehow "already" commits us to Replacement.

5.4 Limitation-of-size

Perhaps the most common attempt to offer an "intrinsic" justification of Replacement comes via the following notion:

> *Limitation-of-size.* Any things form a set, provided that there are not too many of them.

This principle will immediately vindicate Replacement. After all, any set formed by Replacement cannot be any larger than any set from which it was formed. Stated precisely: suppose you form a set $\tau[A] = \{\tau(x) : x \in A\}$ using Replacement; then $\tau[A] \preceq A$; so if the elements of A were not too numerous to form a set, their images are not too numerous to form $\tau[A]$.

The obvious difficulty with invoking *Limitation-of-size* to justify Replacement is that we have *not* yet laid down any principle like *Limitation-of-size*. Moreover, when we told our story about the cumulative-iterative conception of set in chapters 1 to 2, nothing ever *hinted* in the direction of *Limitation-of-size*. This, indeed, is precisely why Boolos at one point wrote: "Perhaps one may conclude that there are at least two thoughts 'behind' set theory" (1989, p. 19). On the one hand, the ideas surrounding the cumulative-iterative conception of set are meant to vindicate **Z**. On the other hand, *Limitation-of-size* is meant to vindicate Replacement.

But the issue it is not just that we have thus far been *silent* about *Limitation-of-size*. Rather, the issue is that *Limitation-of-size* (as just formulated) seems to sit quite badly with the cumulative-iterative notion of set. After all, it mentions nothing about the idea of sets as formed in *stages*.

This is really not much of a surprise, given the history of these "two thoughts" (i.e., the cumulative-iterative conception of set, and *Limitation-of-size*). These "two thoughts" ultimately amount to two rather different projects for blocking the set-theoretic paradoxes. The cumulative-iterative notion of set blocks Russell's paradox by saying, roughly: *we should never have expected a Russell set to exist, because it would not be "formed" at any stage.* By contrast, *Limitation-of-size* is meant to rule out the Russell set, by saying, roughly: *we should never have expected a Russell set to exist, because it would have been too big.*

Put like this, then, let's be blunt: considered as a reply to the paradoxes, *Limitation-of-size* stands in need of *much* more justification. Consider, for example, this version of Russell's Paradox: *no pug sniffs exactly the pugs which don't sniff themselves* (see section 1.2). If you ask "why is there no such pug?", it is not a good answer to be told that such a pug would have to sniff too many pugs. So why would it be a good intuitive explanation, of the non-existence of a Russell set, that it would have to be "too big" to exist?

In short, it's forgivable if you are a bit mystified concerning the "intuitive" motivation for *Limitation-of-size*.

5.5 Replacement and "Absolute Infinity"

We will now put *Limitation-of-size* behind us, and explore a different family of (intrinsic) attempts to justify Replacement, which do take seriously the idea of the sets as formed in stages.

When we first outlined the iterative process, we offered some principles which explained what happens at each stage. These were *Stages-are-key*, *Stages-are-ordered*, and *Stages-accumulate*. Later, we added some principles which told us something about the number of stages: *Stages-keep-going* told us that the process of set-formation never ends, and *Stages-hit-infinity* told us that the process goes through an infinite-th stage.

It is reasonable to suggest that these two latter principles fall out of some a broader principle, like:

> *Stages-are-inexhaustible*. There are absolutely infinitely many stages; the hierarchy is as tall as it could possibly be.

Obviously this is an informal principle. But even if it is not immediately *entailed* by the cumulative-iterative conception of set, it certainly seems *consonant* with it. At the very least, and unlike *Limitation-of-size*, it retains the idea that sets are formed stage-by-stage.

The hope, now, is to leverage *Stages-are-inexhaustible* into a justification of Replacement. So let us see how this might be done.

In section 3.2, we saw that it is easy to construct a well-ordering which (morally) should be isomorphic to $\omega + \omega$. Otherwise put, we can easily imagine a stage-by-stage iterative process, whose order-type (morally) is $\omega + \omega$. As such, if we have accepted *Stages-are-inexhaustible*, then we should surely accept that there is at least an $\omega + \omega$-th stage of the hierarchy, i.e., $V_{\omega+\omega}$, for the hierarchy surely *could* continue thus far.

This thought generalizes as follows: for any well-ordering, the process of building the iterative hierarchy should run at least as far as that well-ordering. And we could guarantee this, just by treating Theorem 3.26 as an *axiom*. This would tell us that any well-ordering is isomorphic to a von Neumann ordinal. Since each von Neumann ordinal will be equal to its own rank, Theorem 3.26 will then tell us that, whenever we can describe a well-ordering in our set theory, the iterative process of set building must outrun that well-ordering.

This idea certainly seems like a corollary of *Stages-are-inexhaustible*. Unfortunately, if our aim is to extract Replacement from this idea, then we face a simple, technical, barrier: Replacement is strictly stronger than Theorem 3.26. (This observation is made by Potter (2004, §13.2); we will prove it in section 5.8.)

The upshot is that, if we are going to understand *Stages-are-inexhaustible* in such a way as to yield Replacement, then it cannot *merely* say that the hierarchy outruns any well-ordering. It must make a stronger claim than that. To this end, Shoenfield (1977) proposed a very natural strengthening of the idea, as follows: the hierarchy is not *cofinal* with any set.[2] In slightly more detail: if τ is a mapping which sends sets to stages of the hierarchy, the image of any set A under τ does not exhaust the hierarchy. Otherwise put (schematically):

> *Stages-are-super-cofinal.* If A is a set and $\tau(x)$ is a stage for every $x \in A$, then there is a stage which comes after each $\tau(x)$ for $x \in A$.

It is obvious that **ZF** proves a suitably formalised version of *Stages-are-super-cofinal.* Conversely, we can informally argue that *Stages-are-super-cofinal* justifies Replacement.[3] For suppose $(\forall x \in A)\exists! y\, \varphi(x, y)$. Then for each $x \in A$, let $\sigma(x)$ be the y such that $\varphi(x, y)$, and let $\tau(x)$ be the stage at which $\sigma(x)$ is first formed. By *Stages-are-super-cofinal*, there is a stage V such that $(\forall x \in A)\tau(x) \in V$. Now since each $\tau(x) \in V$ and $\sigma(x) \subseteq \tau(x)$, by Separation we can obtain $\{y \in V : (\exists x \in A)\sigma(x) = y\} = \{y : (\exists x \in A)\varphi(x, y)\}$.

Problem 5.1. Formalize *Stages-are-super-cofinal* within **ZF**.

So *Stages-are-super-cofinal* vindicates Replacement. And it is at least plausible that *Stages-are-inexhaustible* vindicates *Stages-are-super-cofinal*. For suppose *Stages-are-super-cofinal* fails. So the hierarchy is cofinal with some set A, i.e., we have a map τ such that for any stage S there is some $x \in A$ such that $S \in \tau(x)$. In that case, we do have a way to get a handle on the supposed "absolute infinity" of the hierarchy: it is *exhausted* by the range of τ applied to A. And that compromises the thought that the hierarchy is "absolutely infinite". Contraposing: *Stages-are-inexhaustible* entails *Stages-are-super-cofinal*, which in turn justifies Replacement.

This represents a genuinely promising attempt to provide an intrinsic justification for Replacement. But whether it ultimately works, or not, we will have to leave to you to decide.

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[2]Gödel seems to have proposed a similar thought; see Potter (2004, p. 223). For discussion of Gödel and Shoenfield, see Incurvati (2020, 90–5).

[3]It would be harder to prove Replacement using some formalisation of *Stages-are-super-cofinal*, since **Z** on its own is not strong enough to define the stages, so it is not clear how one would formalise *Stages-are-super-cofinal*. One option, though, is to work in some extension of **LT**, as discussed in section 5.3.

5.6 Replacement and Reflection

Our last attempt to justify Replacement, via *Stages-are-inexhaustible*, begins with a deep and lovely result:[4]

Theorem 5.2 (Reflection Schema). *For any formula φ:*

$$\forall\alpha\exists\beta > \alpha(\forall x_1 \ldots, x_n \in V_\beta)(\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi^{V_\beta}(x_1, \ldots, x_n))$$

As in Definition 5.1, φ^{V_β} is the result of restricting every quantifier in φ to the set V_β. So, intuitively, Reflection says this: if φ is true in the entire hierarchy, then φ is true in arbitrarily many *initial segments* of the hierarchy.

Montague (1961) and Lévy (1960) showed that (suitable formulations of) Replacement and Reflection are equivalent, modulo **Z**, so that adding either gives you **ZF**. (We prove these results in section 5.7.) Given this equivalence, one might hope to justify Reflection and Replacement via *Stages-are-inexhaustible* as follows: given *Stages-are-inexhaustible*, the hierarchy should be very, very tall; so tall, in fact, that nothing we can say about it is sufficient to bound its height. And we can understand this as the thought that, if any sentence φ is true in the entire hierarchy, then it is true in arbitrarily many initial segments of the hierarchy. And that is just Reflection.

Again, this seems like a genuinely promising attempt to provide an intrinsic justification for Replacement. But there is much too much to say about it here. You must now decide for yourself whether it succeeds.[5]

5.7 Appendix: Results surrounding Replacement

In this section, we will prove Reflection within **ZF**. We will also prove a sense in which Reflection is equivalent to Replacement. And we will prove an interesting consequence of all this, concerning the strength of Reflection/Replacement. *Warning: this is easily the most advanced bit of mathematics in this textbook.*

We'll start with a lemma which, for brevity, employs the notational device of *overlining* to deal with sequences of variables or objects. So: "$\overline{a}_k$" abbreviates "$a_{k_1}, \ldots, a_{k_n}$", where n is determined by context.

Lemma 5.3. *For each $1 \leq i \leq k$, let $\varphi_i(\overline{v}_i, x)$ be a formula. Then for each α there is some $\beta > \alpha$ such that, for any $\overline{a}_1, \ldots, \overline{a}_k \in V_\beta$ and each $1 \leq i \leq k$:*

$$\exists x\varphi_i(\overline{a}_i, x) \rightarrow (\exists x \in V_\beta)\varphi_i(\overline{a}_i, x)$$

Proof. We define a term μ as follows: $\mu(\overline{a}_1, \ldots, \overline{a}_k)$ is the least stage, V, which satisfies all of the following conditionals, for $1 \leq i \leq k$:

$$\exists x\varphi_i(\overline{a}_i, x) \rightarrow (\exists x \in V)\varphi_i(\overline{a}_i, x))$$

---

[4]A reminder: all formulas can have parameters (unless explicitly stated otherwise).

[5]Though you might like to continue by reading Incurvati (2020, 95–100).

It is easy to confirm that $\mu(\bar{a}_1, \ldots, \bar{a}_k)$ exists for all $\bar{a}_1, \ldots, \bar{a}_k$. Now, using Replacement and our recursion theorem, define:

$$S_0 = V_{\alpha+1}$$
$$S_{n+1} = S_n \cup \bigcup\{\mu(\bar{a}_1, \ldots, \bar{a}_k) : \bar{a}_1, \ldots, \bar{a}_k \in S_n\}$$
$$S = \bigcup_{m < \omega} S_n.$$

Each S_n, and hence S itself, is a stage after V_α. Now fix $\bar{a}_1, \ldots, \bar{a}_k \in S$; so there is some $n < \omega$ such that $\bar{a}_1, \ldots, \bar{a}_k \in S_n$. Fix some $1 \le i \le k$, and suppose that $\exists x \varphi_i(\bar{a}_i, x)$. So $(\exists x \in \mu(\bar{a}_1, \ldots, \bar{a}_k))\varphi_i(\bar{a}_i, x)$ by construction, so $(\exists x \in S_{n+1})\varphi_i(\bar{a}_i, x)$ and hence $(\exists x \in S)\varphi_i(\bar{a}_i, x)$. So S is our V_β. $\qquad\square$

We can now prove Theorem 5.2 quite straightforwardly:

Proof. Fix α. Without loss of generality, we can assume φ's only connectives are $\exists$, $\neg$ and $\wedge$ (since these are expressively adequate). Let $\psi_1, \ldots, \psi_k$ enumerate each of φ's subformulas according to complexity, so that $\psi_k = \varphi$. By Lemma 5.3, there is a $\beta > \alpha$ such that, for any $\bar{a}_i \in V_\beta$ and each $1 \le i \le k$:

$$\exists x \psi_i(\bar{a}_i, x) \to (\exists x \in V_\beta)\psi_i(\bar{a}_i, x) \qquad (*)$$

By induction on complexity of ψ_i, we will show that $\psi_i(\bar{a}_i) \leftrightarrow \psi_i^{V_\beta}(\bar{a}_i)$, for any $\bar{a}_i \in V_\beta$. If ψ_i is atomic, this is trivial. The biconditional also establishes that, when ψ_i is a negation or conjunction of subformulas satisfying this property, ψ_i itself satisfies this property. So the only interesting case concerns quantification. Fix $\bar{a}_i \in V_\beta$; then:

$$
\begin{aligned}
(\exists x \psi_i(\bar{a}_i, x))^{V_\beta} \text{ iff } & (\exists x \in V_\beta)\psi_i^{V_\beta}(\bar{a}_i, x) && \text{by definition} \\
\text{ iff } & (\exists x \in V_\beta)\psi_i(\bar{a}_i, x) && \text{by hypothesis} \\
\text{ iff } & \exists x \psi_i(\bar{a}_i, x) && \text{by } (*)
\end{aligned}
$$

This completes the induction; the result follows as $\psi_k = \varphi$. $\qquad\square$

We have proved Reflection in **ZF**. Our proof essentially followed Montague (1961). We now want to prove in **Z** that Reflection entails Replacement. The proof follows Lévy (1960), but with a simplification.

Since we are working in **Z**, we cannot present Reflection in exactly the form given above. After all, we formulated Reflection using the "V_α" notation, and that cannot be defined in **Z** (see section 4.5). So instead we will offer an apparently weaker formulation of Replacement, as follows:

Weak-Reflection. For any formula φ, there is a transitive set S such that 0, 1, and any parameters to φ are elements of S, and $(\forall \bar{x} \in S)(\varphi \leftrightarrow \varphi^S)$.

To use this to prove Replacement, we will first follow Lévy (1960, first part of Theorem 2) and show that we can "reflect" two formulas at once:

Lemma 5.4 (in Z + Weak-Reflection.). *For any formulas ψ, χ, there is a* *transitive set S such that 0 and 1 (and any parameters to the formulas) are* *elements of S, and $(\forall \overline{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$.*

Proof. Let φ be the formula $(z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)$.

Here we use an abbreviation; we should spell out "$z = 0$" as "$\forall t\, t \notin z$" and "$z = 1$" as "$\forall s(s \in z \leftrightarrow \forall t\, t \notin s)$". But since $0, 1 \in S$ and S is transitive, these formulas are *absolute* for S; that is, they will apply to the same object whether we restrict their quantifiers to S.[6]

By Weak-Reflection, we have some appropriate S such that:

$$(\forall z, \overline{x} \in S)(\varphi \leftrightarrow \varphi^S)$$
$$\text{i.e. } (\forall z, \overline{x} \in S)(((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)) \leftrightarrow$$
$$((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi))^S)$$
$$\text{i.e. } (\forall z, \overline{x} \in S)(((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)) \leftrightarrow$$
$$((z = 0 \wedge \psi^S) \vee (z = 1 \wedge \chi^S)))$$
$$\text{i.e. } (\forall \overline{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$$

The second claim entails the third because "$z = 0$" and "$z = 1$" are absolute for S; the fourth claim follows since $0 \neq 1$. □

We can now obtain Replacement, just by following and simplifying Lévy (1960, Theorem 6):

Theorem 5.5 (in Z + Weak-Reflection). *For any formula $\varphi(v, w)$, and any* *A, if $(\forall x \in A)\exists! y\, \varphi(x, y)$, then $\{y : (\exists x \in A)\varphi(x, y)\}$ exists.*

Proof. Fix A such that $(\forall x \in A)\exists! y\, \varphi(x, y)$, and define formulas:

$$\psi \text{ is } (\varphi(x, z) \wedge A = A)$$
$$\chi \text{ is } \exists y\, \varphi(x, y)$$

Using Lemma 5.4, since A is a parameter to ψ, there is a transitive S such that $0, 1, A \in S$ (along with any other parameters), and such that:

$$(\forall x, z \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$$

So in particular:

$$(\forall x, z \in S)(\varphi(x, z) \leftrightarrow \varphi^S(x, z))$$
$$(\forall x \in S)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi^S(x, y))$$

---

[6]More formally, letting ξ be either of these formulas, $\xi(z) \leftrightarrow \xi^S(z)$.

Combining these, and observing that $A \subseteq S$ since $A \in S$ and S is transitive:

$$(\forall x \in A)(\exists y \varphi(x,y) \leftrightarrow (\exists y \in S)\varphi(x,y))$$

Now $(\forall x \in A)(\exists! y \in S)\varphi(x,y)$, because $(\forall x \in A)\exists! y \, \varphi(x,y)$. Now Separation yields $\{y \in S : (\exists x \in A)\varphi(x,y)\} = \{y : (\exists x \in A)\varphi(x,y)\}$. $\qquad\square$

5.8   Appendix: Finite axiomatizability

We close this chapter by extracting some results from Replacement. The first result is due to Montague (1961); note that it is not a proof *within* **ZF**, but a proof *about* **ZF**:

Theorem 5.6. **ZF** *is not finitely axiomatizable. More generally: if* **T** *is finite and* **T** $\vdash$ **ZF**, *then* **T** *is inconsistent.*

   (Here, we tacitly restrict ourselves to first-order sentences whose only non-logical primitive is $\in$, *and we write* **T** $\vdash$ **ZF** *to indicate that* **T** $\vdash \varphi$ *for all* $\varphi \in$ **ZF**.*)*

Proof. Fix finite **T** such that **T** $\vdash$ **ZF**. So, **T** proves Reflection, i.e. Theorem 5.2. Since **T** is finite, we can rewrite it as a single conjunction, θ. Reflecting with this formula, **T** $\vdash \exists\beta(\theta \leftrightarrow \theta^{V_\beta})$. Since trivially **T** $\vdash \theta$, we find that **T** $\vdash \exists\beta \, \theta^{V_\beta}$.

   Now, let $\psi(X)$ abbreviate:

$$\theta^X \wedge X \text{ is transitive} \wedge (\forall Y \in X)(Y \text{ is transitive} \to \neg\theta^Y)$$

roughly this says: X is a transitive model of θ, and $\in$-minimal in this regard. Now, recalling that **T** $\vdash \exists\beta \, \theta^{V_\beta}$, by basic facts about ranks within **ZF** and hence within **T**, we have:

$$\mathbf{T} \vdash \exists M \psi(M). \qquad\qquad (*)$$

Using the first conjunct of $\psi(X)$, whenever **T** $\vdash \sigma$, we have that **T** $\vdash \forall X(\psi(X) \to \sigma^X)$. So, by (*):

$$\mathbf{T} \vdash \forall X(\psi(X) \to (\exists N \psi(N))^X)$$

Using this, and (*) again:

$$\mathbf{T} \vdash \exists M(\psi(M) \wedge (\exists N \psi(N))^M)$$

In particular, then:

$$\mathbf{T} \vdash \exists M(\psi(M) \wedge (\exists N \in M)((N \text{ is transitive})^N \wedge (\theta^N)^M))$$

So, by elementary reasoning concerning transitivity:

$$\mathbf{T} \vdash \exists M(\psi(M) \wedge (\exists N \in M)(N \text{ is transitive} \wedge \theta^N))$$

So that **T** is inconsistent.[7]   □

Here is a similar result, noted by Potter (2004, 223):

Proposition 5.7. *Let* **T** *extend* **Z** *with finitely many new axioms. If* **T** ⊢ *sth:replacement:finiteaxiomatizab*
ZF*, then* **T** *is inconsistent. (Here we use the same tacit restrictions as for* *finiteextensionofZ*
Theorem 5.6.)

Proof. Use θ for the conjunction of all of **T**'s axioms *except* for the (infinitely many) instances of Separation. Defining ψ from θ as in Theorem 5.6, we can show that **T** ⊢ $\exists M \psi(M)$.

As in Theorem 5.6, we can establish the schema that, whenever **T** ⊢ σ, we have that **T** ⊢ $\forall X(\psi(X) \to \sigma^X)$. We then finish our proof, exactly as in Theorem 5.6.

However, establishing the schema involves a little more work than in Theorem 5.6. After all, the Separation-instances are in **T**, but they are not conjuncts of θ. However, we can overcome this obstacle by proving that **T** ⊢ $\forall X(X$ is transitive $\to \sigma^X)$, for every Separation-instance σ. We leave this to the reader.   □

Problem 5.2. Show that, for every Separation-instance σ, we have: **Z** ⊢ $\forall X(X$ is transitive $\to \sigma^X)$. (We used this schema in Proposition 5.7.)

Problem 5.3. Show that, for every $\varphi \in$ **Z**, we have **ZF** ⊢ $\varphi^{V_{\omega+\omega}}$.

Problem 5.4. Confirm the remaining schematic results invoked in the proofs of Theorem 5.6 and Proposition 5.7.

As remarked in section 5.5, this shows that Replacement is strictly stronger than Theorem 3.26. Or, slightly more strictly: if **Z** + "every well-ordering is isomorphic to a unique ordinal" is consistent, then it fails to prove some Replacement-instance.

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[7]This "elementary reasoning" involves proving certain "absoluteness facts" for transitive sets.

Chapter 6

Ordinal Arithmetic

6.1 Introduction

In chapter 3, we developed a theory of ordinal numbers. We saw in chapter 4 that we can think of the ordinals as a spine around which the remainder of the hierarchy is constructed. But that is not the only role for the ordinals. There is also the task of performing ordinal arithmetic.

We already gestured at this, back in section 3.2, when we spoke of ω, $\omega + 1$ and $\omega + \omega$. At the time, we spoke informally; the time has come to spell it out properly. However, we should mention that there is not much philosophy in this chapter; just technical developments, coupled with a (mildly) interesting observation that we can do the same thing in two different ways.

6.2 Ordinal Addition

Suppose we want to add α and β. We can simply put a copy of β immediately after a copy of α. (We need to take *copies*, since we know from Proposition 3.22 that either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.) The intuitive effect of this is to run through an α-sequence of steps, *and then* to run through a β-sequence. The resulting sequence will be well-ordered; so by Theorem 3.26 it is isomorphic to a (unique) ordinal. That ordinal can be regarded as the *sum* of α and β.

That is the intuitive idea behind ordinal addition. To define it rigorously, we start with the idea of taking *copies* of sets. The idea here is to use arbitrary tags, 0 and 1, to keep track of which object came from where:

Definition 6.1. The *disjoint sum* of A and B is $A \sqcup B = (A \times \{0\}) \cup (B \times \{1\})$.

We next define an ordering on pairs of ordinals:

Definition 6.2. For any ordinals $\alpha_1, \alpha_2, \beta_1, \beta_2$, say that:

$$\langle \alpha_1, \alpha_2 \rangle \lhd \langle \beta_1, \beta_2 \rangle \text{ iff either } \alpha_2 \in \beta_2$$
$$\text{or both } \alpha_2 = \beta_2 \text{ and } \alpha_1 \in \beta_1$$

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This is a *reverse lexicographic* ordering, since you order by the second element, then by the first. Now recall that we wanted to define $\alpha + \beta$ as the order type of a copy of α followed by a copy of β. To achieve that, we say:

Definition 6.3. For any ordinals α, β, their sum is $\alpha + \beta = \mathrm{ord}(\alpha \sqcup \beta, \lhd)$.

Note that we slightly abused notation here; strictly we should write "$\{\langle x, y \rangle \in \alpha \sqcup \beta : x \lhd y\}$" in place of "$\lhd$". For brevity, though, we will continue to abuse notation in this way in what follows.

The following result, together with Theorem 3.26, confirms that our definition is well-formed:

Lemma 6.4. $\langle \alpha \sqcup \beta, \lhd \rangle$ *is a well-order, for any ordinals α and β.*

Proof. Obviously $\lhd$ is connected on $\alpha \sqcup \beta$. To show it is well-founded, fix a non-empty $X \subseteq \alpha \sqcup \beta$. Let Y be the subset of X whose second coordinate is as small as possible, i.e. $Y = \{\langle \gamma, i \rangle \in X : (\forall \langle \delta, j \rangle \in X) i \leq j\}$. Now choose the element of Y with smallest first coordinate. $\square$

So we have a nice, explicit definition of ordinal addition. Here is an unsurprising fact (recall that $1 = \{0\}$, by Definition 2.7):

Proposition 6.5. $\alpha + 1 = \alpha^+$, *for any ordinal α.*

Proof. Consider the isomorphism f from $\alpha^+ = \alpha \cup \{\alpha\}$ to $\alpha \sqcup 1 = (\alpha \times \{0\}) \sqcup (\{0\} \times \{1\})$ given by $f(\gamma) = \langle \gamma, 0 \rangle$ for $\gamma \in \alpha$, and $f(\alpha) = \langle 0, 1 \rangle$. $\square$

Moreover, it is easy to show that addition obeys certain recursive conditions:

Lemma 6.6. *For any ordinals α, β, we have:*

$$\alpha + 0 = \alpha$$
$$\alpha + (\beta + 1) = (\alpha + \beta) + 1$$
$$\alpha + \beta = \mathrm{lsub}_{\delta < \beta}(\alpha + \delta) \qquad \text{if β is a limit ordinal}$$

Proof. We check case-by-case; first:

$$\alpha + 0 = \mathrm{ord}((\alpha \times \{0\}) \cup (0 \times \{1\}), \lhd)$$
$$= \mathrm{ord}((\alpha \times \{0\}) \cup \{0\}, \lhd)$$
$$= \alpha$$
$$\alpha + (\beta + 1) = \mathrm{ord}((\alpha \times \{0\}) \cup (\beta^+ \times \{1\}), \lhd)$$
$$= \mathrm{ord}((\alpha \times \{0\}) \cup (\beta \times \{1\}), \lhd) + 1$$
$$= (\alpha + \beta) + 1$$

Now let $\beta \neq \emptyset$ be a limit. If $\delta < \beta$ then also $\delta + 1 < \beta$, so $\alpha + \delta$ is a proper initial segment of $\alpha + \beta$. So $\alpha + \beta$ is a strict upper bound on $X = \{\alpha + \delta : \delta < \beta\}$. Moreover, if $\alpha \leq \gamma < \alpha + \beta$, then clearly $\gamma = \alpha + \delta$ for some $\delta < \beta$. So $\alpha + \beta = \mathrm{lsub}_{\delta < \beta}(\alpha + \delta)$. $\square$

But here is a striking fact. To define ordinal addition, we could *instead* have simply used the Transfinite Recursion Theorem, and laid down the recursion equations, exactly as given in Lemma 6.6 (though using "β^+" rather than "$\beta + 1$").

There are, then, two different ways to define operations on the ordinals. We can define them *synthetically*, by explicitly constructing a well-ordered set and considering its order type. Or we can define them *recursively*, just by laying down the recursion equations. Done correctly, though, the outcome is identical. For Theorem 3.26 guarantees that these recursion equations pin down *unique* ordinals.

In many ways, ordinal arithmetic behaves just like addition of the natural numbers. For example, we can prove the following:

Lemma 6.7. *If α, β, γ are ordinals, then:*

1. *if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$*

2. *if $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$*

3. *$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$, i.e., addition is associative*

4. *If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$*

Proof. We prove (3), leaving the rest as an exercise. The proof is by Simple Transfinite Induction on γ, using Lemma 6.6. When $\gamma = 0$:

$$(\alpha + \beta) + 0 = \alpha + \beta = \alpha + (\beta + 0)$$

When $\gamma = \delta + 1$, suppose for induction that $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$; now using Lemma 6.6 three times:

$$\begin{aligned}
(\alpha + \beta) + (\delta + 1) &= ((\alpha + \beta) + \delta) + 1 \\
&= (\alpha + (\beta + \delta)) + 1 \\
&= \alpha + ((\beta + \delta) + 1) \\
&= \alpha + (\beta + (\delta + 1))
\end{aligned}$$

When γ is a limit ordinal, suppose for induction that if $\delta \in \gamma$ then $(\alpha+\beta)+\delta = \alpha + (\beta + \delta)$; now:

$$\begin{aligned}
(\alpha + \beta) + \gamma &= \operatorname*{lsub}_{\delta < \gamma}((\alpha + \beta) + \delta) \\
&= \operatorname*{lsub}_{\delta < \gamma}(\alpha + (\beta + \delta)) \\
&= \alpha + \operatorname*{lsub}_{\delta < \gamma}(\beta + \delta) \\
&= \alpha + (\beta + \gamma) \qquad \qquad \square
\end{aligned}$$

Problem 6.1. Prove the remainder of Lemma 6.7.

In these ways, ordinal addition should be very familiar. But, there is a crucial way in which ordinal addition is *not* like addition on the natural numbers.

Proposition 6.8. *Ordinal addition is not commutative;* $1 + \omega = \omega < \omega + 1$.

Proof. Note that $1 + \omega = \mathrm{lsub}_{n < \omega}(1 + n) = \omega \in \omega \cup \{\omega\} = \omega^+ = \omega + 1$. $\qquad\square$

Whilst this may initially come as a surprise, *it shouldn't.* On the one hand, when you consider $1 + \omega$, you are thinking about the order type you get by putting an extra element *before* all the natural numbers. Reasoning as we did with Hilbert's Hotel in **??**, intuitively, this extra first element shouldn't make any difference to the overall order type. On the other hand, when you consider $\omega + 1$, you are thinking about the order type you get by putting an extra element *after* all the natural numbers. And that's a radically different beast!

6.3 Using Ordinal Addition

Using addition on the ordinals, we can explicitly calculate the ranks of various sets, in the sense of Definition 4.15:

Lemma 6.9. *If* $\mathrm{rank}(A) = \alpha$ *and* $\mathrm{rank}(B) = \beta$*, then:*

1. $\mathrm{rank}(\wp(A)) = \alpha + 1$

2. $\mathrm{rank}(\{A, B\}) = \max(\alpha, \beta) + 1$

3. $\mathrm{rank}(A \cup B) = \max(\alpha, \beta)$

4. $\mathrm{rank}(\langle A, B \rangle) = \max(\alpha, \beta) + 2$

5. $\mathrm{rank}(A \times B) \leq \max(\alpha, \beta) + 2$

6. $\mathrm{rank}(\bigcup A) = \alpha$ *when* α *is empty or a limit;* $\mathrm{rank}(\bigcup A) = \gamma$ *when* $\alpha = \gamma + 1$

Proof. Throughout, we invoke Proposition 4.20 repeatedly.

(1). If $x \subseteq A$ then $\mathrm{rank}(x) \leq \mathrm{rank}(A)$. So $\mathrm{rank}(\wp(A)) \leq \alpha + 1$. Since $A \in \wp(A)$ in particular, $\mathrm{rank}(\wp(A)) = \alpha + 1$.

(2). By Proposition 4.20

(3). By Proposition 4.20.

(4). By (2), twice.

(5). Note that $A \times B \subseteq \wp(\wp(A \cup B))$, and invoke (4).

(6). If $\alpha = \gamma + 1$, there is some $c \in A$ with $\mathrm{rank}(c) = \gamma$, and no element of A has higher rank; so $\mathrm{rank}(\bigcup A) = \gamma$. If α is a limit ordinal, then A has elements with rank arbitrarily close to (but strictly less than) α, so that $\bigcup A$ also has elements with rank arbitrarily close to (but strictly less than) α, so that $\mathrm{rank}(\bigcup A) = \alpha$. $\qquad\square$

We leave it as an exercise to show why (5) involves an *in*equality.

Problem 6.2. Produce sets A and B such that $\mathrm{rank}(A \times B) = \max(\mathrm{rank}(A), \mathrm{rank}(B))$. Produce sets A and B such that $\mathrm{rank}(A \times B) \max(\mathrm{rank}(A), \mathrm{rank}(B)) + 2$. Are any other ranks possible?

We are also now in a position to show that several reasonable notions of what it might mean to describe an ordinal as "finite" or "infinite" coincide:

Lemma 6.10. *For any ordinal α, the following are equivalent:*

1. $\alpha \notin \omega$, *i.e., α is not a natural number*

2. $\omega \leq \alpha$

3. $1 + \alpha = \alpha$

4. $\alpha \approx \alpha + 1$, *i.e., α and $\alpha + 1$ are equinumerous*

5. α *is Dedekind infinite*

So we have five provably equivalent ways to understand what it takes for an ordinal to be (in)finite.

Proof. (1) $\Rightarrow$ (2). By Trichotomy.

(2) $\Rightarrow$ (3). Fix $\alpha \geq \omega$. By Transfinite Induction, there is some least ordinal γ (possibly 0) such that there is a limit ordinal β with $\alpha = \beta + \gamma$. Now:

$$1 + \alpha = 1 + (\beta + \gamma) = (1 + \beta) + \gamma = \underset{\delta < \beta}{\mathrm{lsub}}(1 + \delta) + \gamma = \beta + \gamma = \alpha.$$

(3) $\Rightarrow$ (4). There is clearly a bijection $f \colon (\alpha \sqcup 1) \to (1 \sqcup \alpha)$. If $1 + \alpha = \alpha$, there is an isomorphism $g \colon (1 \sqcup \alpha) \to \alpha$. Now consider $g \circ f$.

(4) $\Rightarrow$ (5). If $\alpha \approx \alpha + 1$, there is a bijection $f \colon (\alpha \sqcup 1) \to \alpha$. Define $g(\gamma) = f(\gamma, 0)$ for each $\gamma < \alpha$; this injection witnesses that α is Dedekind infinite, since $f(0, 1) \in \alpha \setminus \mathrm{ran}(g)$.

(5) $\Rightarrow$ (1). This is Proposition 2.8. $\qquad\square$

6.4 Ordinal Multiplication

We now turn to ordinal multiplication, and we approach this much like ordinal addition. So, suppose we want to multiply α by β. To do this, you might imagine a rectangular grid, with width α and height β; the product of α and β is now the result of moving along each row, then moving through the next row... until you have moved through the entire grid. Otherwise put, the product of α and β arises by replacing *each* element in β with a copy of α.

To make this formal, we simply use the reverse lexicographic ordering on the Cartesian product of α and β:

Definition 6.11. For any ordinals α, β, their product $\alpha \cdot \beta = \mathrm{ord}(\alpha \times \beta, \lhd)$.

We must again confirm that this is a well-formed definition:

Lemma 6.12. $\langle \alpha \times \beta, \lhd \rangle$ *is a well-order, for any ordinals α and β.*

Proof. Exactly as for Lemma 6.4. $\qquad\square$

And it is not hard to prove that multiplication behaves thus:

Lemma 6.13. *For any ordinals α, β:*

$$\alpha \cdot 0 = 0$$
$$\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$$
$$\alpha \cdot \beta = \operatorname*{lsub}_{\delta < \beta}(\alpha \cdot \delta) \qquad \text{when } \beta \text{ is a limit ordinal.}$$

Proof. Left as an exercise. $\qquad\square$

Indeed, just as in the case of addition, we could have defined ordinal multiplication via these recursion equations, rather than offering a direct definition. Equally, as with addition, certain behaviour is familiar:

Lemma 6.14. *If α, β, γ are ordinals, then:*

1. *if $\alpha \neq 0$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$;*

2. *if $\alpha \neq 0$ and $\alpha \cdot \beta = \alpha \cdot \gamma$, then $\beta = \gamma$;*

3. *$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$;*

4. *If $\alpha \leq \beta$, then $\alpha \cdot \gamma \leq \beta \cdot \gamma$;*

5. *$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.*

Proof. Left as an exercise. $\qquad\square$

You can prove (or look up) other results, to your heart's content. But, given Proposition 6.8, the following should not come as a surprise:

Proposition 6.15. *Ordinal multiplication is not commutative: $2 \cdot \omega = \omega < \omega \cdot 2$*

Proof. $2 \cdot \omega = \operatorname{lsub}_{n<\omega}(2 \cdot n) = \omega \in \operatorname{lsub}_{n<\omega}(\omega + n) = \omega + \omega = \omega \cdot 2.$ $\qquad\square$

Again, the intuitive rationale is quite straightforward. To compute $2 \cdot \omega$, you replace each natural number with two entities. You would get the same order type if you simply inserted all the "half" numbers into the natural numbers, i.e., you considered the natural ordering on $\{n/2 : n \in \omega\}$. And, put like that, the order type is plainly the same as that of ω itself. But, to compute $\omega \cdot 2$, you place down two copies of ω, one after the other.

Problem 6.3. Prove Lemma 6.12, Lemma 6.13, and Lemma 6.14

6.5 Ordinal Exponentiation

We now move to ordinal exponentiation. Sadly, there is no *nice* synthetic definition for ordinal exponentiation.

Sure, there *are* explicit synthetic definitions. Here is one. Let $\mathrm{finfun}(\alpha, \beta)$ be the set of all functions $f \colon \alpha \to \beta$ such that $\{\gamma \in \alpha : f(\gamma) \neq 0\}$ is equinumerous with some natural number. Define a well-ordering on $\mathrm{finfun}(\alpha, \beta)$ by $f \sqsubset g$ iff $f \neq g$ and $f(\gamma_0) < g(\gamma_0)$, where $\gamma_0 = \max\{\gamma \in \alpha : f(\gamma) \neq g(\gamma)\}$. Then we can define $\alpha^{(\beta)}$ as $\mathrm{ord}(\mathrm{finfun}(\alpha, \beta), \sqsubset)$. Potter employs this explicit definition, and then immediately explains:

> The choice of this ordering is determined purely by our desire to obtain a definition of ordinal exponentiation which obeys the appropriate recursive condition..., and it is much harder to picture than either the ordered sum or the ordered product. (Potter, 2004, p. 199)

Quite. We explained addition as "a copy of α followed by a copy of β", and multiplication as "a β-sequence of copies of α". But we have nothing pithy to say about $\mathrm{finfun}(\alpha, \gamma)$. So instead, we'll offer the definition of ordinal exponentiation just *by* transfinite recursion, i.e.:

Definition 6.16.

$$\alpha^{(0)} = 1$$
$$\alpha^{(\beta+1)} = \alpha^{(\beta)} \cdot \alpha$$
$$\alpha^{(\beta)} = \bigcup_{\delta < \beta} \alpha^{(\delta)} \qquad \text{when } \beta \text{ is a limit ordinal}$$

If we were working *as* set theorists, we might want to explore some of the properties of ordinal exponentiation. But we have nothing much more to add, except to note the unsurprising fact that ordinal exponentiation does not commute. Thus $2^{(\omega)} = \bigcup_{\delta < \omega} 2^{(\delta)} = \omega$, whereas $\omega^{(2)} = \omega \cdot \omega$. But then, we should not *expect* exponentiation to commute, since it does not commute with natural numbers: $2^{(3)} = 8 < 9 = 3^{(2)}$.

Problem 6.4. Using Transfinite Induction, prove that, if we define $\alpha^{(\beta)} = \mathrm{ord}(\mathrm{finfun}(\alpha, \beta), \sqsubset)$, we obtain the recursion equations of Definition 6.16.

Chapter 7

Cardinals

7.1 Cantor's Principle

Cast your mind back to section 3.5. We were discussing well-ordered sets, and suggested that it would be nice to have objects which go proxy for well-orders. With this is mind, we introduced ordinals, and then showed in Corollary 3.28 that these behave as we would want them to, i.e.:

$$\text{ord}(A, <) = \text{ord}(B, \lessdot) \text{ iff } \langle A, < \rangle \cong \langle B, \lessdot \rangle.$$

Cast your mind back even further, to **??**. There, working naïvely, we introduced the notion of the "size" of a set. Specifically, we said that two sets are equinumerous, $A \approx B$, just in case there is a bijection $f \colon A \to B$. This is an intrinsically simpler notion than that of a well-ordering: we are only interested in bijections, and not (as with order-isomorphisms) whether the bijections "preserve any structure".

This all gives rise to an obvious thought. Just as we introduced certain objects, *ordinals*, to calibrate well-orders, we can introduce certain objects, *cardinals*, to calibrate size. That is the aim of this chapter.

Before we say what these cardinals will be, we should lay down a principle which they ought to satisfy. Writing $|X|$ for the cardinality of the set X, we would want them to obey:

$$|A| = |B| \text{ iff } A \approx B.$$

We'll call this *Cantor's* Principle, since Cantor was probably the first to have it very clearly in mind. (We'll say more about its relationship to *Hume's* Principle in section 7.5.) So our aim is to define $|X|$, for each X, in such a way that it delivers Cantor's Principle.

7.2 Cardinals as Ordinals

In fact, our theory of cardinals will just make (shameless) use of our theory of ordinals. That is: we will just define cardinals as certain specific ordinals. In particular, we will offer the following:

Definition 7.1. If A can be well-ordered, then $|A|$ is the least ordinal γ such that $A \approx \gamma$. For any ordinal γ, we say that γ is a *cardinal* iff $\gamma = |\gamma|$.

We just used the phrase "A can be well-ordered". As is almost always the case in mathematics, the modal locution here is just a hand-waving gloss on an existential claim: to say "A can be well-ordered" is just to say "there is a relation which well-orders A".

But there is a snag with Definition 7.1. We would like it to be the case that *every* set has a size, i.e., that $|A|$ exists for every A. The definition we just gave, though, begins with a conditional: "*If* A can be well-ordered...". If there is some set A which cannot be well-ordered, then our definition will simply fail to define an object $|A|$.

So, to use Definition 7.1, we need a guarantee that every set can be well-ordered. Sadly, though, this guarantee is unavailable in **ZF**. So, if we want to use Definition 7.1, there is no alternative but to add a new axiom, such as:

Axiom (Well-Ordering). Every set can be well-ordered.

We will discuss whether the Well-Ordering Axiom is acceptable in chapter 9. From now on, though, we will simply help ourselves to it. And, using it, it is quite straightforward to prove that cardinals (as defined in Definition 7.1) exist and behave nicely:

Lemma 7.2. *For every set A:*

1. *$|A|$ exists and is unique;*

2. *$|A| \approx A$;*

3. *$|A|$ is a cardinal, i.e., $|A| = ||A||$;*

Proof. Fix A. By Well-Ordering, there is a well-ordering $\langle A, R \rangle$. By Theorem 3.26, $\langle A, R \rangle$ is isomorphic to a unique ordinal, β. So $A \approx \beta$. By Transfinite Induction, there is a uniquely least ordinal, γ, such that $A \approx \gamma$. So $|A| = \gamma$, establishing (1) and (2). To establish (3), note that if $\delta \in \gamma$ then $\delta \prec A$, by our choice of γ, so that also $\delta \prec \gamma$ since equinumerosity is an equivalence relation (**??**). So $\gamma = |\gamma|$. $\square$

The next result guarantees Cantor's Principle, and more besides. (Note that cardinals inherit their ordering from the ordinals, i.e., $\mathfrak{a} < \mathfrak{b}$ iff $\mathfrak{a} \in \mathfrak{b}$. In formulating this, we will use Fraktur letters for objects we know to be cardinals. This is fairly standard. A common alternative is to use Greek letters, since cardinals are ordinals, but to choose them from the middle of the alphabet, e.g.: κ, λ.):

Lemma 7.3. *For any sets A and B:*

$$A \approx B \text{ iff } |A| = |B|$$
$$A \preceq B \text{ iff } |A| \leq |B|$$
$$A \prec B \text{ iff } |A| < |B|$$

Proof. We will prove the left-to-right direction of the second claim (the other cases are similar, and left as an exercise). So, consider the following diagram:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\updownarrow & & \updownarrow \\
|A| & \dashrightarrow & |B|
\end{array}
$$

The double-headed arrows indicate bijections, whose existence is guaranteed by Lemma 7.2. In assuming that $A \preceq B$, there is an injection $A \to B$. Now, chasing the arrows around from $|A|$ to A to B to $|B|$, we obtain an injection $|A| \to |B|$ (the dashed arrow). $\square$

We can also use Lemma 7.3 to re-prove Schröder–Bernstein. This is the claim that if $A \preceq B$ and $B \preceq A$ then $A \approx B$. We stated this as **??**, but first proved it—with some effort—in **??**. Now consider:

Re-proof of Schröder-Bernstein. If $A \preceq B$ and $B \preceq A$, then $|A| \leq |B|$ and $|B| \leq |A|$ by Lemma 7.3. So $|A| = |B|$ and $A \approx B$ by Trichotomy and Lemma 7.3. $\square$

Whilst this is a very simple proof, it implicitly relies on both Replacement (to secure Theorem 3.26) and on Well-Ordering (to guarantee Lemma 7.3). By contrast, the proof of **??** was much more self-standing (indeed, it can be carried out in $\mathbf{Z}^-$).

7.3 ZFC: A Milestone

With the addition of Well-Ordering, we have reached the final theoretical milestone. We now have all the axioms required for **ZFC**. In detail:

Definition 7.4. The theory **ZFC** has these axioms: Extensionality, Union, Pairs, Powersets, Infinity, Foundation, Well-Ordering and all instances of the Separation and Replacement schemes. Otherwise put, **ZFC** adds Well-Ordering to **ZF**.

ZFC stands for *Zermelo-Fraenkel* set theory with *Choice*. Now this might seem slightly odd, since the axiom we added was called "Well-Ordering", not "Choice". But, when we later formulate Choice, it will turn out that Well-Ordering is equivalent (modulo **ZF**) to Choice (see Theorem 9.6). So which to take as our "basic" axiom is a matter of indifference. And the name "**ZFC**" is entirely standard in the literature.

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7.4 Finite, Enumerable, Non-enumerable

Now that we have been introduced to cardinals, it is worth spending a little time talking about different varieties of cardinals; specifically, finite, enumerable, and non-enumerable cardinals.

Our first two results entail that the finite cardinals will be exactly the finite ordinals, which we defined as our *natural numbers* back in Definition 2.7:

Proposition 7.5. *Let $n, m \in \omega$. Then $n = m$ iff $n \approx m$.*

Proof. *Left-to-right* is trivial. To prove *right-to-left*, suppose $n \approx m$ although $n \neq m$. By Trichotomy, either $n \in m$ or $m \in n$; suppose $n \in m$ without loss of generality. Then $n \subsetneq m$ and there is a bijection $f \colon m \to n$, so that m is Dedekind infinite, contradicting Proposition 2.8. $\square$

Corollary 7.6. *If $n \in \omega$, then n is a cardinal.*

Proof. Immediate. $\square$

It also follows that several reasonable notions of what it might mean to describe a cardinal as "finite" or "infinite" coincide:

Theorem 7.7. *For any set A, the following are equivalent:*

1. *$|A| \notin \omega$, i.e., A is not a natural number;*

2. *$\omega \leq |A|$;*

3. *A is Dedekind infinite.*

Proof. From Lemma 6.10, Lemma 7.3, and Corollary 7.6. $\square$

This licenses the following *definition* of some notions which we used rather informally in **??**:

Definition 7.8. We say that A is *finite* iff $|A|$ is a natural number, i.e., $|A| \in \omega$. Otherwise, we say that A is *infinite*.

But note that this definition is presented against the background of **ZFC**. After all, we needed Well-Ordering to guarantee that every set has a cardinality. And indeed, without Well-Ordering, there can be a set which is neither finite nor Dedekind infinite. We will return to this sort of issue in chapter 9. For now, we continue to rely upon Well-Ordering.

Let us now turn from the finite cardinals to the infinite cardinals. Here are two elementary points:

Corollary 7.9. *ω is the least infinite cardinal.*

Proof. ω is a cardinal, since ω is Dedekind infinite and if $\omega \approx n$ for any $n \in \omega$ then n would be Dedekind infinite, contradicting Proposition 2.8. Now ω is the least infinite cardinal by definition. $\square$

Corollary 7.10. *Every infinite cardinal is a limit ordinal.*

Proof. Let α be an infinite successor ordinal, so $\alpha = \beta + 1$ for some β. By Proposition 7.5, β is also infinite, so $\beta \approx \beta + 1$ by Lemma 6.10. Now $|\beta| = |\beta + 1| = |\alpha|$ by Lemma 7.3, so that $\alpha \neq |\alpha|$. $\qquad\square$

Now, as early as **??**, we flagged we can distinguish between enumerable and non-enumerable infinite sets. That definition naturally leads to the following:

Proposition 7.11. *A is enumerable iff $|A| \leq \omega$, and A is non-enumerable iff $\omega < |A|$.*

Proof. By Trichotomy, the two claims are equivalent, so it suffices to prove that A is enumerable iff $|A| \leq \omega$. For *right-to-left*: if $|A| \leq \omega$, then $A \preceq \omega$ by Lemma 7.3 and Corollary 7.9. For *left-to-right*: suppose A is enumerable; then by **??** there are three possible cases:

1. if $A = \emptyset$, then $|A| = 0 \in \omega$, by Corollary 7.6 and Lemma 7.3.

2. if $n \approx A$, then $|A| = n \in \omega$, by Corollary 7.6 and Lemma 7.3.

3. if $\omega \approx A$, then $|A| = \omega$, by Corollary 7.9.

So in all cases, $|A| \leq \omega$. $\qquad\square$

Indeed, ω has a special place. Whilst there are many countable ordinals:

Corollary 7.12. *ω is the only enumerable infinite cardinal.*

Proof. Let $\mathfrak{a}$ be an enumerable infinite cardinal. Since $\mathfrak{a}$ is infinite, $\omega \leq \mathfrak{a}$. Since $\mathfrak{a}$ is an enumerable cardinal, $\mathfrak{a} = |\mathfrak{a}| \leq \omega$. So $\mathfrak{a} = \omega$ by Trichotomy. $\qquad\square$

Of course, there are infinitely many cardinals. So we might ask: *How many cardinals are there?* The following results show that we might want to reconsider that question.

Proposition 7.13. *If every member of X is a cardinal, then $\bigcup X$ is a cardinal.*

sth:cardinals:classing: unioncardinalscardinal

Proof. It is easy to check that $\bigcup X$ is an ordinal. Let $\alpha \in \bigcup X$ be an ordinal; then $\alpha \in \mathfrak{b} \in X$ for some cardinal $\mathfrak{b}$. Since $\mathfrak{b}$ is a cardinal, $\alpha \prec \mathfrak{b}$. Since $\mathfrak{b} \subseteq \bigcup X$, we have $\mathfrak{b} \preceq \bigcup X$, and so $\alpha \not\approx \bigcup X$. Generalising, $\bigcup X$ is a cardinal. $\square$

Theorem 7.14. *There is no largest cardinal.*

sth:cardinals:classing: lem:NoLargestCardinal

Proof. For any cardinal $\mathfrak{a}$, Cantor's Theorem (**??**) and Lemma 7.2 entail that $\mathfrak{a} < |\wp(\mathfrak{a})|$. $\qquad\square$

Theorem 7.15. *The set of all cardinals does not exist.*

Proof. For reductio, suppose $C = \{\mathfrak{a} : \mathfrak{a}$ is a cardinal$\}$. Now $\bigcup C$ is a cardinal by Proposition 7.13, so by Theorem 7.14 there is a cardinal $\mathfrak{b} > \bigcup C$. By definition $\mathfrak{b} \in C$, so $\mathfrak{b} \subseteq \bigcup C$, so that $\mathfrak{b} \leq \bigcup C$, a contradiction. $\qquad\square$

You should compare this with both Russell's Paradox and Burali-Forti.

7.5 Appendix: Hume's Principle

In section 7.1, we described Cantor's Principle. This was:

$$|A| = |B| \text{ iff } A \approx B.$$

This is very similar to what is now called *Hume's Principle*, which says:

$$\#x\, F(x) = \#x\, G(x) \text{ iff } F \sim G$$

where '$F \sim G$' abbreviates that there are exactly as many Fs as Gs, i.e., the Fs can be put into a bijection with the Gs, i.e.:

$$\exists R(\forall v \forall y(Rvy \to (Fv \wedge Gy)) \wedge$$
$$\forall v(Fv \to \exists! y\, Rvy) \wedge$$
$$\forall y(Gy \to \exists! v\, Rvy))$$

But there is a type-difference between Hume's Principle and Cantor's Principle. In the statement of Cantor's Principle, the variables "A" and "B" are first-order terms which stand for *sets*. In the statement of Hume's Principle, "F", "G" and "R" are *not* first-order terms; rather, they are in *predicate position*. (Maybe they stand for *properties*.) So we might gloss Hume's Principle in English as: the number of Fs is the number of Gs iff the Fs are bijective with the Gs. This is called *Hume's Principle*, because Hume once wrote this:

> When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal. (Hume, 1740, Pt.III Bk.1 §1)

And Hume's Principle was brought to contemporary mathematico-logical prominence by Frege (1884, §63), who quoted this passage from Hume, before (in effect) sketching (what we have called) Hume's Principle.

You should note the structural similarity between Hume's Principle and Basic Law V. We formulated this in section 1.6 as follows:

$$\epsilon x\, F(x) = \epsilon x\, G(x) \text{iff } \forall x\, (F(x) \leftrightarrow G(x)).$$

And, at this point, some commentary and comparison might help.

There are two ways to take a principle like Hume's Principle or Basic Law V: *predicatively* or *impredicatively* (recall section 1.3). On the impredicative reading of Basic Law V, for each F, the object $\epsilon x\, F(x)$ falls within the domain of

quantification that we used in formulating Basic Law V itself. Similarly, on the impredicative reading of Hume's Principle, for each F, the object $\#x\,F(x)$ falls within the domain of quantification that we used in formulating Hume's Principle. By contrast, on the *predicative* understanding, the objects $\epsilon x\,F(x)$ and $\#x\,F(x)$ would be entities from some *different* domain.

Now, if we read Basic Law V impredicatively, it leads to inconsistency, via Naïve Comprehension (for the details, see). Much like Naïve Comprehension, it can be rendered consistent by reading it *predicatively*. But it probably will not do everything that we wanted it to.

Hume's Principle, however, *can* consistently be read impredicatively. And, read thus, it is quite powerful.

To illustrate: consider the predicate "$x \neq x$", which obviously nothing satisfies. Hume's Principle now yields an object $\#x(x \neq x)$. We might treat this as the number 0. Now, on the *impredicative* understanding—but *only* on the impredicative understanding—this entity 0 falls within our original domain of quantification. So we can sensibly apply Hume's Principle with the predicate "$x = 0$" to obtain an object $\#x(x = 0)$. We might treat this as the number 1. Moreover, Hume's Principle entails that $0 \neq 1$, since there cannot be a bijection from the non-self-identical objects to the objects identical with 0 (there are none of the former, but one of the latter). Now, working impredicatively again, 1 falls within our original domain of quantification. So we can sensibly apply Hume's Principle with the predicate "$(x = 0 \lor x = 1)$" to obtain an object $\#x(x = 0 \lor x = 1)$. We might treat this as the number 2, and we can show that $0 \neq 2$ and $1 \neq 2$ and so on.

In short, taken impredicatively, Hume's Principle entails that there are *infinitely many objects*. And this has encouraged *neo-Fregean logicists* to take Hume's Principle as the foundation for arithmetic.

Frege *himself*, though, did not take Hume's Principle as his foundation for arithmetic. Instead, Frege proved Hume's Principle from an explicit definition: $\#x\,F(x)$ is defined as the extension of the concept $F \sim \Phi$. In modern terms, we might attempt to render this as $\#x\,F(x) = \{G : F \sim G\}$; but this will pull us back into the problems of Naïve Comprehension.

 /

Chapter 8

Cardinal Arithmetic

8.1 Defining the Basic Operations

Since we do not need to keep track of order, cardinal arithmetic is rather easier to define than ordinal arithmetic. We will define addition, multiplication, and exponentiation simultaneously.

Definition 8.1. When $\mathfrak{a}$ and $\mathfrak{b}$ are cardinals:

$$\mathfrak{a} \oplus \mathfrak{b} = |\mathfrak{a} \sqcup \mathfrak{b}|$$
$$\mathfrak{a} \otimes \mathfrak{b} = |\mathfrak{a} \times \mathfrak{b}|$$
$$\mathfrak{a}^{\mathfrak{b}} = |{}^{\mathfrak{b}}\mathfrak{a}|$$

where ${}^{X}Y = \{f : f \text{ is a function } X \to Y\}$. (It is easy to show that ${}^{X}Y$ exists for any sets X and Y; we leave this as an exercise.)

Problem 8.1. Prove in $\mathbf{Z}^{-}$ that ${}^{X}Y$ exists for any sets X and Y. Working in $\mathbf{ZF}$, compute rank(${}^{X}Y$) from rank(X) and rank(Y), in the manner of Lemma 6.9.

It might help to explain this definition. Concerning addition: this uses the notion of disjoint sum, $\sqcup$, as defined in Definition 6.1; and it is easy to see that this definition gives the right verdict for finite cases. Concerning multiplication: **??** tells us that if A has n members and B has m members then $A \times B$ has $n \cdot m$ members, so our definition simply generalises the idea to transfinite multiplication. Exponentiation is similar: we are simply generalising the thought from the finite to the transfinite. Indeed, in certain ways, transfinite cardinal arithmetic looks much more like "ordinary" arithmetic than does transfinite ordinal arithmetic:

Proposition 8.2. $\oplus$ *and* $\otimes$ *are commutative and associative.*

Proof. For commutativity, by Lemma 7.3 it suffices to observe that $(\mathfrak{a} \sqcup \mathfrak{b}) \approx (\mathfrak{b} \sqcup \mathfrak{a})$ and $(\mathfrak{a} \times \mathfrak{b}) \approx (\mathfrak{b} \times \mathfrak{a})$. We leave associativity as an exercise. $\square$

Problem 8.2. Prove that $\oplus$ and $\otimes$ are associative.

Proposition 8.3. *A is infinite iff* $|A| \oplus 1 = 1 \oplus |A| = |A|$.

Proof. As in Theorem 7.7, from Lemma 6.10 and Lemma 7.3. $\qquad\qquad\square$

This explains why we need to use different symbols for ordinal versus cardinal addition/multiplication: these are genuinely *different* operations. This next pair of results shows that ordinal versus cardinal exponentiation are also different operations. (Recall that Definition 2.7 entails that $2 = \{0, 1\}$):

Lemma 8.4. $|\wp(A)| = 2^{|A|}$, *for any* A.

Proof. For each subset $B \subseteq A$, let $\chi_B \in {}^A2$ be given by:

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Now let $f(B) = \chi_B$; this defines a bijection $f \colon \wp(A) \to {}^A2$. So $\wp(A) \approx {}^A2$. Hence $\wp(A) \approx {}^{|A|}2$, so that $|\wp(A)| = \left|{}^{|A|}2\right| = 2^{|A|}$. $\qquad\square$

This snappy proof essentially subsumes the discussion of **??**. There, we showed how to "reduce" the uncountability of $\wp(\omega)$ to the uncountability of the set of infinite binary strings, $\mathbb{B}^\omega$. In effect, $\mathbb{B}^\omega$ is just ${}^\omega2$; and the preceding proof showed that the reasoning we went through in **??** will go through using any set A in place of ω. The result also yields a quick fact about cardinal exponentiation:

Corollary 8.5. $\mathfrak{a} < 2^\mathfrak{a}$ *for any cardinal* $\mathfrak{a}$.

Proof. From Cantor's Theorem (**??**) and Lemma 8.4. $\qquad\qquad\square$

So $\omega < 2^\omega$. But note: this is a result about *cardinal* exponentiation. It should be contrasted with *ordinal* exponentiation, since in the latter case $\omega = 2^{(\omega)}$ (see section 6.5).

Whilst we are on the topic of cardinal exponentiation, we can also be a bit more precise about the "way" in which $\mathbb{R}$ is non-enumerable.

Theorem 8.6. $|\mathbb{R}| = 2^\omega$

Proof skeleton. There are plenty of ways to prove this. The most straightforward is to argue that $\wp(\omega) \preceq \mathbb{R}$ and $\mathbb{R} \preceq \wp(\omega)$, and then use Schröder-Bernstein to infer that $\mathbb{R} \approx \wp(\omega)$, and Lemma 8.4 to infer that $|\mathbb{R}| = 2^\omega$. We leave it as an (illuminating) exercise to define injections $f \colon \wp(\omega) \to \mathbb{R}$ and $g \colon \mathbb{R} \to \wp(\omega)$. $\square$

Problem 8.3. Complete the proof of Theorem 8.6, by showing that $\wp(\omega) \preceq \mathbb{R}$ and $\mathbb{R} \preceq \wp(\omega)$.

8.2 Simplifying Addition and Multiplication

It turns out that transfinite cardinal addition and multiplication is *extremely* easy. This follows from the fact that cardinals are (certain) ordinals, and so well-ordered, and so can be manipulated in a certain way. Showing this, though, is *not* so easy. To start, we need a tricksy definition:

Definition 8.7. We define a *canonical ordering*, $\lhd$, on pairs of ordinals, by stipulating that $\langle \alpha_1, \alpha_2 \rangle \lhd \langle \beta_1, \beta_2 \rangle$ iff either:

1. $\max(\alpha_1, \alpha_2) < \max(\beta_1, \beta_2)$; or

2. $\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)$ and $\alpha_1 < \beta_1$; or

3. $\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)$ and $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$

Lemma 8.8. $\langle \alpha \times \alpha, \lhd \rangle$ *is a well-order, for any ordinal* α.

Proof. Evidently $\lhd$ is connected on $\alpha \times \alpha$. For suppose that neither $\langle \alpha_1, \alpha_2 \rangle$ nor $\langle \beta_1, \beta_2 \rangle$ is $\lhd$-less than the other. Then $\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)$ and $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, so that $\langle \alpha_1, \alpha_2 \rangle = \langle \beta_1, \beta_2 \rangle$.

To show well-ordering, let $X \subseteq \alpha \times \alpha$ be non-empty. Since α is an ordinal, some δ is the least member of $\{ \max(\gamma_1, \gamma_2) : \langle \gamma_1, \gamma_2 \rangle \in X \}$. Now discard all pairs from $\{ \langle \gamma_1, \gamma_2 \rangle \in X : \max(\gamma_1, \gamma_2) = \delta \}$ except those with least first coordinate; from among these, the pair with least second coordinate is the $\lhd$-least element of X. $\square$

Now for a teensy, simple observation:

Proposition 8.9. *If* $\alpha \approx \beta$, *then* $\alpha \times \alpha \approx \beta \times \beta$.

Proof. Just let $f \colon \alpha \to \beta$ induce $\langle \gamma_1, \gamma_2 \rangle \mapsto \langle f(\gamma_1), f(\gamma_2) \rangle$. $\square$

And now we will put all this to work, in proving a crucial lemma:

Lemma 8.10. $\alpha \approx \alpha \times \alpha$, *for any infinite ordinal* α

Proof. For reductio, let α be the least infinite ordinal for which this is false. **??** shows that $\omega \approx \omega \times \omega$, so $\omega \in \alpha$. Moreover, α is a cardinal: suppose otherwise, for reductio; then $|\alpha| \in \alpha$, so that $|\alpha| \approx |\alpha| \times |\alpha|$, by hypothesis; and $|\alpha| \approx \alpha$ by definition; so that $\alpha \approx \alpha \times \alpha$ by Proposition 8.9.

Now, for each $\langle \gamma_1, \gamma_2 \rangle \in \alpha \times \alpha$, consider the segment:

$$\mathrm{Seg}(\gamma_1, \gamma_2) = \{ \langle \delta_1, \delta_2 \rangle \in \alpha \times \alpha : \langle \delta_1, \delta_2 \rangle \lhd \langle \gamma_1, \gamma_2 \rangle \}$$

Letting $\gamma = \max(\gamma_1, \gamma_2)$, note that $\langle \gamma_1, \gamma_2 \rangle \lhd \langle \gamma + 1, \gamma + 1 \rangle$. So, when γ is infinite, observe:

$$\mathrm{Seg}(\gamma_1, \gamma_2) \precsim ((\gamma + 1) \cdot (\gamma + 1))$$
$$\approx (\gamma \cdot \gamma), \text{ by Lemma 6.10 and Proposition 8.9}$$
$$\approx \gamma, \text{ by the induction hypothesis}$$
$$\prec \alpha, \text{ since } \alpha \text{ is a cardinal}$$

So $\text{ord}(\alpha \times \alpha, \lhd) \leq \alpha$, and hence $\alpha \times \alpha \preceq \alpha$. Since of course $\alpha \preceq \alpha \times \alpha$, the result follows by Schröder-Bernstein. $\qquad\square$

Finally, we get to our simplifying result:

Theorem 8.11. *If $\mathfrak{a}, \mathfrak{b}$ are infinite cardinals, then:*

$$\mathfrak{a} \otimes \mathfrak{b} = \mathfrak{a} \oplus \mathfrak{b} = max(\mathfrak{a}, \mathfrak{b}).$$

Proof. Without loss of generality, suppose $\mathfrak{a} = \max(\mathfrak{a}, \mathfrak{b})$. Then invoking Lemma 8.10, $\mathfrak{a} \otimes \mathfrak{a} = \mathfrak{a} \leq \mathfrak{a} \oplus \mathfrak{b} \leq \mathfrak{a} \oplus \mathfrak{a} \leq \mathfrak{a} \otimes \mathfrak{a}$. $\qquad\square$

Similarly, if $\mathfrak{a}$ is infinite, an $\mathfrak{a}$-sized union of $\leq \mathfrak{a}$-sized sets has size $\leq \mathfrak{a}$:

Proposition 8.12. *Let $\mathfrak{a}$ be an infinite cardinal. For each ordinal $\beta \in \mathfrak{a}$, let X_β be a set with $|X_\beta| \leq \mathfrak{a}$. Then $\left| \bigcup_{\beta \in \mathfrak{a}} X_\beta \right| \leq \mathfrak{a}$.*

Proof. For each $\beta \in \mathfrak{a}$, fix an injection $f_\beta \colon X_\beta \to \mathfrak{a}$.[1] Define an injection $g \colon \bigcup_{\beta \in \mathfrak{a}} X_\beta \to \mathfrak{a} \times \mathfrak{a}$ by $g(v) = \langle \beta, f_\beta(v) \rangle$, where $v \in X_\beta$ and $v \notin X_\gamma$ for any $\gamma \in \beta$. Now $\bigcup_{\beta \in \mathfrak{a}} X_\beta \preceq \mathfrak{a} \times \mathfrak{a} \approx \mathfrak{a}$ by Theorem 8.11. $\qquad\square$

8.3 Some Simplification with Cardinal Exponentiation

Whilst defining $\lhd$ was a little involved, the upshot is a useful result concerning cardinal addition and multiplication, Theorem 8.11. Transfinite exponentiation, however, cannot be simplified so straightforwardly. To explain why, we start with a result which extends a familiar pattern from the finitary case (though its proof is at a high level of abstraction):

Proposition 8.13. $\mathfrak{a}^{\mathfrak{b} \oplus \mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}} \otimes \mathfrak{a}^{\mathfrak{c}}$ *and* $(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b} \otimes \mathfrak{c}}$*, for any cardinals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$.*

Proof. For the first claim, consider a function $f \colon (\mathfrak{b} \sqcup \mathfrak{c}) \to \mathfrak{a}$. Now "split this", by defining $f_\mathfrak{b}(\beta) = f(\beta, 0)$ for each $\beta \in \mathfrak{b}$, and $f_\mathfrak{c}(\gamma) = f(\gamma, 1)$ for each $\gamma \in \mathfrak{c}$. The map $f \mapsto (f_\mathfrak{b} \times f_\mathfrak{c})$ is a bijection ${}^{\mathfrak{b} \sqcup \mathfrak{c}}\mathfrak{a} \to ({}^{\mathfrak{b}}\mathfrak{a} \times {}^{\mathfrak{c}}\mathfrak{a})$.

For the second claim, consider a function $f \colon \mathfrak{c} \to ({}^{\mathfrak{b}}\mathfrak{a})$; so for each $\gamma \in \mathfrak{c}$ we have some function $f(\gamma) \colon \mathfrak{b} \to \mathfrak{a}$. Now define $f^*(\beta, \gamma) = (f(\gamma))(\beta)$ for each $\langle \beta, \gamma \rangle \in \mathfrak{b} \times \mathfrak{c}$. The map $f \mapsto f^*$ is a bijection ${}^{\mathfrak{c}}({}^{\mathfrak{b}}\mathfrak{a}) \to {}^{\mathfrak{b} \otimes \mathfrak{c}}\mathfrak{a}$. $\qquad\square$

Now, what we would *like* is an easy way to compute $\mathfrak{a}^{\mathfrak{b}}$ when we are dealing with infinite cardinals. Here is a nice step in this direction:

Proposition 8.14. *If $2 \leq \mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b}$ is infinite, then $\mathfrak{a}^{\mathfrak{b}} = 2^{\mathfrak{b}}$*

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[1]How are these "fixed"? See section 9.5.

Proof.

$$\begin{aligned}
2^{\mathfrak{b}} &\leq \mathfrak{a}^{\mathfrak{b}}, \text{ as } 2 \leq \mathfrak{a} \\
&\leq (2^{\mathfrak{a}})^{\mathfrak{b}}, \text{ by Lemma 8.4} \\
&= 2^{\mathfrak{a} \otimes \mathfrak{b}}, \text{ by Proposition 8.13} \\
&= 2^{\mathfrak{b}}, \text{ by Theorem 8.11} \qquad \qquad \square
\end{aligned}$$

We should not really expect to be able to simplify this any further, since $\mathfrak{b} < 2^{\mathfrak{b}}$ by Lemma 8.4. However, this does not tell us what to say about $\mathfrak{a}^{\mathfrak{b}}$ when $\mathfrak{b} < \mathfrak{a}$. Of course, if $\mathfrak{b}$ is *finite*, we know what to do.

Proposition 8.15. *If $\mathfrak{a}$ is infinite and $n \in \omega$ then $\mathfrak{a}^n = \mathfrak{a}$*

Proof. $\mathfrak{a}^n = \mathfrak{a} \otimes \mathfrak{a} \otimes \ldots \otimes \mathfrak{a} = \mathfrak{a}$, by Theorem 8.11. $\qquad \square$

Additionally, in some other cases, we can control the size of $\mathfrak{a}^{\mathfrak{b}}$:

Proposition 8.16. *If $2 \leq \mathfrak{b} < \mathfrak{a} \leq 2^{\mathfrak{b}}$ and $\mathfrak{b}$ is infinite, then $\mathfrak{a}^{\mathfrak{b}} = 2^{\mathfrak{b}}$*

Proof. $2^{\mathfrak{b}} \leq \mathfrak{a}^{\mathfrak{b}} \leq (2^{\mathfrak{b}})^{\mathfrak{b}} = 2^{\mathfrak{b} \otimes \mathfrak{b}} = 2^{\mathfrak{b}}$, reasoning as in Proposition 8.14. $\qquad \square$

But, beyond this point, things become rather more subtle.

8.4 The Continuum Hypothesis

The previous result hints (correctly) that cardinal exponentiation would be quite *easy*, if infinite cardinals are guaranteed to "play straightforwardly" with powers of 2, i.e., (by Lemma 8.4) with taking powersets. But we cannot assume that infinite cardinals *do* play straightforwardly powersets.

To start unpacking this, we introduce some nice notation.

Definition 8.17. Where $\mathfrak{a}^{\oplus}$ is the least cardinal strictly greater than $\mathfrak{a}$, we define two infinite sequences:

$$\begin{aligned}
\aleph_0 &= \omega & \beth_0 &= \omega \\
\aleph_{\alpha+1} &= (\aleph_\alpha)^{\oplus} & \beth_{\alpha+1} &= 2^{\beth_\alpha} \\
\aleph_\alpha &= \bigcup_{\beta < \alpha} \aleph_\beta & \beth_\alpha &= \bigcup_{\beta < \alpha} \beth_\beta & \text{when } \alpha \text{ is a limit ordinal.}
\end{aligned}$$

The definition of $\mathfrak{a}^{\oplus}$ is in order, since Theorem 7.14 tells us that, for each cardinal $\mathfrak{a}$, there is some cardinal greater than $\mathfrak{a}$, and Transfinite Induction guarantees that there is a *least* cardinal greater than $\mathfrak{a}$. The rest of the definition of $\mathfrak{a}$ is provided by transfinite recursion.

Cantor introduced this "$\aleph$" notation; this is *aleph*, the first letter in the Hebrew alphabet and the first letter in the Hebrew word for "infinite". Peirce

introduced the "$\beth$" notation; this is *beth*, which is the second letter in the Hebrew alphabet.[2] Now, these notations provide us with infinite cardinals.

Proposition 8.18. $\aleph_\alpha$ *and* $\beth_\alpha$ *are cardinals, for every ordinal* α.

Proof. Both results hold by a simple transfinite induction. $\aleph_0 = \beth_0 = \omega$ is a cardinal by Corollary 7.9. Assuming $\aleph_\alpha$ and $\beth_\alpha$ are both cardinals, $\aleph_{\alpha+1}$ and $\beth_{\alpha+1}$ are explicitly defined as cardinals. And the union of a set of cardinals is a cardinal, by Proposition 7.13. $\square$

Moreover, every infinite cardinal is an $\aleph$:

Proposition 8.19. *If* $\mathfrak{a}$ *is an infinite cardinal, then* $\mathfrak{a} = \aleph_\gamma$ *for some unique* γ.

Proof. By transfinite induction on cardinals. For induction, suppose that if $\mathfrak{b} < \mathfrak{a}$ then $\mathfrak{b} = \aleph_{\gamma_\mathfrak{b}}$. If $\mathfrak{a} = \mathfrak{b}^\oplus$ for some $\mathfrak{b}$, then $\mathfrak{a} = (\aleph_{\gamma_\mathfrak{b}})^\oplus = \aleph_{\gamma_\mathfrak{b}+1}$. If $\mathfrak{a}$ is not the successor of any cardinal, then since cardinals are ordinals $\mathfrak{a} = \bigcup_{\mathfrak{b} < \mathfrak{a}} \mathfrak{b} = \bigcup_{\mathfrak{b} < \mathfrak{a}} \aleph_{\gamma_\mathfrak{b}}$, so $\mathfrak{a} = \aleph_\gamma$ where $\gamma = \bigcup_{\mathfrak{b} < \mathfrak{a}} \gamma_\mathfrak{b}$. $\square$

Since every infinite cardinal is an $\aleph$, this prompts us to ask: is every infinite cardinal a $\beth$? Certainly if that *were* the case, then the infinite cardinals would "play straightforwardly" with the operation of taking powersets. Indeed, we would have the following:

Generalized Continuum Hypothesis (GCH). $\aleph_\alpha = \beth_\alpha$, for all α.

Moreover, if GCH held, then we could make some considerable simplifications with cardinal exponentiation. In particular, we could show that when $\mathfrak{b} < \mathfrak{a}$, the value of $\mathfrak{a}^\mathfrak{b}$ is trapped by $\mathfrak{a} \leq \mathfrak{a}^\mathfrak{b} \leq \mathfrak{a}^\oplus$. We could then go on to give precise conditions which determine which of the two possibilities obtains (i.e., whether $\mathfrak{a} = \mathfrak{a}^\mathfrak{b}$ or $\mathfrak{a}^\mathfrak{b} = \mathfrak{a}^\oplus$).[3]

But GCH is a *hypothesis*, not a *theorem*. In fact, Gödel (1938) proved that if **ZFC** is consistent, then so is **ZFC** + GCH. But it later turned out that we can equally add ¬GCH to **ZFC**. Indeed, consider the simplest non-trivial *instance* of GCH, namely:

Continuum Hypothesis (CH). $\aleph_1 = \beth_1$.

Cohen (1963) proved that if **ZFC** is consistent then so is **ZFC** + ¬CH. So the Continuum Hypothesis is independent from **ZFC**.

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[2]Peirce used this notation in a letter to Cantor of December 1900. Unfortunately, Peirce also gave a bad argument there that $\beth_\alpha$ does not exist for $\alpha \geq \omega$.

[3]The condition is dictated by *cofinality*.

The Continuum Hypothesis is so-called, since "the continuum" is another name for the real line, $\mathbb{R}$. Theorem 8.6 tells us that $|\mathbb{R}| = \beth_1$. So the Continuum Hypothesis states that there is no cardinal between the cardinality of the natural numbers, $\aleph_0 = \beth_0$, and the cardinality of the continuum, $\beth_1$.

Given the *independence* of (G)CH from **ZFC**, what should say about their *truth*? Well, there is *much* to say. Indeed, and much fertile recent work in set theory has been directed at investigating these issues. But two very quick points are certainly worth emphasising.

First: it does not *immediately* follow from these formal independence results that either GCH or CH is *indeterminate* in truth value. After all, maybe we just need to add more axioms, which strike us as natural, and which will settle the question one way or another. Gödel himself suggested that this was the right response.

Second: the independence of CH from **ZFC** is certainly *striking*, but it is certainly not *incredible* (in the literal sense). The point is simply that, for all **ZFC** tells us, moving from cardinals to their successors may involve a less blunt tool than simply taking powersets.

With those two observations made, if you want to know more, you will now have to turn to the various philosophers and mathematicians with horses in the race.[4]

8.5 $\aleph$-Fixed Points

In chapter 4, we suggested that Replacement stands in need of justification, because it forces the hierarchy to be rather tall. Having done some cardinal arithmetic, we can give a little illustration of the height of the hierarchy.

Evidently $0 < \aleph_0$, and $1 < \aleph_1$, and $2 < \aleph_2 \ldots$ and, indeed, the difference in size only gets *bigger* with every step. So it is tempting to conjecture that $\kappa < \aleph_\kappa$ for every ordinal κ.

But this conjecture is *false*, given **ZFC**. In fact, we can prove that there are $\aleph$-*fixed-points*, i.e., cardinals κ such that $\kappa = \aleph_\kappa$.

Proposition 8.20. *There is an $\aleph$-fixed-point.*

Proof. Using recursion, define:

$$\kappa_0 = 0$$
$$\kappa_{n+1} = \aleph_{\kappa_n}$$
$$\kappa = \bigcup_{n < \omega} \kappa_n$$

Now κ is a cardinal by Proposition 7.13. But now:

$$\kappa = \bigcup_{n < \omega} \kappa_{n+1} = \bigcup_{n < \omega} \aleph_{\kappa_n} = \bigcup_{\alpha < \kappa} \aleph_\alpha = \aleph_\kappa$$

$\square$

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[4]Though you might want to start by reading Potter (2004, §15.6).

Boolos once wrote an article about exactly the $\aleph$-fixed-point we just constructed. After noting the existence of κ, at the start of his article, he said:

> [κ is] a *pretty big* number, by the lights of those with no previous exposure to set theory, so big, it seems to me, that it calls into question the truth of any theory, one of whose assertions is the claim that there are at least κ objects. (Boolos, 2000, p. 257)

And he ultimately concluded his paper by asking:

> [do] we suspect that, however it may have been at the beginning of the story, by the time we have come thus far the wheels are spinning and we are no longer listening to a description of anything that is the case? (Boolos, 2000, p. 268)

If we have, indeed, outrun "anything that is the case", then we must point the finger of blame directly at Replacement. For it is this axiom which allows our proof to work. In which case, one assumes, Boolos would need to revisit the claim he made, a few decades earlier, that Replacement has "no undesirable" consequences (see section 5.3).

But is the existence of κ so bad? It might help, here, to consider Russell's *Tristram Shandy paradox*. Tristram Shandy documents his life in his diary, but it takes him a year to record a single day. With every passing year, Tristram falls further and further behind: after one year, he has recorded only one day, and has lived 364 days unrecorded days; after two years, he has only recorded two days, and has lived 728 unrecorded days; after three years, he has only recorded three days, and lived 1092 unrecorded days ... [5] Still, if Tristram is *immortal*, Tristram will manage to record every day, for he will record the nth day on the nth year of his life. And so, "at the end of time", Tristram will have a complete diary.

Now: why is this so different from the thought that α is smaller than $\aleph_\alpha$— and indeed, increasingly, desperately smaller—up until κ, at which point, we catch up, and $\kappa = \aleph_\kappa$?

Setting that aside, and assuming we accept **ZFC**, let's close with a little more fun concerning fixed-point constructions. The next three results establish, intuitively, that there is a (non-trivial) point at which the hierarchy is as wide as it is tall:

Proposition 8.21. *There is a $\beth$-fixed-point, i.e., a κ such that $\kappa = \beth_\kappa$.*

sth:card-arithmetic:fix:
bethfixed

Proof. As in Proposition 8.20, using "$\beth$" in place of "$\aleph$". $\square$

Proposition 8.22. $|V_{\omega+\alpha}| = \beth_\alpha$. *If $\omega \cdot \omega \leq \alpha$, then $|V_\alpha| = \beth_\alpha$.*

sth:card-arithmetic:fix:
stagesize

Proof. The first claim holds by a simple transfinite induction. The second claim follows, since if $\omega \cdot \omega \leq \alpha$ then $\omega + \alpha = \alpha$. To establish this, we use facts

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[5]Forgetting about leap years.

set-theory rev: c9d2ed6 (2023-09-14) by OLP / CC–BY

about ordinal arithmetic from chapter 6. First note that $\omega \cdot \omega = \omega \cdot (1 + \omega) = (\omega \cdot 1) + (\omega \cdot \omega) = \omega + (\omega \cdot \omega)$. Now if $\omega \cdot \omega \leq \alpha$, i.e., $\alpha = (\omega \cdot \omega) + \beta$ for some β, then $\omega + \alpha = \omega + ((\omega \cdot \omega) + \beta) = (\omega + (\omega \cdot \omega)) + \beta = (\omega \cdot \omega) + \beta = \alpha$. $\quad\square$

Corollary 8.23. *There is a κ such that $|V_\kappa| = \kappa$.*

Proof. Let κ be a $\beth$-fixed point, as given by Proposition 8.21. Clearly $\omega \cdot \omega < \kappa$. So $|V_\kappa| = \beth_\kappa = \kappa$ by Proposition 8.22. $\quad\square$

There are as many stages beneath V_κ as there are elements of V_κ. Intuitively, then, V_κ is as wide as it is tall. This is very Tristram-Shandy-esque: we move from one stage to the next by taking *powersets*, thereby making our hierarchy *much* bigger with each step. But, "in the end", i.e., at stage κ, the hierarchy's width catches up with its height.

One might ask: *How often does the hierarchy's width match its height?* The answer is: *As often as there are ordinals.* But this needs a little explanation.

We define a term τ as follows. For any A, let:

$$\tau_0(A) = |A|$$
$$\tau_{n+1}(A) = \beth_{\tau_n(A)}$$
$$\tau(A) = \bigcup_{n<\omega} \tau_n(A)$$

As in Proposition 8.21, $\tau(A)$ is a $\beth$-fixed point for any A, and trivially $|A| < \tau(A)$. So now consider this recursive definition:

$$W_0 = 0$$
$$W_{\alpha+1} = \tau(W_\alpha)$$
$$W_\alpha = \bigcup_{\beta<\alpha} W_\beta, \text{ when } \alpha \text{ is a limit}$$

The construction is defined for all ordinals. Intuitively, then, W is "an injection" from the ordinals to $\beth$-fixed points. And, exactly as before, V_{W_α} is as wide as it is tall, for any α.

Chapter 9

Choice

9.1 Introduction

In chapters 7 to 8, we developed a theory of cardinals by treating cardinals as ordinals. That approach depends upon the Axiom of Well-Ordering. It turns out that Well-Ordering is equivalent to another principle—the Axiom of Choice—and there has been serious philosophical discussion of its acceptability. Our question for this chapter are: How is the Axiom used, and can it be justified?

9.2 The Tarski-Scott Trick

In Definition 7.1, we defined cardinals as ordinals. To do this, we assumed the Axiom of Well-Ordering. We did this, for no other reason than that it is the "industry standard".

Before we discuss any of the philosophical issues surrounding Well-Ordering, then, it is important to be clear that we *can* depart from the industry standard, and develop a theory of cardinals *without* assuming Well-Ordering. We can still employ the definitions of $A \approx B$, $A \preceq B$ and $A \prec B$, as they appeared in **??**. We will just need a new notion of *cardinal*.

A naïve thought would be to attempt to define A's cardinality thus:

$$\{x : A \approx x\}.$$

You might want to compare this with Frege's definition of $\#xFx$, sketched at the very end of section 7.5. And, for reasons we gestured at there, this definition fails. Any singleton set is equinumerous with $\{\emptyset\}$. But new singleton sets are formed at every successor stage of the hierarchy (just consider the singleton of the previous stage). So $\{x : A \approx x\}$ does not exist, since it cannot have a rank.

To get around this problem, we use a trick due to Tarski and Scott:[1]

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[1] A reminder: all formulas may have parameters (unless explicitly stated otherwise).

Definition 9.1 (Tarski-Scott). For any formula $\varphi(x)$, let $[x : \varphi(x)]$ be the set of all x, of least possible rank, such that $\varphi(x)$ (or $\emptyset$, if there are no φs).

We should check that this definition is legitimate. Working in **ZF**, Theorem 4.13 guarantees that $\mathrm{rank}(x)$ exists for every x. Now, if there are any entities satisfying φ, then we can let α be the least rank such that $(\exists x \subseteq V_\alpha)\varphi(x)$, i.e., $(\forall \beta \in \alpha)(\forall x \subseteq V_\beta)\neg\varphi(x)$. We can then define $[x : \varphi(x)]$ by Separation as $\{x \in V_{\alpha+1} : \varphi(x)\}$.

Having justified the Tarski-Scott trick, we can now use it to define a notion of cardinality:

Definition 9.2. The TS-cardinality of A is $\mathrm{tsc}(A) = [x : A \approx x]$.

The definition of a TS-cardinal does not use Well-Ordering. But, even without that Axiom, we can show that TS-*cardinals* behave rather like *cardinals* as defined in Definition 7.1. For example, if we restate Lemma 7.3 and Lemma 8.4 in terms of TS-cardinals, the proofs go through just fine in **ZF**, without assuming Well-Ordering.

Whilst we are on the topic, it is worth noting that we can also develop a theory of ordinals using the Tarski-Scott trick. Where $\langle A, < \rangle$ is a well-ordering, let $\mathrm{tso}(A, <) = [\langle X, R \rangle : \langle A, < \rangle \cong \langle X, R \rangle]$. For more on this treatment of cardinals and ordinals, see Potter (2004, chs. 9–12).

9.3 Comparability and Hartogs' Lemma

That's the plus side. Here's the minus side. Without Choice, things get *messy*. To see why, here is a nice result due to Hartogs (1915):

Lemma 9.3 (*in* ZF). *For any set A, there is an ordinal α such that $\alpha \npreceq A$*

Proof. If $B \subseteq A$ and $R \subseteq B^2$, then $\langle B, R \rangle \subseteq V_{\mathrm{rank}(A)+4}$ by Lemma 6.9. So, using Separation, consider:

$$C = \{\langle B, R \rangle \in V_{\mathrm{rank}(A)+5} : B \subseteq A \text{ and } \langle B, R \rangle \text{ is a well-ordering}\}$$

Using Replacement and Theorem 3.26, form the set:

$$\alpha = \{\mathrm{ord}(B, R) : \langle B, R \rangle \in C\}.$$

By Corollary 3.19, α is an ordinal, since it is a transitive set of ordinals. After all, if $\gamma \in \beta \in \alpha$, then $\beta = \mathrm{ord}(B, R)$ for some $B \subseteq R$, whereupon $\gamma = \mathrm{ord}(B_b, R_b)$ for some $b \in B$ by Lemma 3.10, so that $\gamma \in \alpha$.

For reductio, suppose there is an injection $f : \alpha \to A$. Then, where:

$$B = \mathrm{ran}(f)$$
$$R = \{\langle f(\alpha), f(\beta) \rangle \in A \times A : \alpha \in \beta\}.$$

Clearly $\alpha = \mathrm{ord}(B, R)$ and $\langle B, R \rangle \in C$. So $\alpha \in \alpha$, which is a contradiction. $\square$

This entails a deep result:

Theorem 9.4 (*in* ZF). *The following claims are equivalent:*

1. *The Axiom of Well-Ordering*

2. *Either $A \preceq B$ or $B \preceq A$, for any sets A and B*

Proof. (1) $\Rightarrow$ (2). Fix A and B. Invoking (1), there are well-orderings $\langle A, R \rangle$ and $\langle B, S \rangle$. Invoking Theorem 3.26, let $f \colon \alpha \to \langle A, R \rangle$ and $g \colon \beta \to \langle B, S \rangle$ be isomorphisms. By Proposition 3.22, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. If $\alpha \subseteq \beta$, then $g \circ f^{-1} \colon A \to B$ is an injection, and hence $A \preceq B$; similarly, if $\beta \subseteq \alpha$ then $B \preceq A$.

(2) $\Rightarrow$ (1). Fix A; by Lemma 9.3 there is some ordinal β such that $\beta \npreceq A$. Invoking (2), we have $A \preceq \beta$. So there is some injection $f \colon A \to \beta$, and we can use this injection to well-order the elements of A, by defining an order $\{\langle a, b \rangle \in A \times A : f(a) \in f(b)\}$. $\qquad \square$

As an immediate consequence: if Well-Ordering fails, then some sets are *literally incomparable* with regard to their size. So, if Well-Ordering fails, then transfinite cardinal arithmetic will be messy. For example, we will have to abandon the idea that if A and B are infinite then $A \sqcup B \approx A \times B \approx M$, where M is the larger of A and B (see Theorem 8.11). The problem is simple: if we cannot *compare* the size of A and B, then it is nonsensical to ask which is larger.

9.4 The Well-Ordering Problem

Evidently rather a lot hangs on whether we accept Well-Ordering. But the discussion of this principle has tended to focus on an equivalent principle, the Axiom of Choice. So we will now turn our attention to that (and prove the equivalence).

In 1883, Cantor expressed his support for the Axiom of Well-Ordering, calling it "a law of thought which appears to me to be fundamental, rich in its consequences, and particularly remarkable for its general validity" (cited in Potter 2004, p. 243). But Cantor ultimately became convinced that the "Axiom" was in need of proof. So did the mathematical community.

The problem was "solved" by Zermelo in 1904. To explain his solution, we need some definitions.

Definition 9.5. A function f is a *choice function* iff $f(x) \in x$ for all $x \in \mathrm{dom}(f)$. We say that f is a *choice function for A* iff f is a choice function with $\mathrm{dom}(f) = A \setminus \{\emptyset\}$.

Intuitively, for every (non-empty) set $x \in A$, a choice function for A *chooses* a particular element, $f(x)$, from x. The Axiom of Choice is then:

Axiom (Choice). Every set has a choice function.

Zermelo showed that Choice entails well-ordering, and vice versa:

Theorem 9.6 (in ZF). *Well-Ordering and Choice are equivalent.*

Proof. Left-to-right. Let A be a set of sets. Then $\bigcup A$ exists by the Axiom of Union, and so by Well-Ordering there is some $<$ which well-orders $\bigcup A$. Now let $f(x) =$ the $<$-least member of x. This is a choice function for A.

Right-to-left. Fix A. By Choice, there is a choice function, f, for $\wp(A) \setminus \{\emptyset\}$. Using Transfinite Recursion, define a function:

$$g(0) = f(A)$$

$$g(\alpha) = \begin{cases} \text{stop!} & \text{if } A = g[\alpha] \\ f(A \setminus g[\alpha]) & \text{otherwise} \end{cases}$$

The indication to "stop!" is just a shorthand for what would otherwise be a more long-winded definition. That is, when $A = g[\alpha]$ for the first time, let $g(\delta) = A$ for all $\delta \leq \alpha$. Now, in the first instance, we can only be sure that this defines a *term* (see the remarks after Theorem 4.4); but we will show that we indeed have a function.

Since f is a choice function, for each α (when defined) we have $g(\alpha) = f(A \setminus g[\alpha]) \in A \setminus g[\alpha]$; i.e., $g(\alpha) \notin g[\alpha]$. So if $g(\alpha) = g(\beta)$ then $g(\beta) \notin g[\alpha]$, i.e., $\beta \notin \alpha$, and similarly $\alpha \notin \beta$. So $\alpha = \beta$, by Trichotomy. So g is injective.

Next, observe that we do stop!, i.e. that there is some (least) ordinal α such that $A = g[\alpha]$. For suppose otherwise; then as g is injective we would have $\alpha \prec \wp(A) \setminus \{\emptyset\}$ for every ordinal α, contradicting Lemma 9.3. Hence also $\mathrm{ran}(g) = A$.

Assembling these facts, g is a bijection from some ordinal to A. Now g can be used to well-order A. $\square$

So Well-Ordering and Choice stand or fall together. But the question remains: do they stand or fall?

9.5 Countable Choice

It is easy to prove, without any use of Choice/Well-Ordering, that:

Lemma 9.7 (in $\mathbf{Z}^-$). *Every finite set has a choice function.*

Proof. Let $a = \{b_1, \ldots, b_n\}$. Suppose for simplicity that each $b_i \neq \emptyset$. So there are objects $c_1, \ldots, c_n$ such that $c_1 \in b_1, \ldots, c_n \in b_n$. Now by Proposition 2.5, the set $\{\langle b_1, c_1 \rangle, \ldots, \langle b_n, c_n \rangle\}$ exists; and this is a choice function for a. $\square$

But matters get murkier as soon as we consider infinite sets. For example, consider this "minimal" extension to the above:

> *Countable Choice.* Every *countable* set has a choice function.

This is a special case of Choice. And it transpires that this principle was invoked fairly frequently, without an obvious awareness of its use. Here are two nice examples.[2]

Example 9.8. Here is a natural thought: for any set A, either $\omega \preceq A$, or $A \approx n$ for some $n \in \omega$. This is one way to state the intuitive idea, that every set is either finite or infinite. Cantor, and many other mathematicians, made this claim without proving it. Cautious as we are, we proved this in Theorem 7.7. But in that proof we were working in **ZFC**, since we were assuming that any set A can be well-ordered, and hence that $|A|$ is guaranteed to exist. That is: we explicitly assumed Choice.

In fact, Dedekind (1888) offered his own proof of this claim, as follows:

Theorem 9.9 (in $\mathbf{Z}^- +$ Countable Choice). *For any A, either $\omega \preceq A$ or $A \approx n$ for some $n \in \omega$.*

Proof. Suppose $A \not\approx n$ for all $n \in \omega$. Then in particular for each $n < \omega$ there is subset $A_n \subseteq A$ with exactly 2^n elements. Using this sequence $A_0, A_1, A_2, \ldots$, we define for each n:
$$B_n = A_n \setminus \bigcup_{i < n} A_i.$$

Now note the following

$$\left| \bigcup_{i < n} A_n \right| \leq |A_0| + |A_1| + \ldots + |A_{n-1}|$$
$$= 1 + 2 + \ldots + 2^{n-1}$$
$$= 2^n - 1$$
$$< 2^n = |A_n|$$

Hence each B_n has at least one member, c_n. Moreover, the B_ns are pairwise disjoint; so if $c_n = c_m$ then $n = m$. But every $c_n \in A$. So the function $f(n) = c_n$ is an injection $\omega \to A$. $\qquad\square$

Dedekind did not flag that he had used Countable Choice. But, did *you* spot its use? Look again. (Really: *look again.*)

The proof used Countable Choice twice. We used it once, to obtain our sequence of sets A_0, A_1, A_2, ... We then used it again to select our elements c_n from each B_n. Moreover, this use of Choice is ineliminable. Cohen (1966, p. 138) proved that the result fails if we have no version of Choice. That is: it is consistent with **ZF** that there are sets which are *incomparable* with ω.

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[2]Due to Potter (2004, §9.4) and Luca Incurvati.

Example 9.10. In 1878, Cantor stated that a countable union of countable sets is countable. He did not present a proof, perhaps indicating that he took the proof to be obvious. Now, cautious as we are, we proved a more general version of this result in Proposition 8.12. But our proof explicitly assumed Choice. And even the proof of the less general result requires Countable Choice.

Theorem 9.11 (in $\mathbf{Z}^-$ + Countable Choice). *If A_n is countable for each $n \in \omega$, then $\bigcup_{n<\omega} A_n$ is countable.*

Proof. Without loss of generality, suppose that each $A_n \neq \emptyset$. So for each $n \in \omega$ there is a surjection $f_n \colon \omega \to A_n$. Define $f \colon \omega \times \omega \to \bigcup_{n<\omega} A_n$ by $f(m,n) = f_n(m)$. The result follows because $\omega \times \omega$ is countable (**??**) and f is a surjection. $\qquad\square$

Did you spot the use of the Countable Choice? It is used to choose our sequence of functions f_0, f_1, f_2, ...[3] And again, the result fails in the absence of any Choice principle. Specifically, Feferman and Levy (1963) proved that it is consistent with **ZF** that a countable union of countable sets has cardinality $\beth_1$. But here is a much funnier statement of the point, from Russell:

> This is illustrated by the millionaire who bought a pair of socks whenever he bought a pair of boots, and never at any other time, and who had such a passion for buying both that at last he had $\aleph_0$ pairs of boots and $\aleph_0$ pairs of socks... Among boots we can distinguish right and left, and therefore we can make a selection of one out of each pair, namely, we can choose all the right boots or all the left boots; but with socks no such principle of selection suggests itself, and we cannot be sure, unless we assume the multiplicative axiom [i.e., in effect Choice], that there is any class consisting of one sock out of each pair. (Russell, 1919, p. 126)

In short, some form of Choice is needed to prove the following: If you have countably many pairs of socks, then you have (only) countably many socks. And in fact, without Countable Choice (or something equivalent), a countable union of countable sets can fail to be countable.

The moral is that Countable Choice was used repeatedly, without much awareness of its users. The philosophical question is: How could we *justify* Countable Choice?

An attempt at an intuitive justification might invoke an appeal to a super-task. Suppose we make the first choice in $1/2$ a minute, our second choice in $1/4$ a minute, ..., our n-th choice in $1/2^n$ a minute, ... Then within 1 minute, we will have made an ω-sequence of choices, and defined a choice function.

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[3] A similar use of Choice occurred in Proposition 8.12, when we gave the instruction "For each $\beta \in \mathfrak{a}$, fix an injection f_β".

But what, really, could such a thought-experiment tell us? For a start, it relies upon taking this idea of "choosing" rather literally. For another, it seems to bind up mathematics in metaphysical possibility.

More important: it is not going to give us any justification for Choice *tout court*, rather than *mere* Countable Choice. For if we need *every* set to have a choice function, then we'll need to be able to perform a "supertask of arbitrary ordinal length." Bluntly, that idea is laughable.

9.6 Intrinsic Considerations about Choice

The broader question, then, is whether Well-Ordering, or Choice, or indeed the comparability of all sets as regards their size—it doesn't matter which—can be justified.

Here is an attempted *intrinsic* justification. Back in section 2.1, we introduced several principles about the hierarchy. One of these is worth restating:

> *Stages-accumulate.* For any stage S, and for any sets which were formed *before* stage S: a set is formed at stage S whose members are exactly those sets. Nothing else is formed at stage S.

In fact, many authors have suggested that the Axiom of Choice can be justified via (something like) this principle. We will briefly provide a gloss on that approach.

We will start with a simple little result, which offers *yet another* equivalent for Choice:

Theorem 9.12 (in ZF). *Choice is equivalent to the following principle. If the elements of A are disjoint and non-empty, then there is some C such that $C \cap x$ is a singleton for every $x \in A$. (We call such a C a choice set for A.)*

The proof of this result is straightforward, and we leave it as an exercise for the reader.

Problem 9.1. Prove Theorem 9.12. If you struggle, you can find a proof in (Potter, 2004, pp. 242–3).

The essential point is that a choice set for A is just the range of a choice function for A. So, to justify Choice, we can simply try to justify its equivalent formulation, in terms of the existence of choice sets. And we will now try to do exactly that.

Let A's elements be disjoint and non-empty. By *Stages-are-key* (see section 2.1), A is formed at some stage S. Note that all the elements of $\bigcup A$ are available before stage S. Now, by *Stages-accumulate*, for *any* sets which were formed before S, a set is formed whose members are exactly those sets. Otherwise put: every *possible* collections of earlier-available sets will exist at S. But it is certainly *possible* to select objects which could be formed into a choice set

for A; that is just some very specific subset of $\bigcup A$. So: some such choice set exists, as required.

Well, that's a *very* quick attempt to offer a justification of Choice on intrinsic grounds. But, to pursue this idea further, you should read Potter's (2004, §14.8) neat development of it.

9.7 The Banach-Tarski Paradox

We might also attempt to justify Choice, as Boolos attempted to justify Replacement, by appealing to *extrinsic* considerations (see section 5.3). After all, adopting Choice has many desirable consequences: the ability to compare every cardinal; the ability to well-order every set; the ability to treat cardinals as a particular kind of ordinal; etc.

Sometimes, however, it is claimed that Choice has *undesirable* consequences. Mostly, this is due to a result by Banach and Tarski (1924).

Theorem 9.13 (Banach-Tarski Paradox (in ZFC)). *Any ball can be decomposed into finitely many pieces, which can be reassembled (by rotation and transportation) to form two copies of that ball.*

At first glance, this is a bit amazing. Clearly the two balls have *twice* the volume of the original ball. But rigid motions—rotation and transportation—do not change volume. So it looks as if Banach-Tarski allows us to magick new matter into existence.

It gets worse.[4] Similar reasoning shows that a pea can be cut into finitely many pieces, which can then be reassembled (by rotation and transportation) to form an entity the shape and size of Big Ben.

None of this, however, holds in **ZF** on its own.[5] So we face a decision: reject Choice, or learn to live with the "paradox".

We're going to suggest that we should learn to live with the "paradox". Indeed, we don't think it's much of a paradox at all. In particular, we don't see why it is any more or less paradoxical than any of the following results:[6]

1. There are as many points in the interval $(0, 1)$ as in $\mathbb{R}$.
   Proof: consider $\tan(\pi(r - 1/2))$.

2. There are as many points in a line as in a square.
   See **??** and **??**.

3. There are space-filling curves.
   See **??** and **??**.

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[4]See Tomkowicz and Wagon (2016, Theorem 3.12).

[5]Though Banach-Tarski can be proved with principles which are strictly weaker than Choice; see Tomkowicz and Wagon (2016, 303).

[6]Potter (2004, 276–7), Weston (2003, 16), Tomkowicz and Wagon (2016, 31, 308–9), make similar points, using other examples.

None of these three results require Choice. Indeed, we now just regard them as surprising, lovely, bits of mathematics. Maybe we should adopt the same attitude to the Banach-Tarski Paradox.

To be sure, a technical observation is required here; but it only requires keeping a level head. Rigid motions preserve volume. Consequently, the five[7] pieces into which the ball is decomposed cannot all be *measurable*. Roughly put, then, it makes no sense to assign a volume to these individual pieces. You should think of these as unpicturable, "infinite scatterings" of points. Now, maybe it is "weird" to conceive of such "infinitely scattered" sets. But their existence seems to fall out from the injunction, embodied in *Stages-accumulate*, that you should form *all possible* collections of earlier-available sets.

If none of that convinces, here is a final (extrinsic) argument in favour of embracing the Banach-Tarski Paradox. It immediately entails the best math joke of all time:

> *Question.* What's an anagram of "Banach-Tarski"?

> *Answer.* "Banach-Tarski Banach-Tarski".

9.8 Appendix: Vitali's Paradox

To get a real sense of whether the Banach-Tarski construction is acceptable or not, we should examine its *proof*. Unfortunately, that would require much more algebra than we can present here. However, we can offer some quick remarks which might shed some insight on the proof of Banach-Tarski,[8] by focussing on the following result:

Theorem 9.14 (Vitali's Paradox (in ZFC)). *Any circle can be decomposed into countably many pieces, which can be reassembled (by rotation and transportation) to form two copies of that circle.*

Vitali's Paradox is much easier to prove than the Banach–Tarski Paradox. We have called it "Vitali's Paradox", since it follows from Vitali's 1905 construction of an unmeasurable set. But the set-theoretic aspects of the proof of Vitali's Paradox and the Banach-Tarski Paradox are very similar. The essential difference between the results is just that Banach-Tarski considers a *finite* decomposition, whereas Vitali's Paradox considers a *countably infinite* decomposition. As Weston (2003) puts it, Vitali's Paradox "is certainly not nearly as striking as the Banach–Tarski paradox, but it does illustrate that geometric paradoxes can happen even in 'simple' situations."

Vitali's Paradox concerns a two-dimensional figure, a circle. So we will work on the plane, $\mathbb{R}^2$. Let R be the set of (clockwise) rotations of points around

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[7]We stated the Paradox in terms of "finitely many pieces". In fact, Robinson (1947) proved that the decomposition can be achieved with *five* pieces (but no fewer). For a proof, see Tomkowicz and Wagon (2016, pp. 66–7).

[8]For a much fuller treatment, see Weston (2003) or Tomkowicz and Wagon (2016).

the origin by *rational* radian values between $[0, 2\pi)$. Here are some algebraic facts about R (if you don't understand the statement of the result, the proof will make its meaning clear):

Lemma 9.15. *R forms an abelian group under composition of functions.*

Proof. Writing 0_R for the rotation by 0 radians, this is an identity element for R, since $\rho \circ 0_R = 0_R \circ \rho = \rho$ for any $\rho \in R$.

Every element has an inverse. Where $\rho \in R$ rotates by r radians, $\rho^{-1} \in R$ rotates by $2\pi - r$ radians, so that $\rho \circ \rho^{-1} = 0_R$.

Composition is associative: $(\tau \circ \sigma) \circ \rho = \tau \circ (\sigma \circ \rho)$ for any $\rho, \sigma, \tau \in R$

Composition is commutative: $\sigma \circ \rho = \rho \circ \sigma$ for any $\rho, \sigma \in R$. $\qquad\square$

In fact, we can split our group R in half, and then use either half to recover the whole group:

Lemma 9.16. *There is a partition of R into two disjoint sets, R_1 and R_2, both of which are a basis for R.*

Proof. Let R_1 consist of the rotations by rational radian values in $[0, \pi)$; let $R_2 = R \setminus R_1$. By elementary algebra, $\{\rho \circ \rho : \rho \in R_1\} = R$. A similar result can be obtained for R_2. $\qquad\square$

We will use this fact about groups to establish Theorem 9.14. Let $\mathbf{S}$ be the unit circle, i.e., the set of points exactly 1 unit away from the origin of the plane, i.e., $\{\langle r, s \rangle \in \mathbb{R}^2 : \sqrt{r^2 + s^2} = 1\}$. We will split $\mathbf{S}$ into parts by considering the following relation on $\mathbf{S}$:

$$r \sim s \ \textit{iff} \ (\exists \rho \in R)\rho(r) = s.$$

That is, the points of $\mathbf{S}$ are linked by this relation iff you can get from one to the other by a rational-valued rotation about the origin. Unsurprisingly:

Lemma 9.17. *$\sim$ is an equivalence relation.*

Proof. Trivial, using Lemma 9.15. $\qquad\square$

We now invoke Choice to obtain a set, C, containing exactly one member from each equivalence class of $\mathbf{S}$ under $\sim$. That is, we consider a choice function f on the set of equivalence classes,[9]

$$E = \{[r]_\sim : r \in \mathbf{S}\},$$

and let $C = \text{ran}(f)$. For each rotation $\rho \in R$, the set $\rho[C]$ consists of the points obtained by applying the rotation ρ to each point in C. These next two results show that these sets cover the circle completely and without overlap:

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[9]Since R is enumerable, each element of E is enumerable. Since $\mathbf{S}$ is non-enumerable, it follows from Lemma 9.18 and Proposition 8.12 that E is non-enumerable. So this is a use of *un*countable Choice.

Lemma 9.18. $\mathbf{S} = \bigcup_{\rho \in R} \rho[C]$.

Proof. Fix $s \in \mathbf{S}$; there is some $r \in C$ such that $r \in [s]_\sim$, i.e., $r \sim s$, i.e., $\rho(r) = s$ for some $\rho \in R$. $\qquad\square$

Lemma 9.19. *If $\rho_1 \neq \rho_2$ then $\rho_1[C] \cap \rho_2[C] = \emptyset$.*

Proof. Suppose $s \in \rho_1[C] \cap \rho_2[C]$. So $s = \rho_1(r_1) = \rho_2(r_2)$ for some $r_1, r_2 \in C$. Hence $\rho_2^{-1}(\rho_1(r_1)) = r_2$, and $\rho_2^{-1} \circ \rho_1 \in R$, so $r_1 \sim r_2$. So $r_1 = r_2$, as C selects exactly one member from each equivalence class under $\sim$. So $s = \rho_1(r_1) = \rho_2(r_1)$, and hence $\rho_1 = \rho_2$. $\qquad\square$

We now apply our earlier algebraic facts to our circle:

Lemma 9.20. *There is a partition of $\mathbf{S}$ into two disjoint sets, D_1 and D_2,* *such that D_1 can be partitioned into countably many sets which can be rotated to form a copy of $\mathbf{S}$ (and similarly for D_2).*

Proof. Using R_1 and R_2 from Lemma 9.16, let:

$$D_1 = \bigcup_{\rho \in R_1} \rho[C] \qquad\qquad D_2 = \bigcup_{\rho \in R_2} \rho[C]$$

This is a partition of $\mathbf{S}$, by Lemma 9.18, and D_1 and D_2 are disjoint by Lemma 9.19. By construction, D_1 can be partitioned into countably many sets, $\rho[C]$ for each $\rho \in R_1$. And these can be rotated to form a copy of $\mathbf{S}$, since $\mathbf{S} = \bigcup_{\rho \in R} \rho[C] = \bigcup_{\rho \in R_1} (\rho \circ \rho)[C]$ by Lemma 9.16 and Lemma 9.18. The same reasoning applies to D_2. $\qquad\square$

This immediately entails Vitali's Paradox. For we can generate *two* copies of $\mathbf{S}$ from $\mathbf{S}$, just by splitting it up into countably many pieces (the various $\rho[C]$'s) and then rigidly moving them (simply rotate each piece of D_1, and first transport and then rotate each piece of D_2).

Let's recap the proof-strategy. We started with some algebraic facts about the group of rotations on the plane. We used this group to partition $\mathbf{S}$ into equivalence classes. We then arrived at a "paradox", by using Choice to select elements from each class.

We use exactly the same strategy to prove Banach–Tarski. The main difference is that the algebraic facts used to prove Banach–Tarski are significantly more complicated than those used to prove Vitali's Paradox. But those algebraic facts have nothing to do with Choice. We will summarise them quickly.

To prove Banach–Tarski, we start by establishing an analogue of Lemma 9.16: any *free group* can be split into four pieces, which intuitively we can "move around" to recover two copies of the whole group.[10] We then show that we can

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[10]The fact that we can use *four* pieces is due to Robinson (1947). For a recent proof, see Tomkowicz and Wagon (2016, Theorem 5.2). We follow Weston (2003, p. 3) in describing this as "moving" the pieces of the group.

use two particular rotations around the origin of $\mathbb{R}^3$ to generate a free group of rotations, F.[11] (No Choice yet.) We now regard points on the surface of the sphere as "similar" iff one can be obtained from the other by a rotation in F. We then *use Choice* to select exactly one point from each equivalence class of "similar" points. Applying our division of F to the surface of the sphere, as in Lemma 9.20, we split that surface into four pieces, which we can "move around" to obtain two copies of the surface of the sphere. And this establishes (Hausdorff, 1914):

Theorem 9.21 (Hausdorff's Paradox (in ZFC)). *The surface of any sphere can be decomposed into finitely many pieces, which can be reassembled (by rotation and transportation) to form two disjoint copies of that sphere.*

A couple of further algebraic tricks are needed to obtain the full Banach-Tarski Theorem (which concerns not just the sphere's surface, but its interior too). Frankly, however, this is just icing on the algebraic cake. Hence Weston writes:

> [. . .] the result on free groups is the *key step* in the proof of the Banach-Tarski paradox. From this point of view, the Banach-Tarski paradox is not a statement about $\mathbb{R}^3$ so much as it is a statement about the complexity of the group [of translations and rotations in $\mathbb{R}^3$]. (Weston, 2003, p. 16)

That is: whether we can offer a *finite* decomposition (as in Banach–Tarski) or a *countably infinite* decomposition (as in Vitali's Paradox) comes down to certain group-theoretic facts about working in two-dimension or three-dimensions.

Admittedly, this last observation slightly spoils the joke at the end of section 9.7. Since it is two dimensional, "Banach-Tarski" must be divided into a countable *infinity* of pieces, if one wants to rearrange those pieces to form "Banach-Tarski Banach-Tarski". To repair the joke, one must write in three dimensions. We leave this as an exercise for the reader.

One final comment. In section 9.7, we mentioned that the "pieces" of the sphere one obtains cannot be *measurable*, but must be unpicturable "infinite scatterings". The same is true of our use of Choice in obtaining Lemma 9.20. And this is all worth explaining.

Again, we must sketch some background (but this is *just* a sketch; you may want to consult a textbook entry on *measure*). To define a measure for a set X is to assign a value $\mu(E) \in \mathbb{R}$ for each E in some "σ-algebra" on X. Details here are not essential, except that the function μ must obey the principle of countable additivity: the measure of a countable union of disjoint sets is the sum of their individual measures, i.e., $\mu(\bigcup_{n<\omega} X_n) = \sum_{n<\omega} \mu(X_n)$ whenever the X_ns are disjoint. To say that a set is "unmeasurable" is to say that no measure can be suitably assigned. Now, using our R from before:

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[11]See Tomkowicz and Wagon (2016, Theorem 2.1).

Corollary 9.22 (Vitali). *Let μ be a measure such that $\mu(\mathbf{S}) = 1$, and such that $\mu(X) = \mu(Y)$ if X and Y are congruent. Then $\rho[C]$ is unmeasurable for all $\rho \in R$.*

Proof. For reductio, suppose otherwise. So let $\mu(\sigma[C]) = r$ for some $\sigma \in R$ and some $r \in \mathbb{R}$. For any $\rho \in C$, $\rho[C]$ and $\sigma[C]$ are congruent, and hence $\mu(\rho[C]) = r$ for any $\rho \in C$. By Lemma 9.18 and Lemma 9.19, $\mathbf{S} = \bigcup_{\rho \in R} \rho[C]$ is a countable union of pairwise disjoint sets. So countable additivity dictates that $\mu(\mathbf{S}) = 1$ is the sum of the measures of each $\rho[C]$, i.e.,

$$1 = \mu(\mathbf{S}) = \sum_{\rho \in R} \mu(\rho[C]) = \sum_{\rho \in R} r$$

But if $r = 0$ then $\sum_{\rho \in R} r = 0$, and if $r > 0$ then $\sum_{\rho \in R} r = \infty$. $\qquad\square$

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