replacement.1 The Strength of Replacement

sth:replacement:strength:sec

We begin with a simple observation about the strength of Replacement: unless we go beyond **Z**, we cannot prove the existence of any von Neumann ordinal greater than or equal to $\omega + \omega$.

Here is a sketch of why. Working in **ZF**, consider the set $V_{\omega+\omega}$. This set acts as the domain for a *model* for **Z**. To see this, we introduce some notation for the *relativization* of a formula:

sth:replacement:strength:formularelativization

Definition replacement.1. For any set M, and any formula φ, let φ^M be the formula which results by restricting all of φ's quantifiers to M. That is, replace "$\exists x$" with "$(\exists x \in M)$", and replace "$\forall x$" with "$(\forall x \in M)$".

It can be shown that, for every axiom φ of **Z**, we have that $\mathbf{ZF} \vdash \varphi^{V_{\omega+\omega}}$. But $\omega+\omega$ is not *in* $V_{\omega+\omega}$, by **??**. So **Z** is consistent with the non-existence of $\omega+\omega$.

This is why we said, in **??**, that **??** cannot be proved without Replacement. For it is easy, within **Z**, to define an explicit well-ordering which intuitively *should* have order-type $\omega + \omega$. Indeed, we gave an informal example of this in **??**, when we presented the ordering on the natural numbers given by:

$$\begin{aligned} n \lessdot m \text{ iff either } n < m \text{ and } m - n \text{ is even,} \\ \text{or } n \text{ is even and } m \text{ is odd.} \end{aligned}$$

But if $\omega+\omega$ does not exist, this well-ordering is not isomorphic to any ordinal. So **Z** does *not* prove **??**.

Flipping things around: Replacement allows us to prove the existence of $\omega+\omega$, and hence must allow us to prove the existence of $V_{\omega+\omega}$. And not just that. For *any* well-ordering we can define, **??** tells us that there is some α isomorphic with that well-ordering, and hence that V_α exists. In a straightforward way, then, Replacement guarantees that the hierarchy of sets must be *very tall*.

Over the next few sections, and then again in **??**, we'll get a better sense of better just *how* tall Replacement forces the hierarchy to be. The simple point, for now, is that Replacement really *does* stand in need of justification!

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Bibliography