replacement.1 The Strength of Replacement

We begin with a simple observation about the strength of Replacement: unless we go beyond Z, we cannot prove the existence of any von Neumann ordinal greater than or equal to $\omega + \omega$.

Here is a sketch of why. Working in ZF, consider the set $V_{\omega+\omega}$. This set acts as the domain for a model for Z. To see this, we introduce some notation for the relativization of a formula:

Definition replacement.1. For any set $M$, and any formula $\varphi$, let $\varphi^M$ be the formula which results by restricting all of $\varphi$'s quantifiers to $M$. That is, replace “$\exists x$” with “($\exists x \in M$)”, and replace “$\forall x$” with “($\forall x \in M$)”.

It can be shown that, for every axiom $\varphi$ of Z, we have that $\text{ZF} \vdash \varphi^{V_{\omega+\omega}}$. But $\omega + \omega$ is not in $V_{\omega+\omega}$, by ???. So Z is consistent with the non-existence of $\omega + \omega$.

This is why we said, in ???, that ?? cannot be proved without Replacement. For it is easy, within Z, to define an explicit well-ordering which intuitively should have order-type $\omega + \omega$. Indeed, we gave an informal example of this in ???, when we presented the ordering on the natural numbers given by:

$$n \prec m \text{ iff either } n < m \text{ and } m - n \text{ is even,}$$
$$\quad \text{ or } n \text{ is even and } m \text{ is odd.}$$

But if $\omega + \omega$ does not exist, this well-ordering is not isomorphic to any ordinal. So Z does not prove ???.

Flipping things around: Replacement allows us to prove the existence of $\omega + \omega$, and hence must allow us to prove the existence of $V_{\omega+\omega}$. And not just that. For any well-ordering we can define, ?? tells us that there is some $\alpha$ isomorphic with that well-ordering, and hence that $V_{\alpha}$ exists. In a straightforward way, then, Replacement guarantees that the hierarchy of sets must be very tall.

Over the next few sections, and then again in ???, we'll get a better sense of better just how tall Replacement forces the hierarchy to be. The simple point, for now, is that Replacement really does stand in need of justification!

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Bibliography