Chapter udf

Replacement

replacement.1 The Strength of Replacement

sth:replacement:strength:sec

Replacement is the axiom which makes the difference between **ZF** and **Z**. We helped ourselves to it throughout **??**–**??**. In this chapter, we will finally consider the question: is Replacement justified? To make the question sharp, it is worth observing that Replacement is really rather *strong*.

Unless we go beyond **Z**, we cannot prove the existence of any von Neumann ordinal greater than or equal to $\omega + \omega$. Here is a sketch of why. Working in **ZF**, consider the set $V_{\omega+\omega}$. We know from **??** that $\text{rank}(\omega + \omega) = V_{\omega+\omega}$. Now, this set acts as the domain for a *model* for **Z**. Indeed, where φ is any axiom of **Z**, let $\varphi^{V_{\omega+\omega}}$ be the formula which results by restricting all of φ's quantifiers to $V_{\omega+\omega}$ (that is, replace "$\exists x$" with "$(\exists x \in V_{\omega+\omega})$", and replace "$\forall x$" with "$(\forall x \in V_{\omega+\omega})$"). It can be shown that, for every axiom φ of **Z**, we have that $\mathbf{ZF} \vdash \varphi^{V_{\omega+\omega}}$. But $\omega + \omega$ is not *in* $V_{\omega+\omega}$. So **Z** is consistent with the non-existence of $\omega + \omega$.

This is why we said, in **??**, that **??** cannot be proved without Replacement. For it is easy, within **Z**, to define an explicit well-ordering which intuitively *should* have order-type $\omega + \omega$. Indeed, we gave an informal example of this in **??**, when we presented the ordering on the natural numbers given by:

$$\begin{aligned} n \lessdot m \text{ iff either } & |n - m| \text{ is even and } n < m, \\ & \text{or } n \text{ is even and } m \text{ is odd.} \end{aligned}$$

But if $\omega + \omega$ does not exist, this well-ordering is not isomorphic to any ordinal. So **Z** does *not* prove **??**.

Flipping things around: Replacement allows us to prove the existence of $\omega + \omega$, and hence must allow us to prove the existence of $V_{\omega+\omega}$. And not just that. For *any* well-ordering we can define, **??** tells us that there is some α isomorphic with that well-ordering, and hence that V_α exists. In a straightforward way, then, Replacement guarantees that the hierarchy of sets must be *very tall*.

Over the next few sections, and then again in **??**, we'll get a better sense of better just *how* tall Replacement forces the hierarchy to be. The simple point, for now, is that Replacement really *does* stand in need of justification!

replacement.2 Extrinsic Considerations about Replacement

We start by considering an *extrinsic* attempt to justify Replacement. Boolos suggests one, as follows. sth:replacement:extrinsic:sec

> [...] the reason for adopting the axioms of replacement is quite simple: they have many desirable consequences and (apparently) no undesirable ones. In addition to theorems about the iterative conception, the consequences include a satisfactory if not ideal theory of infinite numbers, and a highly desirable result that justifies inductive definitions on well-founded relations. (Boolos, 1971, 229)

The gist of Boolos's idea is that we should justify Replacement by its fruits. And the specific fruits he mentions are the things we have discussed in the past few chapters. Replacement allowed us to prove that the von Neumann ordinals were excellent surrogates for the idea of a well-ordering type (this is our "satisfactory if not ideal theory of infinite numbers"). Replacement also allowed us to define the V_αs, establish the notion of rank, and prove $\in$-Induction (this amounts to our "theorems about the iterative conception"). Finally, Replacement allows us to prove the Transfinite Recursion Theorem (this is the "inductive definitions on well-founded relations").

These are, indeed, desirable consequences. But do these desirable consequences suffice to *justify* Replacement? *No.* Or at least, not straightforwardly.

Here is a simple problem. Whilst we have stated some desirable consequences of Replacement, we could have obtained many of them via other means. This is not as well known as it ought to be. But the brief point is this. Building on work by Montague, Scott, and Derrick, Potter (2004) presents an elegant theory of sets. This is sometimes called **SP**, for "Scott–Potter", and we will stick with that name. Now, in its vanilla form, **SP** is strictly weaker than **ZF**, and does not deliver Replacement. Indeed, $V_{\omega+\omega}$ is an intuitive model of Potter's theory, just as it was of **Z**. However, **SP** is a bit stronger than **Z**. Indeed, it is sufficiently strong to deliver: a perfectly satisfactory theory of ordinals; results which stratify the hierarchy into well-ordered stages; a proof of $\in$-Induction; and a *version* of Transfinite Recursion. In short: although Boolos didn't know this, all of the desirable consequences which he mentions could have been arrived at *without* Replacement.

(Given all of this, why did we follow the conventional route, of teaching you **ZF**, rather than **SP**? There are three reasons for this. First: Potter's approach is rather nonstandard, and we wanted to equip you for reading more standard discussions of set theory. Second: when it comes to dealing with foundations, **SP** may be more philosophically satisfying than **ZF**, but it is harder to work with at first. So, frankly, you will only be in a position to appreciate **SP** *after* you've studied **ZF**. Third: when you are ready to appreciate **SP**, you can simply read Potter 2004.)

Of course, since **SP** is weaker than **ZF**, there are results which **ZF** proves which **SP** leaves open. So one could try to justify Replacement on extrinsic grounds by pointing to one of these results. But, once you know how to use **SP**, it is quite hard to find many examples of things that are (a) settled by Replacement but not otherwise, and (b) are intuitively true. (For more on this, see Potter 2004, §13.2.)

The bottom line is this. To provide a compelling extrinsic justification for Replacement, one would need to find a result which *cannot* be achieved without Replacement. And that's not an easy enterprise.

Let's consider a further problem which arises for any attempt to offer a purely extrinsic justification for Replacement. (This problem is perhaps more fundamental than the first.) Boolos does not just point out that Replacement has many desirable consequences. He also states that Replacement has "(apparently) no undesirable" consequences. But this paranthetical caveat, "apparently," is surely absolutely crucial.

Recall how we ended up here: Naïve Comprehension ran into inconsistency, and we responded to this inconsistency by embracing the cumulative-iterative conception of set. This conception comes equipped with a story which, we hope, assures us of its consistency. But if we cannot justify Replacement from within that story, then we have (as yet) no reason to believe that **ZF** is consistent. Or rather: we have no reason to believe that **ZF** is consistent, apart from the (perhaps merely contingent) fact that no one has discovered a contradiction *yet*. In exactly that sense, Boolos's comment seems to come down to this: "(apparently) **ZF** is consistent". We should demand greater reassurance of consistency than this.

This issue will affect any *purely* extrinsic attempt to justify Replacement, i.e., any justification which is couched solely in terms of the (known) consequences of **ZF**. As such, we will want to look for an *intrinsic* justification of Replacement, i.e., a justification which suggests that the story which we told about sets somehow "already" commits us to Replacement.

replacement.3 Limitation-of-size

sth:replacement:limofsize:sec

Perhaps the most common to offer an "intrinsic" justification of Replacement comes via the following notion:

> *Limitation-of-size.* Any things form a set, provided that there are not too many of them.

This principle will immediately vindicate Replacement. After all, any set formed by Replacement cannot be any larger than any set from which it was formed. Stated precisely: suppose you form a set $\tau[A] = \{\tau(x) : x \in A\}$ using Replacement; then $\tau[A] \preceq A$; so if the elements of A were not too numerous to form a set, their images are not too numerous to form $\tau[A]$.

The obvious difficulty with invoking *Limitation-of-size* to justify Replacement is that we have *not* yet laid down any principle like *Limitation-of-size*. Moreover, when we told our story about the cumulative-iterative conception of set in **??**–**??**, nothing ever *hinted* in the direction of *Limitation-of-size*. This, indeed, is precisely why Boolos at one point wrote: "Perhaps one may conclude that there are at least two thoughts 'behind' set theory" (1989, p. 19). On the one hand, the ideas surrounding the cumulative-iterative conception of set are meant to to vindicate **Z**. On the other hand, *Limitation-of-size* is meant to vindicate Replacement.

But the issue it is not just that we have thus far been *silent* about *Limitation-of-size*. Rather, the issue is that *Limitation-of-size* (as just formulated) seems to sit quite badly with the cumulative-iterative notion of set. After all, it mentions nothing about the idea of sets as formed in *stages*.

This is really not much of a surprise, given the history of these "two thoughts" (i.e., the cumulative-iterative conception of set, and *Limitation-of-size*). These "two thoughts" ultimately amount to two rather different projects for blocking the set-theoretic paradoxes. The cumulative-iterative notion of set blocks Russell's paradox by saying, roughly: *we should never have expected a Russell set to exist, because it would not be "formed" at any stage*. By contrast, *Limitation-of-size* is meant to rule out the Russell set, by saying, roughly: *we should never have expected a Russell set to exist, because it would have been too big*.

Put like this, then, let's be blunt: considered as a reply to the paradoxes, *Limitation-of-size* stands in need of *much* more justification. Consider, for example, this version of Russell's Paradox: *no pug sniffs exactly the pugs which don't sniff themselves*. If one asks "why is there no such pug?" it is not a good answer to be told that such a pug would have to sniff too many pugs. So why would it be a good intuitive explanation, for the non-existence of a Russell set, that it would have to be "too big" to exist?

So it's forgivable if you are a bit mystified concerning the "intuitive" motivation for *Limitation-of-size*.

replacement.4 Replacement and "Absolute Infinity"

We will now put *Limitation-of-size* behind us, and explore a different family of (intrinsic) attempts to justify Replacement, which do take seriously the idea of the sets as formed in stages. sth:replacement:absinf:sec

When we first outlined the iterative process, we offered some principles which explained what happens at each stage. These were *Stages-are-key*, *Stages-are-ordered*, and *Stages-accumulate*. Later, we added some principles which told us something about the number of stages: *Stages-keep-going* told us that the process of set-formation never ends, and *Stages-hit-infinity* told us that the process goes through an infinite-th stage.

It is reasonable to suggest that these two latter principles fall out of some a broader principle, like:

Stages-are-inexhaustible. There are absolutely infinitely many stages; the hierarchy is as tall as it could possibly be.

Obviously this is an informal principle. But even if it is not immediately *entailed* by the cumulative-iterative conception of set, it certainly seems *consonant* with it. At the very least, and unlike *Limitation-of-size*, it retains the idea that sets are formed stage-by-stage.

The hope, now, is to leverage *Stages-are-inexhaustible* into a justification of Replacement. So let us see how this might be done.

In **??**, we saw that it is easy to construct a well-ordering which (morally) should be isomorphic to $\omega + \omega$. Otherwise put, we can easily imagine a stage-by-stage iterative process, whose order-type (morally) is $\omega + \omega$. As such, if we have accepted *Stages-are-inexhaustible*, then we should surely accept that there is at least an $\omega + \omega$-th stage of the hierarchy, i.e., $V_{\omega+\omega}$, for the hierarchy surely *could* continue thus far.

This thought generalizes as follows: for any well-ordering, the process of building the iterative hierarchy should run at least as far as that well-ordering. And we could guarantee this, just by treating **??** as an *axiom*. This would tell us that any well-ordering is isomorphic to a von Neumann ordinal. Since each von Neumann ordinal will be equal to its own rank, **??** will then tell us that, whenever we can describe a well-ordering in our set theory, the iterative process of set building must outrun that well-ordering.

This idea certainly seems like a corollary of *Stages-are-inexhaustible*. Unfortunately, if our aim is to extract Replacement from this idea, then we face a simple, technical, barrier. By a result of Montague (1961), Replacement is strictly stronger than **??**.[1]

The upshot is that, if we are going to understand *Stages-are-inexhaustible* in such a way as to yield Replacement, then it cannot *merely* say that the hierarchy outruns any well-ordering. It must make a stronger claim than that. To this end, Shoenfield (1977) proposed a very natural strengthening of the idea, as follows: the hierarchy is not *cofinal* with any set.[2] In slightly more detail: if τ is a mapping which sends sets to stages of the hierarchy, the image of any set A under τ does not exhaust the hierarchy. Otherwise put (schematically):

Stages-are-super-cofinal. If A is a set and $\tau(x)$ is a stage for every $x \in A$, then there is a stage which comes after each $\tau(x)$ for $x \in A$.

It is obvious that **ZF** proves a suitably formalised version of *Stages-are-super-cofinal*. Conversely, we can informally argue that *Stages-are-super-cofinal* justifies Replacement.[3] For suppose $(\forall x \in A)\exists! y\, \varphi(x, y)$. Then for each $x \in A$,

[1] For more discussion of this general idea, though, see Potter (2004, §13.2) and Incurvati (2010) on the Axiom of Ordinals.

[2] Gödel seems to have proposed a similar thought; see (Potter, 2004, p. 223).

[3] It would be harder to prove Replacement using some formalisation of *Stages-are-super-cofinal*, since **Z** on its own is not strong enough to define the stages, so it is not clear how one would formalise *Stages-are-super-cofinal*. One good option, though, is to work in the theory presented by Potter (2004), which *can* define stages.

let $\sigma(x)$ be the y such that $\varphi(x, y)$, and let $\tau(x)$ be the stage at which $\sigma(x)$ is first formed. By *Stages-are-super-cofinal*, there is a stage V such that $(\forall x \in A)\tau(x) \in V$. Now since each $\tau(x) \in V$ and $\sigma(x) \subseteq \tau(x)$, by Separation we can obtain $\{y \in V : (\exists x \in A)\sigma(x) = y\} = \{y : (\exists x \in A)\varphi(x, y)\}$.

Problem replacement.1. Prove *Stages-are-super-cofinal* within **ZF**.

So *Stages-are-super-cofinal* vindicates Replacement. And it is at least plausible that *Stages-are-inexhaustible* vindicates *Stages-are-super-cofinal*. For suppose *Stages-are-super-cofinal* fails. So the hierarchy is cofinal with some set A, i.e., we have a map τ such that for any stage S there is some $x \in A$ such that $S \in \tau(x)$. In that case, we do have a way to get a handle on the supposed "absolute infinity" of the hierarchy: it is *exhausted* by the range of τ applied to A. And that compromises the thought that the hierarchy is "absolutely infinite". Contraposing: *Stages-are-inexhaustible* entails *Stages-are-super-cofinal*, which in turn justifies Replacement.

This represents a genuinely promising attempt to provide an intrinsic justification for Replacement. But whether it ultimately works, or not, we will have to leave to you to decide.

replacement.5 Replacement and Reflection

A final last attempt to justify Replacement, via *Stages-are-inexhaustible*, is to consider a lovely result: sth:replacement:ref:sec

Theorem replacement.1 (Reflection Schema)**.** *For any formula φ:*[4] *sth:replacement:ref:reflectionschema*

$$\forall\alpha\exists\beta > \alpha(\forall x_1 \dots, x_n \in V_\beta)(\varphi(x_1, \dots, x_n) \leftrightarrow \varphi^{V_\beta}(x_1, \dots, x_n))$$

As before, φ^{V_β} is the result of restricting every quantifier in φ to the set V_β. So, intuitively, Reflection says this: if φ is true in the entire hierarchy, then φ is true in arbitrarily many *initial segments* of the hierarchy.

Montague (1961) and Lévy (1960) showed that (suitable formulations of) Replacement and Reflection are equivalent, modulo **Z**, so that adding either gives you **ZF**. So, given this equivalence, one might hope to justify Reflection and Replacement via *Stages-are-inexhaustible* as follows: given *Stages-are-inexhaustible*, the hierarchy should be very, very tall; so tall, in fact, that nothing we can say about it is sufficient to bound its height. And we can understand this as the thought that, if any sentence φ is true in the entire hierarchy, then it is true in arbitrarily many initial segments of the hierarchy. And that is just Reflection.

Again, this seems like a genuinely promising attempt to provide an intrinsic justification for Replacement. But there is much too much to say about it here. You must now decide for yourself whether it succeeds.

[4]But which may also have parameters

Finally, we will prove that Replacement entails Reflection. This is easily the most advanced bit of mathematics in this textbook (so if you follow it, well done). We'll start with a lemma which, for brevity, employs the notational device of *overlining* to deal with sequences of variables or objects. So: "$\overline{a}_k$" abbreviates "$a_{k_1}, \dots, a_{k_n}$", where n is determined by context.

sth:replacement:ref: lemreflection

Lemma replacement.2. *For each $1 \leq i \leq k$, let $\varphi_i(\overline{v}_i, x)$ be a formula.*[5] *Then for each α there is some $\beta > \alpha$ such that, for any $\overline{a}_1, \dots, \overline{a}_k \in V_\beta$ and each $1 \leq i \leq k$:*

$$\exists x \varphi_i(\overline{a}_i, x) \to (\exists x \in V_\beta)\varphi_i(\overline{a}_i, x)$$

Proof. We define a term μ as follows: $\mu(\overline{a}_1, \dots, \overline{a}_k)$ is the least stage, V, which satisfies all of the following conditionals, for $1 \leq i \leq k$:

$$\exists x \varphi_i(\overline{a}_i, x) \to (\exists x \in V)\varphi_i(\overline{a}_i, x))$$

Using Replacement and our recursion theorem, define:

$$\begin{aligned} S_0 &= V_{\alpha+1} \\ S_{m+1} &= S_m \cup \bigcup \{\mu(\overline{a}_1, \dots, \overline{a}_k) : \overline{a}_1, \dots, \overline{a}_k \in S_m\} \\ S &= \bigcup_{m<\omega} S_m. \end{aligned}$$

Each S_m, and hence S itself, is a stage after V_α. Now fix $\overline{a}_1, \dots, \overline{a}_k \in S$; so there is some $m < \omega$ such that $\overline{a}_1, \dots, \overline{a}_k \in S_m$. Fix some $1 \leq i \leq k$, and suppose that $\exists x \varphi_i(\overline{a}_i, x)$. So $(\exists x \in \mu(\overline{a}_1, \dots, \overline{a}_k))\varphi_i(\overline{a}_i, x)$ by construction, so $(\exists x \in S_{m+1})\varphi_i(\overline{a}_i, x)$ and hence $(\exists x \in S)\varphi_i(\overline{a}_i, x)$. So S is our V_β. □

From here, we can prove Theorem replacement.1 quite straightforwardly:

Proof of Theorem replacement.1. Fix α. Without loss of generality, we can assume φ's only connectives are $\exists$, $\neg$ and $\wedge$ (since these are expressively adequate). Let $\psi_1, \dots, \psi_k$ enumerate each of φ's subformulas according to complexity, so that $\psi_k = \varphi$. By Lemma replacement.2, there is a $\beta > \alpha$ such that, for any $\overline{a}_i \in V_\beta$ and each $1 \leq i \leq k$:

$$\exists x \psi_i(\overline{a}_i, x) \to (\exists x \in V_\beta)\psi_i(\overline{a}_i, x) \tag{*}$$

By induction on complexity of ψ_i, we will show that $\psi_i(\overline{a}_i) \leftrightarrow \psi_i^{V_\beta}(\overline{a}_i)$, for any $\overline{a}_i \in V_\beta$. If ψ_i is atomic, this is trivial. The biconditional also establishes that, when ψ_i is a negation or conjunction of subformulas satisfying this property, ψ_i itself satisfies this property. So the only interesting case concerns quantification. Fix $\overline{a}_i \in V_\beta$; then:

$$\begin{aligned} (\exists x \psi_i(\overline{a}_i, x))^{V_\beta} & \text{ iff } (\exists x \in V_\beta)\psi_i^{V_\beta}(\overline{a}_i, x) && \text{by definition} \\ & \text{ iff } (\exists x \in V_\beta)\psi_i(\overline{a}_i, x) && \text{by the induction hypothesis} \\ & \text{ iff } \exists x \psi_i(\overline{a}_i, x) && \text{by } (*) \end{aligned}$$

[5] Which may have parameters.

This completes the induction; the result follows as $\psi_k = \varphi$. □

We have shown in **ZF** that Reflection holds. The proof essentially followed Montague (1961). We now want to prove in **Z** that Reflection entails Replacement. The proof follows Lévy (1960), but with a simplification.

Since we are working in **Z**, we cannot present Reflection in exactly the form given above. After all, we formulated Reflection using the "V_α" notation, and that cannot be defined in **Z**. So instead we will offer an apparently weaker formulation of Replacement, as follows:

Weak-Reflection. For any formula φ, there is a transitive set S such that 0, 1, and any parameters to φ are elements of S, and $(\forall \overline{x} \in S)(\varphi \leftrightarrow \varphi^S)$.

To use this to prove Replacement, we will first follow Lévy (1960, first part of Theorem 2) and show that we can "reflect" two formulas at once:

Lemma replacement.3 (in $\mathbf{Z}$ + Weak-Reflection.)**.** *For any formulas ψ, χ, there is a transitive set S such that* 0 *and* 1 *(and any parameters to the formulas) are elements of S, and $(\forall \overline{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$.* *sth:replacement:ref:lem:reflect*

Proof. Let φ be the formula $(z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)$.

Here we use an abbreviation; we should spell out "$z = 0$" as "$\forall t\, t \notin z$" and "$z = 1$" as "$\forall s(s \in z \leftrightarrow \forall t\, t \notin s)$". But since $0, 1 \in S$ and S is transitive, these formulas are *absolute* for S; that is, they will apply to the same object whether we restrict their quantifiers to S.[6]

By Weak-Reflection, we have some appropriate S such that:

$$(\forall z, \overline{x} \in S)(\varphi \leftrightarrow \varphi^S)$$
$$(\forall z, \overline{x} \in S)(((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)) \leftrightarrow ((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi))^S)$$
$$(\forall z, \overline{x} \in S)(((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)) \leftrightarrow ((z = 0 \wedge \psi^S) \vee (z = 1 \wedge \chi^S)))$$
$$(\forall \overline{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$$

The second claim entails the third because "$z = 0$" and "$z = 1$" are absolute for S; the fourth claim follows since $0 \neq 1$. □

We now obtain Replacement, simplifying Lévy (1960, Theorem 6):

Theorem replacement.4 (in $\mathbf{Z}$ + Weak-Reflection)**.** *For any formula $\varphi(v, w)$,*[7] *and any A, if $(\forall x \in A)\exists! y\, \varphi(x, y)$, then $\{y : (\exists x \in A)\varphi(x, y\}$ exists.*

Proof. Fix A such that $(\forall x \in A)\exists! y\, \varphi(x, y)$, and define some formulas:

$$\psi \text{ is } (\varphi(x, z) \wedge A = A)$$
$$\chi \text{ is } \exists y\, \varphi(x, y)$$

[6] More formally, letting ξ be either of these formulas, $\xi(z) \leftrightarrow \xi^S(z)$.
[7] Which may contain parameters

Using Lemma replacement.3, since A is a parameter to ψ, there is a transitive S such that $0, 1, A \in S$ (along with any other parameters), and such that:

$$(\forall x, z \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$$

So in particular:

$$(\forall x, z \in S)(\varphi(x, z) \leftrightarrow \varphi^S(x, z))$$
$$(\forall x \in S)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi^S(x, y))$$

Combining these, and observing that $A \subseteq S$ since $A \in S$ and S is transitive:

$$(\forall x \in A)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi(x, y))$$

Now $(\forall x \in A)(\exists! y \in S)\varphi(x, y)$, because $(\forall x \in A)\exists! y\, \varphi(x, y)$. Now Separation yields $\{y \in S : (\exists x \in A)\varphi(x, y)\} = \{y : (\exists x \in A)\varphi(x, y)\}$. $\square$

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