

Chapter udf

Replacement

replacement.1 Introduction

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sec

Replacement is the axiom scheme which makes the difference between **ZF** and **Z**. We helped ourselves to it throughout ??-??. In this chapter, we will finally consider the question: is Replacement justified?

To make the question sharp, it is worth observing that Replacement is really rather *strong*. We will get a sense of just how strong it is, during this chapter (and again in ??). But this will suggest that justification really is required.

We will discuss two kinds of justification. Roughly: an *extrinsic* justification is an attempt to justify an axiom by its fruits; an *intrinsic* justification is an attempt to justify an axiom by suggesting that it is vindicated by the mathematical concepts in question. We will get a greater sense of what this means during this chapter, but it is just the tip of an iceberg. For more, see in particular [Maddy \(1988a and 1988b\)](#).

replacement.2 The Strength of Replacement

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We begin with a simple observation about the strength of Replacement: unless we go beyond **Z**, we cannot prove the existence of any von Neumann ordinal greater than or equal to $\omega + \omega$.

Here is a sketch of why. Working in **ZF**, consider the set $V_{\omega+\omega}$. This set acts as the domain for a *model* for **Z**. To see this, we introduce some notation for the *relativization* of a formula:

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formularelativization

Definition replacement.1. For any set M , and any formula φ , let φ^M be the formula which results by restricting all of φ 's quantifiers to M . That is, replace " $\exists x$ " with " $(\exists x \in M)$ ", and replace " $\forall x$ " with " $(\forall x \in M)$ ".

It can be shown that, for every axiom φ of **Z**, we have that $\mathbf{ZF} \vdash \varphi^{V_{\omega+\omega}}$. But $\omega + \omega$ is not *in* $V_{\omega+\omega}$, by ??. So **Z** is consistent with the non-existence of $\omega + \omega$.

This is why we said, in ??, that ?? cannot be proved without Replacement. For it is easy, within **Z**, to define an explicit well-ordering which intuitively

should have order-type $\omega + \omega$. Indeed, we gave an informal example of this in ??, when we presented the ordering on the natural numbers given by:

$$n \prec m \text{ iff either } n < m \text{ and } m - n \text{ is even,} \\ \text{or } n \text{ is even and } m \text{ is odd.}$$

But if $\omega + \omega$ does not exist, this well-ordering is not isomorphic to any ordinal. So **Z** does *not* prove ??.

Flipping things around: Replacement allows us to prove the existence of $\omega + \omega$, and hence must allow us to prove the existence of $V_{\omega+\omega}$. And not just that. For *any* well-ordering we can define, ?? tells us that there is some α isomorphic with that well-ordering, and hence that V_α exists. In a straightforward way, then, Replacement guarantees that the hierarchy of sets must be *very tall*.

Over the next few sections, and then again in ??, we'll get a better sense of better just *how* tall Replacement forces the hierarchy to be. The simple point, for now, is that Replacement really *does* stand in need of justification!

replacement.3 Extrinsic Considerations about Replacement

We start by considering an *extrinsic* attempt to justify Replacement. Boolos [sth:replacement:extrinsic:sec](#) suggests one, as follows.

[...] the reason for adopting the axioms of replacement is quite simple: they have many desirable consequences and (apparently) no undesirable ones. In addition to theorems about the iterative conception, the consequences include a satisfactory if not ideal theory of infinite numbers, and a highly desirable result that justifies inductive definitions on well-founded relations. (Boolos, 1971, 229)

The gist of Boolos's idea is that we should justify Replacement by its fruits. And the specific fruits he mentions are the things we have discussed in the past few chapters. Replacement allowed us to prove that the von Neumann ordinals were excellent surrogates for the idea of a well-ordering type (this is our "satisfactory if not ideal theory of infinite numbers"). Replacement also allowed us to define the V_α s, establish the notion of rank, and prove \in -Induction (this amounts to our "theorems about the iterative conception"). Finally, Replacement allows us to prove the Transfinite Recursion Theorem (this is the "inductive definitions on well-founded relations").

These are, indeed, desirable consequences. But do these desirable consequences suffice to *justify* Replacement? *No*. Or at least, not straightforwardly.

Here is a simple problem. Whilst we have stated some desirable consequences of Replacement, we could have obtained many of them via other means. This is not as well known as it ought to be, though, so we should pause to explain the situation.

There is a simple theory of sets, Level Theory, or **LT** for short.¹ **LT**'s axioms are just Extensionality, Separation, and the claim that every set is a subset of some *level*, where “level” is cunningly defined so that the levels behave like our friends, the V_α s. So **ZF** proves **LT**; but **LT** is *much* weaker than **ZF**. In fact, **LT** does not give you Pairs, Powersets, Infinity, or Replacement. Let **Zr** be the result of adding Infinity and Powersets to **LT**; this delivers Pairs too, so, **Zr** is at least as strong as **Z**. But, in fact, **Zr** is strictly stronger than **Z**, since it adds the claim that every set has a rank (hence my suggestion that we call it **Zr**). Indeed, **Zr** delivers: a perfectly satisfactory theory of ordinals; results which stratify the hierarchy into well-ordered stages; a proof of \in -Induction; and a *version* of Transfinite Recursion.

In short: although Boolos didn't know this, all of the desirable consequences which he mentions could have been arrived at *without* Replacement; he simply needed to use **Zr** rather than **Z**.

(Given all of this, why did we follow the conventional route, of teaching you **ZF**, rather than **LT** and **Zr**? There are two reasons. First: for purely historical reasons, starting with **LT** is rather nonstandard; we wanted to equip you to be able to read more standard discussions of set theory. Second: when you are ready to appreciate **LT** and **Zr**, you can simply read [Potter 2004](#) and [Button forthcoming](#).)

Of course, since **Zr** is strictly weaker than **ZF**, there are results which **ZF** proves which **Zr** leaves open. So one could try to justify Replacement on extrinsic grounds by pointing to one of these results. But, once you know how to use **Zr**, it is quite hard to find many examples of things that are (a) settled by Replacement but not otherwise, and (b) are intuitively true. (For more on this, see [Potter 2004](#), §13.2.)

The bottom line is this. To provide a compelling extrinsic justification for Replacement, one would need to find a result which *cannot* be achieved without Replacement. And that's not an easy enterprise.

Let's consider a further problem which arises for any attempt to offer a purely extrinsic justification for Replacement. (This problem is perhaps more fundamental than the first.) Boolos does not just point out that Replacement has many desirable consequences. He also states that Replacement has “(apparently) no undesirable” consequences. But this parenthetical caveat, “apparently,” is surely absolutely crucial.

Recall how we ended up here: Naïve Comprehension ran into inconsistency, and we responded to this inconsistency by embracing the cumulative-iterative conception of set. This conception comes equipped with a story which, we hope, assures us of its consistency. But if we cannot justify Replacement from within that story, then we have (as yet) no reason to believe that **ZF** is consistent. Or rather: we have no reason to believe that **ZF** is consistent, apart from the (perhaps merely contingent) fact that no one has discovered a contradiction

¹The first versions of **LT** are offered by [Montague \(1965\)](#) and [Scott \(1974\)](#); this was simplified, and given a book-length treatment, by [Potter \(2004\)](#); and [Button \(forthcoming\)](#) has recently simplified **LT** further.

yet. In exactly that sense, Boolos’s comment seems to come down to this: “(apparently) **ZF** is consistent”. We should demand greater reassurance of consistency than this.

This issue will affect any *purely* extrinsic attempt to justify Replacement, i.e., any justification which is couched solely in terms of the (known) consequences of **ZF**. As such, we will want to look for an *intrinsic* justification of Replacement, i.e., a justification which suggests that the story which we told about sets somehow “already” commits us to Replacement.

replacement.4 Limitation-of-size

Perhaps the most common attempt to offer an “intrinsic” justification of Replacement comes via the following notion:

[sth:replacement:limofsize:sec](#)

Limitation-of-size. Any things form a set, provided that there are not too many of them.

This principle will immediately vindicate Replacement. After all, any set formed by Replacement cannot be any larger than any set from which it was formed. Stated precisely: suppose you form a set $\tau[A] = \{\tau(x) : x \in A\}$ using Replacement; then $\tau[A] \preceq A$; so if the **elements** of A were not too numerous to form a set, their images are not too numerous to form $\tau[A]$.

The obvious difficulty with invoking *Limitation-of-size* to justify Replacement is that we have *not* yet laid down any principle like *Limitation-of-size*. Moreover, when we told our story about the cumulative-iterative conception of set in ??-??, nothing ever *hinted* in the direction of *Limitation-of-size*. This, indeed, is precisely why Boolos at one point wrote: “Perhaps one may conclude that there are at least two thoughts ‘behind’ set theory” (1989, p. 19). On the one hand, the ideas surrounding the cumulative-iterative conception of set are meant to vindicate **Z**. On the other hand, *Limitation-of-size* is meant to vindicate Replacement.

But the issue it is not just that we have thus far been *silent* about *Limitation-of-size*. Rather, the issue is that *Limitation-of-size* (as just formulated) seems to sit quite badly with the cumulative-iterative notion of set. After all, it mentions nothing about the idea of sets as formed in *stages*.

This is really not much of a surprise, given the history of these “two thoughts” (i.e., the cumulative-iterative conception of set, and *Limitation-of-size*). These “two thoughts” ultimately amount to two rather different projects for blocking the set-theoretic paradoxes. The cumulative-iterative notion of set blocks Russell’s paradox by saying, roughly: *we should never have expected a Russell set to exist, because it would not be “formed” at any stage*. By contrast, *Limitation-of-size* is meant to rule out the Russell set, by saying, roughly: *we should never have expected a Russell set to exist, because it would have been too big*.

Put like this, then, let’s be blunt: considered as a reply to the paradoxes, *Limitation-of-size* stands in need of *much* more justification. Consider, for

example, this version of Russell’s Paradox: *no pug sniffs exactly the pugs which don’t sniff themselves* (see ??). If you ask “why is there no such pug?”, it is not a good answer to be told that such a pug would have to sniff too many pugs. So why would it be a good intuitive explanation, of the non-existence of a Russell set, that it would have to be “too big” to exist?

In short, it’s forgivable if you are a bit mystified concerning the “intuitive” motivation for *Limitation-of-size*.

replacement.5 Replacement and “Absolute Infinity”

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We will now put *Limitation-of-size* behind us, and explore a different family of (intrinsic) attempts to justify Replacement, which do take seriously the idea of the sets as formed in stages.

When we first outlined the iterative process, we offered some principles which explained what happens at each stage. These were *Stages-are-key*, *Stages-are-ordered*, and *Stages-accumulate*. Later, we added some principles which told us something about the number of stages: *Stages-keep-going* told us that the process of set-formation never ends, and *Stages-hit-infinity* told us that the process goes through an infinite-th stage.

It is reasonable to suggest that these two latter principles fall out of some a broader principle, like:

Stages-are-inexhaustible. There are absolutely infinitely many stages; the hierarchy is as tall as it could possibly be.

Obviously this is an informal principle. But even if it is not immediately *entailed* by the cumulative-iterative conception of set, it certainly seems *consonant* with it. At the very least, and unlike *Limitation-of-size*, it retains the idea that sets are formed stage-by-stage.

The hope, now, is to leverage *Stages-are-inexhaustible* into a justification of Replacement. So let us see how this might be done.

In ??, we saw that it is easy to construct a well-ordering which (morally) should be isomorphic to $\omega + \omega$. Otherwise put, we can easily imagine a stage-by-stage iterative process, whose order-type (morally) is $\omega + \omega$. As such, if we have accepted *Stages-are-inexhaustible*, then we should surely accept that there is at least an $\omega + \omega$ -th stage of the hierarchy, i.e., $V_{\omega+\omega}$, for the hierarchy surely *could* continue thus far.

This thought generalizes as follows: for any well-ordering, the process of building the iterative hierarchy should run at least as far as that well-ordering. And we could guarantee this, just by treating ?? as an *axiom*. This would tell us that any well-ordering is isomorphic to a von Neumann ordinal. Since each von Neumann ordinal will be equal to its own rank, ?? will then tell us that, whenever we can describe a well-ordering in our set theory, the iterative process of set building must outrun that well-ordering.

This idea certainly seems like a corollary of *Stages-are-inexhaustible*. Unfortunately, if our aim is to extract Replacement from this idea, then we face

a simple, technical, barrier: Replacement is strictly stronger than **??**. (This observation is made by [Potter \(2004, §13.2\)](#); we will prove it in [section replacement.8](#).)

The upshot is that, if we are going to understand *Stages-are-inexhaustible* in such a way as to yield Replacement, then it cannot *merely* say that the hierarchy outruns any well-ordering. It must make a stronger claim than that. To this end, [Shoenfield \(1977\)](#) proposed a very natural strengthening of the idea, as follows: the hierarchy is not *cofinal* with any set.² In slightly more detail: if τ is a mapping which sends sets to stages of the hierarchy, the image of any set A under τ does not exhaust the hierarchy. Otherwise put (schematically):

Stages-are-super-cofinal. If A is a set and $\tau(x)$ is a stage for every $x \in A$, then there is a stage which comes after each $\tau(x)$ for $x \in A$.

It is obvious that **ZF** proves a suitably formalised version of *Stages-are-super-cofinal*. Conversely, we can informally argue that *Stages-are-super-cofinal* justifies Replacement.³ For suppose $(\forall x \in A)\exists!y\varphi(x, y)$. Then for each $x \in A$, let $\sigma(x)$ be the y such that $\varphi(x, y)$, and let $\tau(x)$ be the stage at which $\sigma(x)$ is first formed. By *Stages-are-super-cofinal*, there is a stage V such that $(\forall x \in A)\tau(x) \in V$. Now since each $\tau(x) \in V$ and $\sigma(x) \subseteq \tau(x)$, by Separation we can obtain $\{y \in V : (\exists x \in A)\sigma(x) = y\} = \{y : (\exists x \in A)\varphi(x, y)\}$.

Problem replacement.1. Formalize *Stages-are-super-cofinal* within **ZF**.

So *Stages-are-super-cofinal* vindicates Replacement. And it is at least plausible that *Stages-are-inexhaustible* vindicates *Stages-are-super-cofinal*. For suppose *Stages-are-super-cofinal* fails. So the hierarchy is cofinal with some set A , i.e., we have a map τ such that for any stage S there is some $x \in A$ such that $S \in \tau(x)$. In that case, we do have a way to get a handle on the supposed “absolute infinity” of the hierarchy: it is *exhausted* by the range of τ applied to A . And that compromises the thought that the hierarchy is “absolutely infinite”. Contraposing: *Stages-are-inexhaustible* entails *Stages-are-super-cofinal*, which in turn justifies Replacement.

This represents a genuinely promising attempt to provide an intrinsic justification for Replacement. But whether it ultimately works, or not, we will have to leave to you to decide.

replacement.6 Replacement and Reflection

Our last attempt to justify Replacement, via *Stages-are-inexhaustible*, begins with a deep and lovely result:⁴

[sth:replacement:ref:sec](#)

²Gödel seems to have proposed a similar thought; see [Potter \(2004, p. 223\)](#). For discussion of Gödel and [Shoenfield](#), see [Incurvati \(2020, 90–5\)](#).

³It would be harder to prove Replacement using some formalisation of *Stages-are-super-cofinal*, since **Z** on its own is not strong enough to define the stages, so it is not clear how one would formalise *Stages-are-super-cofinal*. One option, though, is to work in some extension of **LT**, as discussed in [section replacement.3](#).

⁴A reminder: all formulas can have parameters (unless explicitly stated otherwise).

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reflectionschema

Theorem replacement.2 (Reflection Schema). For any formula φ :

$$\forall\alpha\exists\beta > \alpha(\forall x_1 \dots, x_n \in V_\beta)(\varphi(x_1, \dots, x_n) \leftrightarrow \varphi^{V_\beta}(x_1, \dots, x_n))$$

As in [Definition replacement.1](#), φ^{V_β} is the result of restricting every quantifier in φ to the set V_β . So, intuitively, Reflection says this: if φ is true in the entire hierarchy, then φ is true in arbitrarily many *initial segments* of the hierarchy.

[Montague \(1961\)](#) and [Lévy \(1960\)](#) showed that (suitable formulations of) Replacement and Reflection are equivalent, modulo **ZF**, so that adding either gives you **ZF**. (We prove these results in [section replacement.7](#).) Given this equivalence, one might hope to justify Reflection and Replacement via *Stages-are-inexhaustible* as follows: given *Stages-are-inexhaustible*, the hierarchy should be very, very tall; so tall, in fact, that nothing we can say about it is sufficient to bound its height. And we can understand this as the thought that, if any sentence φ is true in the entire hierarchy, then it is true in arbitrarily many initial segments of the hierarchy. And that is just Reflection.

Again, this seems like a genuinely promising attempt to provide an intrinsic justification for Replacement. But there is much too much to say about it here. You must now decide for yourself whether it succeeds.⁵

replacement.7 Appendix: Results surrounding Replacement

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In this section, we will prove Reflection within **ZF**. We will also prove a sense in which Reflection is equivalent to Replacement. And we will prove an interesting consequence of all this, concerning the strength of Reflection/Replacement. *Warning: this is easily the most advanced bit of mathematics in this textbook.*

We'll start with a lemma which, for brevity, employs the notational device of *overlining* to deal with sequences of variables or objects. So: " \bar{a}_k " abbreviates " a_{k_1}, \dots, a_{k_n} ", where n is determined by context.

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lemreflection

Lemma replacement.3. For each $1 \leq i \leq k$, let $\varphi_i(\bar{v}_i, x)$ be a formula. Then for each α there is some $\beta > \alpha$ such that, for any $\bar{a}_1, \dots, \bar{a}_k \in V_\beta$ and each $1 \leq i \leq k$:

$$\exists x \varphi_i(\bar{a}_i, x) \rightarrow (\exists x \in V_\beta) \varphi_i(\bar{a}_i, x)$$

Proof. We define a term μ as follows: $\mu(\bar{a}_1, \dots, \bar{a}_k)$ is the least stage, V , which satisfies all of the following conditionals, for $1 \leq i \leq k$:

$$\exists x \varphi_i(\bar{a}_i, x) \rightarrow (\exists x \in V) \varphi_i(\bar{a}_i, x)$$

⁵Though you might like to continue by reading [Incurvati \(2020, 95–100\)](#).

It is easy to confirm that $\mu(\bar{a}_1, \dots, \bar{a}_k)$ exists for all $\bar{a}_1, \dots, \bar{a}_k$. Now, using Replacement and our recursion theorem, define:

$$\begin{aligned} S_0 &= V_{\alpha+1} \\ S_{n+1} &= S_n \cup \bigcup \{ \mu(\bar{a}_1, \dots, \bar{a}_k) : \bar{a}_1, \dots, \bar{a}_k \in S_n \} \\ S &= \bigcup_{m < \omega} S_m. \end{aligned}$$

Each S_n , and hence S itself, is a stage after V_α . Now fix $\bar{a}_1, \dots, \bar{a}_k \in S$; so there is some $n < \omega$ such that $\bar{a}_1, \dots, \bar{a}_k \in S_n$. Fix some $1 \leq i \leq k$, and suppose that $\exists x \varphi_i(\bar{a}_i, x)$. So $(\exists x \in \mu(\bar{a}_1, \dots, \bar{a}_k)) \varphi_i(\bar{a}_i, x)$ by construction, so $(\exists x \in S_{n+1}) \varphi_i(\bar{a}_i, x)$ and hence $(\exists x \in S) \varphi_i(\bar{a}_i, x)$. So S is our V_β . \square

We can now prove [Theorem replacement.2](#) quite straightforwardly:

Proof. Fix α . Without loss of generality, we can assume φ 's only connectives are \exists , \neg and \wedge (since these are expressively adequate). Let ψ_1, \dots, ψ_k enumerate each of φ 's subformulas according to complexity, so that $\psi_k = \varphi$. By [Lemma replacement.3](#), there is a $\beta > \alpha$ such that, for any $\bar{a}_i \in V_\beta$ and each $1 \leq i \leq k$:

$$\exists x \psi_i(\bar{a}_i, x) \rightarrow (\exists x \in V_\beta) \psi_i(\bar{a}_i, x) \quad (*)$$

By induction on complexity of ψ_i , we will show that $\psi_i(\bar{a}_i) \leftrightarrow \psi_i^{V_\beta}(\bar{a}_i)$, for any $\bar{a}_i \in V_\beta$. If ψ_i is atomic, this is trivial. The biconditional also establishes that, when ψ_i is a negation or conjunction of subformulas satisfying this property, ψ_i itself satisfies this property. So the only interesting case concerns quantification. Fix $\bar{a}_i \in V_\beta$; then:

$$\begin{aligned} (\exists x \psi_i(\bar{a}_i, x))^{V_\beta} &\text{ iff } (\exists x \in V_\beta) \psi_i^{V_\beta}(\bar{a}_i, x) && \text{by definition} \\ &\text{ iff } (\exists x \in V_\beta) \psi_i(\bar{a}_i, x) && \text{by hypothesis} \\ &\text{ iff } \exists x \psi_i(\bar{a}_i, x) && \text{by } (*) \end{aligned}$$

This completes the induction; the result follows as $\psi_k = \varphi$. \square

We have proved Reflection in **ZF**. Our proof essentially followed [Montague \(1961\)](#). We now want to prove in **Z** that Reflection entails Replacement. The proof follows [Lévy \(1960\)](#), but with a simplification.

Since we are working in **Z**, we cannot present Reflection in exactly the form given above. After all, we formulated Reflection using the “ V_α ” notation, and that cannot be defined in **Z** (see ??). So instead we will offer an apparently weaker formulation of Replacement, as follows:

Weak-Reflection. For any formula φ , there is a transitive set S such that 0, 1, and any parameters to φ are elements of S , and $(\forall \bar{x} \in S)(\varphi \leftrightarrow \varphi^S)$.

To use this to prove Replacement, we will first follow Lévy (1960, first part of Theorem 2) and show that we can “reflect” two formulas at once:

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lem:reflect

Lemma replacement.4 (in Z + Weak-Reflection). *For any formulas ψ, χ , there is a transitive set S such that 0 and 1 (and any parameters to the formulas) are elements of S , and $(\forall \bar{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$.*

Proof. Let φ be the formula $(z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)$.

Here we use an abbreviation; we should spell out “ $z = 0$ ” as “ $\forall t t \notin z$ ” and “ $z = 1$ ” as “ $\forall s (s \in z \leftrightarrow \forall t t \notin s)$ ”. But since $0, 1 \in S$ and S is transitive, these formulas are *absolute* for S ; that is, they will apply to the same object whether we restrict their quantifiers to S .⁶

By Weak-Reflection, we have some appropriate S such that:

$$\begin{aligned} & (\forall z, \bar{x} \in S)(\varphi \leftrightarrow \varphi^S) \\ \text{i.e. } & (\forall z, \bar{x} \in S)((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi) \leftrightarrow \\ & ((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi))^S) \\ \text{i.e. } & (\forall z, \bar{x} \in S)((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi) \leftrightarrow \\ & ((z = 0 \wedge \psi^S) \vee (z = 1 \wedge \chi^S))) \\ \text{i.e. } & (\forall \bar{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S)) \end{aligned}$$

The second claim entails the third because “ $z = 0$ ” and “ $z = 1$ ” are absolute for S ; the fourth claim follows since $0 \neq 1$. \square

We can now obtain Replacement, just by following and simplifying Lévy (1960, Theorem 6):

Theorem replacement.5 (in Z + Weak-Reflection). *For any formula $\varphi(v, w)$, and any A , if $(\forall x \in A)\exists!y \varphi(x, y)$, then $\{y : (\exists x \in A)\varphi(x, y)\}$ exists.*

Proof. Fix A such that $(\forall x \in A)\exists!y \varphi(x, y)$, and define formulas:

$$\begin{aligned} \psi & \text{ is } (\varphi(x, z) \wedge A = A) \\ \chi & \text{ is } \exists y \varphi(x, y) \end{aligned}$$

Using Lemma replacement.4, since A is a parameter to ψ , there is a transitive S such that $0, 1, A \in S$ (along with any other parameters), and such that:

$$(\forall x, z \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$$

So in particular:

$$\begin{aligned} & (\forall x, z \in S)(\varphi(x, z) \leftrightarrow \varphi^S(x, z)) \\ & (\forall x \in S)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi^S(x, y)) \end{aligned}$$

⁶More formally, letting ξ be either of these formulas, $\xi(z) \leftrightarrow \xi^S(z)$.

Combining these, and observing that $A \subseteq S$ since $A \in S$ and S is transitive:

$$(\forall x \in A)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi(x, y))$$

Now $(\forall x \in A)(\exists! y \in S)\varphi(x, y)$, because $(\forall x \in A)\exists! y \varphi(x, y)$. Now Separation yields $\{y \in S : (\exists x \in A)\varphi(x, y)\} = \{y : (\exists x \in A)\varphi(x, y)\}$. \square

replacement.8 Appendix: Finite axiomatizability

We close this chapter by extracting some results from Replacement. The first result is due to [Montague \(1961\)](#); note that it is not a proof *within* \mathbf{ZF} , but a proof *about* \mathbf{ZF} :

Theorem replacement.6. \mathbf{ZF} is not finitely axiomatizable. More generally: if \mathbf{T} is finite and $\mathbf{T} \vdash \mathbf{ZF}$, then \mathbf{T} is inconsistent.

(Here, we tacitly restrict ourselves to first-order sentences whose only non-logical primitive is \in , and we write $\mathbf{T} \vdash \mathbf{ZF}$ to indicate that $\mathbf{T} \vdash \varphi$ for all $\varphi \in \mathbf{ZF}$.)

Proof. Fix finite \mathbf{T} such that $\mathbf{T} \vdash \mathbf{ZF}$. So, \mathbf{T} proves Reflection, i.e. [Theorem replacement.2](#). Since \mathbf{T} is finite, we can rewrite it as a single conjunction, θ . Reflecting with this formula, $\mathbf{T} \vdash \exists \beta (\theta \leftrightarrow \theta^{V_\beta})$. Since trivially $\mathbf{T} \vdash \theta$, we find that $\mathbf{T} \vdash \exists \beta \theta^{V_\beta}$.

Now, let $\psi(X)$ abbreviate:

$$\theta^X \wedge X \text{ is transitive} \wedge (\forall Y \in X)(Y \text{ is transitive} \rightarrow \neg \theta^Y)$$

roughly this says: X is a transitive model of θ , and \in -minimal in this regard. Now, recalling that $\mathbf{T} \vdash \exists \beta \theta^{V_\beta}$, by basic facts about ranks within \mathbf{ZF} and hence within \mathbf{T} , we have:

$$\mathbf{T} \vdash \exists M \psi(M). \quad (*)$$

Using the first conjunct of $\psi(X)$, whenever $\mathbf{T} \vdash \sigma$, we have that $\mathbf{T} \vdash \forall X (\psi(X) \rightarrow \sigma^X)$. So, by (*):

$$\mathbf{T} \vdash \forall X (\psi(X) \rightarrow (\exists N \psi(N))^X)$$

Using this, and (*) again:

$$\mathbf{T} \vdash \exists M (\psi(M) \wedge (\exists N \psi(N))^M)$$

In particular, then:

$$\mathbf{T} \vdash \exists M (\psi(M) \wedge (\exists N \in M)((N \text{ is transitive})^N \wedge (\theta^N)^M))$$

So, by elementary reasoning concerning transitivity:

$$\mathbf{T} \vdash \exists M (\psi(M) \wedge (\exists N \in M)(N \text{ is transitive} \wedge \theta^N))$$

So that \mathbf{T} is inconsistent.⁷ □

Here is a similar result, noted by [Potter \(2004, 223\)](#):

*replacement:finiteaxiomatizability:
finiteextensionofZ*

Proposition replacement.7. *Let \mathbf{T} extend \mathbf{Z} with finitely many new axioms. If $\mathbf{T} \vdash \mathbf{ZF}$, then \mathbf{T} is inconsistent. (Here we use the same tacit restrictions as for [Theorem replacement.6](#).)*

Proof. Use θ for the conjunction of all of \mathbf{T} 's axioms *except* for the (infinitely many) instances of Separation. Defining ψ from θ as in [Theorem replacement.6](#), we can show that $\mathbf{T} \vdash \exists M \psi(M)$.

As in [Theorem replacement.6](#), we can establish the schema that, whenever $\mathbf{T} \vdash \sigma$, we have that $\mathbf{T} \vdash \forall X (\psi(X) \rightarrow \sigma^X)$. We then finish our proof, exactly as in [Theorem replacement.6](#).

However, establishing the schema involves a little more work than in [Theorem replacement.6](#). After all, the Separation-instances are in \mathbf{T} , but they are not conjuncts of θ . However, we can overcome this obstacle by proving that $\mathbf{T} \vdash \forall X (X \text{ is transitive} \rightarrow \sigma^X)$, for every Separation-instance σ . We leave this to the reader. □

Problem replacement.2. Show that, for every Separation-instance σ , we have: $\mathbf{Z} \vdash \forall X (X \text{ is transitive} \rightarrow \sigma^X)$. (We used this schema in [Proposition replacement.7](#).)

Problem replacement.3. Show that, for every $\varphi \in \mathbf{Z}$, we have $\mathbf{ZF} \vdash \varphi^{V_{\omega+\omega}}$.

Problem replacement.4. Confirm the remaining schematic results invoked in the proofs of [Theorem replacement.6](#) and [Proposition replacement.7](#).

As remarked in [section replacement.5](#), this shows that Replacement is strictly stronger than $??$. Or, slightly more strictly: if $\mathbf{Z} +$ “every well-ordering is isomorphic to a unique ordinal” is consistent, then it fails to prove some Replacement-instance.

Photo Credits

⁷This “elementary reasoning” involves proving certain “absoluteness facts” for transitive sets.

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