In this section, we will prove Reflection within $\textbf{ZF}$. We will also prove a sense in which Reflection is equivalent to Replacement. And we will prove an interesting consequence of all this, concerning the strength of Reflection/Replacement.

Warning: this is easily the most advanced bit of mathematics in this textbook.

We’ll start with a lemma which, for brevity, employs the notational device of overlining to deal with sequences of variables or objects. So: “$\overline{a_k}$” abbreviates “$a_{k_1}, \ldots, a_{k_n}$”, where $n$ is determined by context.

**Lemma replacement.1.** For each $1 \leq i \leq k$, let $\varphi_i(\overline{a_i}, x)$ be a formula. Then for each $\alpha$ there is some $\beta > \alpha$ such that, for any $\overline{a_1}, \ldots, \overline{a_k} \in V_\alpha$ and each $1 \leq i \leq k$:

$$\exists x \varphi_i(\overline{a_i}, x) \rightarrow (\exists x \in V_\beta) \varphi_i(\overline{a_i}, x)$$

*Proof.* We define a term $\mu$ as follows: $\mu(\overline{a_1}, \ldots, \overline{a_k})$ is the least stage, $V_\beta$, which satisfies all of the following conditionals, for $1 \leq i \leq k$:

$$\exists x \varphi_i(\overline{a_i}, x) \rightarrow (\exists x \in V_\beta) \varphi_i(\overline{a_i}, x)$$

It is easy to confirm that $\mu(\overline{a_1}, \ldots, \overline{a_k})$ exists for all $\overline{a_1}, \ldots, \overline{a_k}$. Now, using Replacement and our recursion theorem, define:

$$S_0 = V_{\alpha+1}$$

$$S_{n+1} = S_n \cup \{ \mu(\overline{a_1}, \ldots, \overline{a_k}) : \overline{a_1}, \ldots, \overline{a_k} \in S_n \}$$

$$S = \bigcup_{m < \omega} S_m.$$

Each $S_n$, and hence $S$ itself, is a stage after $V_\alpha$. Now fix $\overline{a_1}, \ldots, \overline{a_k} \in S$; so there is some $n < \omega$ such that $\overline{a_1}, \ldots, \overline{a_k} \in S_n$. Fix some $1 \leq i \leq k$, and suppose that $\exists x \varphi_i(\overline{a_i}, x)$. So $(\exists x \in \mu(\overline{a_1}, \ldots, \overline{a_k})) \varphi_i(\overline{a_i}, x)$ by construction, so $(\exists x \in S) \varphi_i(\overline{a_i}, x)$ and hence $(\exists x \in S) \varphi_i(\overline{a_i}, x)$. So $S$ is our $V_\beta$. \qed

We can now prove ?? quite straightforwardly:

*Proof.* Fix $\alpha$. Without loss of generality, we can assume $\varphi$’s only connectives are $\exists$, $\neg$ and $\land$ (since these are expressively adequate). Let $\psi_1, \ldots, \psi_k$ enumerate each of $\varphi$’s subformulas according to complexity, so that $\psi_k = \varphi$. By **Lemma replacement.1**, there is a $\beta > \alpha$ such that, for any $\overline{a_i} \in V_\beta$ and each $1 \leq i \leq k$:

$$\exists x \psi_i(\overline{a_i}, x) \rightarrow (\exists x \in V_\beta) \psi_i(\overline{a_i}, x)$$

(*)

By induction on complexity of $\psi_i$, we will show that $\psi_i(\overline{a_i}) \leftrightarrow \psi_i^V(\overline{a_i})$, for any $\overline{a_i} \in V_\beta$. If $\psi_i$ is atomic, this is trivial. The biconditional also establishes that, when $\psi_i$ is a negation or conjunction of subformulas satisfying this...
property, \( \psi_i \) itself satisfies this property. So the only interesting case concerns quantification. Fix \( \pi_i \in V_\beta \); then:

\[
(\exists x \psi_i(\pi_i, x))_{V^\beta} \iff (\exists x \in V_\beta)\psi_i(\pi_i, x)
\]

by hypothesis

\[
\iff \exists x \psi_i(\pi_i, x)
\]

by \((*)\)

This completes the induction; the result follows as \( \psi_k = \phi \).

We have proved Reflection in \( \text{ZF} \). Our proof essentially followed Montague (1961). We now want to prove in \( \text{Z} \) that Reflection entails Replacement. The proof follows Lévy (1960), but with a simplification.

Since we are working in \( \text{Z} \), we cannot present Reflection in exactly the form given above. After all, we formulated Reflection using the “\( V_\alpha \)” notation, and that cannot be defined in \( \text{Z} \) (see ??). So instead we will offer an apparently weaker formulation of Replacement, as follows:

**Weak-Reflection.** For any formula \( \varphi \), there is a transitive set \( S \) such that 0, 1, and any parameters to \( \varphi \) are elements of \( S \), and \((\forall \pi \in S)(\varphi \leftrightarrow \varphi^S)\).

To use this to prove Replacement, we will first follow Lévy (1960, first part of Theorem 2) and show that we can “reflect” two formulas at once:

**Lemma replacement.2 (in \( \text{Z} + \text{Weak-Reflection} \).)** For any formulas \( \psi, \chi \), there is a transitive set \( S \) such that 0 and 1 (and any parameters to the formulas) are elements of \( S \), and \((\forall \pi \in S)((\psi \leftrightarrow \psi^S) \land (\chi \leftrightarrow \chi^S))\).

**Proof.** Let \( \varphi \) be the formula \((z = 0 \land \psi) \lor (z = 1 \land \chi)\).

Here we use an abbreviation; we should spell out “\( z = 0 \)” as “\( \forall t t \notin z \)” and “\( z = 1 \)” as “\( \forall s(s \in z \leftrightarrow \exists t t \notin s) \)”.

But since 0, 1 \( \in S \) and \( S \) is transitive, these formulas are absolute for \( S \); that is, they will apply to the same object whether we restrict their quantifiers to \( S \).\footnote{More formally, letting \( \xi \) be either of these formulas, \( \xi(z) \leftrightarrow \xi^S(z) \).}

By Weak-Reflection, we have some appropriate \( S \) such that:

\[
(\forall z, \pi \in S)(\varphi \leftrightarrow \varphi^S)
\]

i.e. \((\forall z, \pi \in S)(((z = 0 \land \psi) \lor (z = 1 \land \chi)) \leftrightarrow ((z = 0 \land \psi^S) \lor (z = 1 \land \chi^S)))\)

i.e. \((\forall z, \pi \in S)(((z = 0 \land \psi) \lor (z = 1 \land \chi)) \leftrightarrow ((z = 0 \land \psi^S) \lor (z = 1 \land \chi^S)))\)
We can now obtain Replacement, just by following and simplifying Lévy (1960, Theorem 6):

**Theorem replacement.3 (in Z + Weak-Reflection).** For any formula $\varphi(v, w)$, and any $A$, if $(\forall x \in A) \exists! y \varphi(x, y)$, then \{ $y : (\exists x \in A) \varphi(x, y)$ \} exists.

**Proof.** Fix $A$ such that $(\forall x \in A) \exists! y \varphi(x, y)$, and define formulas:

\[
\psi \text{ is } (\varphi(x, z) \wedge A = A) \\
\chi \text{ is } \exists y \varphi(x, y)
\]

Using Lemma replacement.2, since $A$ is a parameter to $\psi$, there is a transitive $S$ such that $0, 1, A \in S$ (along with any other parameters), and such that:

\[
(\forall x, z \in S)(\psi \leftrightarrow S_\psi(x, z)) \wedge (\chi \leftrightarrow S_\chi(x, y))
\]

So in particular:

\[
(\forall x, z \in S)(\varphi(x, z) \leftrightarrow S_\psi(x, z)) \\
(\forall x \in S)(\exists y \varphi(x, y) \leftrightarrow S_\chi(x, y))
\]

Combining these, and observing that $A \subseteq S$ since $A \in S$ and $S$ is transitive:

\[
(\forall x \in A) (\exists y \varphi(x, y) \leftrightarrow (\exists y \in S) \varphi(x, y))
\]

Now $(\forall x \in A) (\exists! y \in S) \varphi(x, y)$, because $(\forall x \in A) \exists! y \varphi(x, y)$. Now Separation yields \{ $y \in S' : (\exists x \in A) \varphi(x, y)$ \} = \{ $y : (\exists x \in A) \varphi(x, y)$ \}. \qed

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**Bibliography**
