replacement.1 Replacement and Reflection

Our last attempt to justify Replacement, via Stages-are-inexhaustible, begins with a deep and lovely result:\footnote{A reminder: all formulas can have parameters (unless explicitly stated otherwise).}

**Theorem replacement.1 (Reflection Schema).** For any formula $\varphi$:

$$\forall \alpha \exists \beta > \alpha (\forall x_1, \ldots, x_n \in V_{\beta})(\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi^{V_{\beta}}(x_1, \ldots, x_n))$$

As in \textit{??}, $\varphi^{V_{\beta}}$ is the result of restricting every quantifier in $\varphi$ to the set $V_{\beta}$. So, intuitively, Reflection says this: if $\varphi$ is true in the entire hierarchy, then $\varphi$ is true in arbitrarily many initial segments of the hierarchy.

Montague (1961) and Lévy (1960) showed that (suitable formulations of) Replacement and Reflection are equivalent, modulo Z, so that adding either gives you ZF. (We prove these results in \textit{??}.) Given this equivalence, one might hope to justify Reflection and Replacement via Stages-are-inexhaustible as follows: given Stages-are-inexhaustible, the hierarchy should be very, very tall; so tall, in fact, that nothing we can say about it is sufficient to bound its height. And we can understand this as the thought that, if any sentence $\varphi$ is true in the entire hierarchy, then it is true in arbitrarily many initial segments of the hierarchy. And that is just Reflection.

Again, this seems like a genuinely promising attempt to provide an intrinsic justification for Replacement. But there is much too much to say about it here. You must now decide for yourself whether it succeeds.\footnote{Though you might like to continue by reading Incurvati (2020, 95–100).}

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**Bibliography**

