

replacement.1 Replacement and Reflection

sth:replacement:ref:sec Our last attempt to justify Replacement, via *Stages-are-inexhaustible*, begins with a deep and lovely result:¹

sth:replacement:ref:reflectionschema **Theorem replacement.1 (Reflection Schema).** *For any formula φ :*

$$\forall\alpha\exists\beta > \alpha(\forall x_1 \dots, x_n \in V_\beta)(\varphi(x_1, \dots, x_n) \leftrightarrow \varphi^{V_\beta}(x_1, \dots, x_n))$$

As in ??, φ^{V_β} is the result of restricting every quantifier in φ to the set V_β . So, intuitively, Reflection says this: if φ is true in the entire hierarchy, then φ is true in arbitrarily many *initial segments* of the hierarchy.

Montague (1961) and Lévy (1960) showed that (suitable formulations of) Replacement and Reflection are equivalent, modulo **Z**, so that adding either gives you **ZF**. (We prove these results in ??.) Given this equivalence, one might hope to justify Reflection and Replacement via *Stages-are-inexhaustible* as follows: given *Stages-are-inexhaustible*, the hierarchy should be very, very tall; so tall, in fact, that nothing we can say about it is sufficient to bound its height. And we can understand this as the thought that, if any sentence φ is true in the entire hierarchy, then it is true in arbitrarily many initial segments of the hierarchy. And that is just Reflection.

Again, this seems like a genuinely promising attempt to provide an intrinsic justification for Replacement. But there is much too much to say about it here. You must now decide for yourself whether it succeeds.²

Photo Credits

Bibliography

- Incurvati, Luca. 2020. *Conceptions of Set and the Foundations of Mathematics*. Cambridge: Cambridge University Press.
- Lévy, Azriel. 1960. Axiom schemata of strong infinity in axiomatic set theory. *Pacific Journal of Mathematics* 10(1): 223–38.
- Montague, Richard. 1961. Semantic closure and non-finite axiomatizability I. In *Infinitistic Methods: Proceedings of the Symposium on Foundations of Mathematics (Warsaw 1959)*, 45–69. New York: Pergamon.

¹A reminder: all formulas can have parameters (unless explicitly stated otherwise).

²Though you might like to continue by reading Incurvati (2020, 95–100).