replacement.1 Replacement and Reflection

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A final last attempt to justify Replacement, via *Stages-are-inexhaustible*, is to consider a lovely result:

sth:replacement:ref:reflectionschema

Theorem replacement.1 (Reflection Schema). *For any formula φ:*[1]

$$\forall\alpha\exists\beta > \alpha(\forall x_1 \ldots, x_n \in V_\beta)(\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi^{V_\beta}(x_1, \ldots, x_n))$$

As before, φ^{V_β} is the result of restricting every quantifier in φ to the set V_β. So, intuitively, Reflection says this: if φ is true in the entire hierarchy, then φ is true in arbitrarily many *initial segments* of the hierarchy.

Montague (1961) and Lévy (1960) showed that (suitable formulations of) Replacement and Reflection are equivalent, modulo **Z**, so that adding either gives you **ZF**. So, given this equivalence, one might hope to justify Reflection and Replacement via *Stages-are-inexhaustible* as follows: given *Stages-are-inexhaustible*, the hierarchy should be very, very tall; so tall, in fact, that nothing we can say about it is sufficient to bound its height. And we can understand this as the thought that, if any sentence φ is true in the entire hierarchy, then it is true in arbitrarily many initial segments of the hierarchy. And that is just Reflection.

Again, this seems like a genuinely promising attempt to provide an intrinsic justification for Replacement. But there is much too much to say about it here. You must now decide for yourself whether it succeeds.

Finally, we will prove that Replacement entails Reflection. This is easily the most advanced bit of mathematics in this textbook (so if you follow it, well done). We'll start with a lemma which, for brevity, employs the notational device of *overlining* to deal with sequences of variables or objects. So: "$\overline{a}_k$" abbreviates "$a_{k_1}, \ldots, a_{k_n}$", where n is determined by context.

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Lemma replacement.2. *For each $1 \leq i \leq k$, let $\varphi_i(\overline{v}_i, x)$ be a formula.*[2] *Then for each α there is some $\beta > \alpha$ such that, for any $\overline{a}_1, \ldots, \overline{a}_k \in V_\beta$ and each $1 \leq i \leq k$:*

$$\exists x\varphi_i(\overline{a}_i, x) \rightarrow (\exists x \in V_\beta)\varphi_i(\overline{a}_i, x)$$

Proof. We define a term μ as follows: $\mu(\overline{a}_1, \ldots, \overline{a}_k)$ is the least stage, V, which satisfies all of the following conditionals, for $1 \leq i \leq k$:

$$\exists x\varphi_i(\overline{a}_i, x) \rightarrow (\exists x \in V)\varphi_i(\overline{a}_i, x))$$

Using Replacement and our recursion theorem, define:

$$\begin{aligned} S_0 &= V_{\alpha+1} \\ S_{m+1} &= S_m \cup \bigcup\{\mu(\overline{a}_1, \ldots, \overline{a}_k) : \overline{a}_1, \ldots, \overline{a}_k \in S_m\} \\ S &= \bigcup_{m<\omega} S_m. \end{aligned}$$

[1] But which may also have parameters

[2] Which may have parameters.

Each S_m, and hence S itself, is a stage after V_α. Now fix $\overline{a}_1, \ldots, \overline{a}_k \in S$; so there is some $m < \omega$ such that $\overline{a}_1, \ldots, \overline{a}_k \in S_m$. Fix some $1 \leq i \leq k$, and suppose that $\exists x \varphi_i(\overline{a}_i, x)$. So $(\exists x \in \mu(\overline{a}_1, \ldots, \overline{a}_k))\varphi_i(\overline{a}_i, x)$ by construction, so $(\exists x \in S_{m+1})\varphi_i(\overline{a}_i, x)$ and hence $(\exists x \in S)\varphi_i(\overline{a}_i, x)$. So S is our V_β. □

From here, we can prove Theorem replacement.1 quite straightforwardly:

Proof of Theorem replacement.1. Fix α. Without loss of generality, we can assume φ's only connectives are $\exists$, $\neg$ and $\wedge$ (since these are expressively adequate). Let $\psi_1, \ldots, \psi_k$ enumerate each of φ's subformulas according to complexity, so that $\psi_k = \varphi$. By Lemma replacement.2, there is a $\beta > \alpha$ such that, for any $\overline{a}_i \in V_\beta$ and each $1 \leq i \leq k$:

$$\exists x \psi_i(\overline{a}_i, x) \to (\exists x \in V_\beta)\psi_i(\overline{a}_i, x) \tag{*}$$

By induction on complexity of ψ_i, we will show that $\psi_i(\overline{a}_i) \leftrightarrow \psi_i^{V_\beta}(\overline{a}_i)$, for any $\overline{a}_i \in V_\beta$. If ψ_i is atomic, this is trivial. The biconditional also establishes that, when ψ_i is a negation or conjunction of subformulas satisfying this property, ψ_i itself satisfies this property. So the only interesting case concerns quantification. Fix $\overline{a}_i \in V_\beta$; then:

$$\begin{aligned}
(\exists x \psi_i(\overline{a}_i, x))^{V_\beta} &\text{ iff } (\exists x \in V_\beta)\psi_i^{V_\beta}(\overline{a}_i, x) && \text{by definition}\\
&\text{ iff } (\exists x \in V_\beta)\psi_i(\overline{a}_i, x) && \text{by the induction hypothesis}\\
&\text{ iff } \exists x \psi_i(\overline{a}_i, x) && \text{by (*)}
\end{aligned}$$

This completes the induction; the result follows as $\psi_k = \varphi$. □

We have shown in **ZF** that Reflection holds. The proof essentially followed Montague (1961). We now want to prove in **Z** that Reflection entails Replacement. The proof follows Lévy (1960), but with a simplification.

Since we are working in **Z**, we cannot present Reflection in exactly the form given above. After all, we formulated Reflection using the "V_α" notation, and that cannot be defined in **Z**. So instead we will offer an apparently weaker formulation of Replacement, as follows:

Weak-Reflection. For any formula φ, there is a transitive set S such that 0, 1, and any parameters to φ are elements of S, and $(\forall \overline{x} \in S)(\varphi \leftrightarrow \varphi^S)$.

To use this to prove Replacement, we will first follow Lévy (1960, first part of Theorem 2) and show that we can "reflect" two formulas at once:

Lemma replacement.3 (in **Z** + Weak-Reflection.)**.** *For any formulas ψ, χ, there is a transitive set S such that* 0 *and* 1 *(and any parameters to the formulas) are elements of S, and* $(\forall \overline{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$. *sth:replacement:ref:lem:reflect*

Proof. Let φ be the formula $(z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)$.

Here we use an abbreviation; we should spell out "$z = 0$" as "$\forall t\, t \notin z$" and "$z = 1$" as "$\forall s(s \in z \leftrightarrow \forall t\, t \notin s)$". But since $0, 1 \in S$ and S is transitive, these formulas are *absolute* for S; that is, they will apply to the same object whether we restrict their quantifiers to S.[3]

By Weak-Reflection, we have some appropriate S such that:

$$(\forall z, \overline{x} \in S)(\varphi \leftrightarrow \varphi^S)$$
$$(\forall z, \overline{x} \in S)(((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)) \leftrightarrow ((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi))^S)$$
$$(\forall z, \overline{x} \in S)(((z = 0 \wedge \psi) \vee (z = 1 \wedge \chi)) \leftrightarrow ((z = 0 \wedge \psi^S) \vee (z = 1 \wedge \chi^S)))$$
$$(\forall \overline{x} \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$$

The second claim entails the third because "$z = 0$" and "$z = 1$" are absolute for S; the fourth claim follows since $0 \neq 1$. □

We now obtain Replacement, simplifying Lévy (1960, Theorem 6):

Theorem replacement.4 (in $\mathbf{Z}$ + Weak-Reflection)**.** *For any formula $\varphi(v, w)$,*[4] *and any A, if $(\forall x \in A)\exists! y\, \varphi(x, y)$, then $\{y : (\exists x \in A)\varphi(x, y\}$ exists.*

Proof. Fix A such that $(\forall x \in A)\exists! y\, \varphi(x, y)$, and define some formulas:

$$\psi \text{ is } (\varphi(x, z) \wedge A = A)$$
$$\chi \text{ is } \exists y\, \varphi(x, y)$$

Using Lemma replacement.3, since A is a parameter to ψ, there is a transitive S such that $0, 1, A \in S$ (along with any other parameters), and such that:

$$(\forall x, z \in S)((\psi \leftrightarrow \psi^S) \wedge (\chi \leftrightarrow \chi^S))$$

So in particular:

$$(\forall x, z \in S)(\varphi(x, z) \leftrightarrow \varphi^S(x, z))$$
$$(\forall x \in S)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi^S(x, y))$$

Combining these, and observing that $A \subseteq S$ since $A \in S$ and S is transitive:

$$(\forall x \in A)(\exists y \varphi(x, y) \leftrightarrow (\exists y \in S)\varphi(x, y))$$

Now $(\forall x \in A)(\exists! y \in S)\varphi(x, y)$, because $(\forall x \in A)\exists! y\, \varphi(x, y)$. Now Separation yields $\{y \in S : (\exists x \in A)\varphi(x, y)\} = \{y : (\exists x \in A)\varphi(x, y)\}$. □

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Bibliography

Lévy, Azriel. 1960. Axiom schemata of strong infinity in axiomatic set theory. *Pacific Journal of Mathematics* 10(1): 223–38.

[3] More formally, letting ξ be either of these formulas, $\xi(z) \leftrightarrow \xi^S(z)$.

[4] Which may contain parameters

Montague, Richard. 1961. Semantic closure and non-finite axiomatizability I. In *Infinitistic Methods: Proceedings of the Symposium on Foundations of Mathematics (Warsaw 1959)*, 45–69. New York: Pergamon.

