replacement.1 Appendix: Finite axiomatizability

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We close this chapter by extracting some results from Replacement. The first result is due to Montague (1961); note that it is not a proof *within* **ZF**, but a proof *about* **ZF**:

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Theorem replacement.1. **ZF** *is not finitely axiomatizable. More generally: if* **T** *is finite and* $\mathbf{T} \vdash \mathbf{ZF}$*, then* **T** *is inconsistent.*

(Here, we tacitly restrict ourselves to first-order sentences whose only non-logical primitive is $\in$*, and we write* $\mathbf{T} \vdash \mathbf{ZF}$ *to indicate that* $\mathbf{T} \vdash \varphi$ *for all* $\varphi \in \mathbf{ZF}$*.)*

Proof. Fix finite **T** such that $\mathbf{T} \vdash \mathbf{ZF}$. So, **T** proves Reflection, i.e. **??**. Since **T** is finite, we can rewrite it as a single conjunction, θ. Reflecting with this formula, $\mathbf{T} \vdash \exists\beta(\theta \leftrightarrow \theta^{V_\beta})$. Since trivially $\mathbf{T} \vdash \theta$, we find that $\mathbf{T} \vdash \exists\beta\, \theta^{V_\beta}$.

Now, let $\psi(X)$ abbreviate:

$$\theta^X \land X \text{ is transitive} \land (\forall Y \in X)(Y \text{ is transitive} \rightarrow \neg\theta^Y)$$

roughly this says: X is a transitive model of θ, and $\in$-minimal in this regard. Now, recalling that $\mathbf{T} \vdash \exists\beta\, \theta^{V_\beta}$, by basic facts about ranks within **ZF** and hence within **T**, we have:

$$\mathbf{T} \vdash \exists M \psi(M). \tag{*}$$

Using the first conjunct of $\psi(X)$, whenever $\mathbf{T} \vdash \sigma$, we have that $\mathbf{T} \vdash \forall X(\psi(X) \rightarrow \sigma^X)$. So, by (*):

$$\mathbf{T} \vdash \forall X(\psi(X) \rightarrow (\exists N \psi(N))^X)$$

Using this, and (*) again:

$$\mathbf{T} \vdash \exists M(\psi(M) \land (\exists N \psi(N))^M)$$

In particular, then:

$$\mathbf{T} \vdash \exists M(\psi(M) \land (\exists N \in M)((N \text{ is transitive})^N \land (\theta^N)^M))$$

So, by elementary reasoning concerning transitivity:

$$\mathbf{T} \vdash \exists M(\psi(M) \land (\exists N \in M)(N \text{ is transitive} \land \theta^N))$$

So that **T** is inconsistent.[1] □

Here is a similar result:

[1] This "elementary reasoning" involves proving certain "absoluteness facts" for transitive sets.

Proposition replacement.2. *Let* $\mathbf{T}$ *extend* $\mathbf{Z}$ *with finitely many new axioms. If* $\mathbf{T} \vdash \mathbf{ZF}$*, then* $\mathbf{T}$ *is inconsistent. (Here we use the same tacit restrictions as for Theorem replacement.1.)*

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Proof. Use θ for the conjunction of all of $\mathbf{T}$'s axioms *except* for the (infinitely many) instances of Separation. Defining ψ from θ as in Theorem replacement.1, we can show that $\mathbf{T} \vdash \exists M \psi(M)$.

As in Theorem replacement.1, we can establish the schema that, whenever $\mathbf{T} \vdash \sigma$, we have that $\mathbf{T} \vdash \forall X(\psi(X) \rightarrow \sigma^X)$. We then finish our proof, exactly as in Theorem replacement.1.

However, establishing the schema involves a little more work than in Theorem replacement.1. After all, the Separation-instances are in $\mathbf{T}$, but they are not conjuncts of θ. However, we can overcome this obstacle by proving that $\mathbf{T} \vdash \forall X(X \text{ is transitive} \rightarrow \sigma^X)$, for every Separation-instance σ. We leave this to the reader. □

Problem replacement.1. Show that, for every Separation-instance σ, we have: $\mathbf{Z} \vdash \forall X(X \text{ is transitive} \rightarrow \sigma^X)$. (We used this schema in Proposition replacement.2.)

Problem replacement.2. Show that, for every $\varphi \in \mathbf{Z}$, we have $\mathbf{ZF} \vdash \varphi^{V_{\omega+\omega}}$.

Problem replacement.3. Confirm the remaining schematic results invoked in the proofs of Theorem replacement.1 and Proposition replacement.2.

As remarked in **??**, this shows that Replacement is strictly stronger than **??**. Or, slightly more strictly: if $\mathbf{Z}$ + "every well-ordering is isomorphic to a unique ordinal" is consistent, then it fails to prove some Replacement-instance.

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Bibliography

Montague, Richard. 1961. Semantic closure and non-finite axiomatizability I. In *Infinitistic Methods: Proceedings of the Symposium on Foundations of Mathematics (Warsaw 1959)*, 45–69. New York: Pergamon.