ordinals.1 Well-Orderings

The fundamental notion is as follows:

**Definition ordinals.1.** The relation $<$ *well-orders* $A$ iff it meets these two conditions:

1. $<$ is connected, i.e., for all $a, b \in A$, either $a < b$ or $a = b$ or $b < a$;
2. every non-empty subset of $A$ has a $<$-minimal element, i.e., if $\emptyset \neq X \subseteq A$ then $(\exists m \in X)(\forall z \in X) z \not< m$

It is easy to see that three examples we just considered were indeed well-ordering relations.

**Problem ordinals.1.** presented three example orderings on the natural numbers. Check that each is a well-ordering.

Here are some elementary but extremely important observations concerning well-ordering.

**Proposition ordinals.2.** If $<$ well-orders $A$, then every non-empty subset of $A$ has a $<$-least member, and $<$ is irreflexive, asymmetric and transitive.

**Proof.** If $X$ is a non-empty subset of $A$, it has a $<$-minimal element $m$, i.e., $(\forall z \in X) z \not< m$. Since $<$ is connected, $(\forall z \in X) m \leq z$. So $m$ is $<$-least.

For irreflexivity, fix $a \in A$; since $\{a\}$ has a $<$-least element, $a \not< a$. For transitivity, if $a < b < c$, then since $\{a, b, c\}$ has a $<$-least element, $a < c$.

Asymmetry follows from irreflexivity and transitivity

**Proposition ordinals.3.** If $<$ well-orders $A$, then for any formula $\varphi(x)$:

$$\text{if } (\forall a \in A)((\forall b < a)\varphi(b) \rightarrow \varphi(a)), \text{ then } (\forall a \in A)\varphi(a).$$

**Proof.** We will prove the contrapositive. Suppose $\neg((\forall a \in A)\varphi(a)$, i.e., that $X = \{x \in A : \neg \varphi(x)\} \neq \emptyset$. Then $X$ has an $<$-minimal element, $a$. So $(\forall b < a)\varphi(b)$ but $\neg \varphi(a)$.

This last property should remind you of the principle of strong induction on the naturals, i.e.: if $(\forall n \in \omega)((\forall m < n)\varphi(m) \rightarrow \varphi(n)), \text{ then } (\forall n \in \omega)\varphi(n)$. And this property makes well-ordering into a very *robust* notion.

Photo Credits

Bibliography

\[ ^1\text{which may have parameters} \]