ordinals.1  Well-Orderings

The fundamental notion is as follows:

Definition ordinals.1. The relation $<$ well-orders $A$ iff it meets these two conditions:

1. $<$ is connected, i.e., for all $a, b \in A$, either $a < b$ or $a = b$ or $b < a$;

2. every non-empty subset of $A$ has a $<$-minimal element, i.e., if $\emptyset \neq X \subseteq A$ then $(\exists m \in X)(\forall z \in X)z \not< m$

It is easy to see that three examples we just considered were indeed well-ordering relations.

Problem ordinals.1. present three example orderings on the natural numbers. Check that each is a well-ordering.

Here are some elementary but extremely important observations concerning well-ordering.

Proposition ordinals.2. If $<$ well-orders $A$, then every non-empty subset of $A$ has a unique $<$-least member, and $<$ is irreflexive, asymmetric and transitive.

Proof. If $X$ is a non-empty subset of $A$, it has a $<$-minimal element $m$, i.e., $(\forall z \in X)z \not< m$. Since $<$ is connected, $(\forall z \in X)m \leq z$. So $m$ is the $<$-least element of $X$.

For irreflexivity, fix $a \in A$: the $<$-least element of $\{a\}$ is $a$, so $a \not< a$. For transitivity, if $a < b < c$, then since $\{a, b, c\}$ has a $<$-least element, $a < c$. Asymmetry follows from irreflexivity and transitivity.

Proposition ordinals.3. If $<$ well-orders $A$, then for any formula $\varphi(x)$:

$$(\forall a \in A)((\forall b < a)\varphi(b) \rightarrow \varphi(a)), \text{ then } (\forall a \in A)\varphi(a).$$

Proof. We will prove the contrapositive. Suppose $\neg(\forall a \in A)\varphi(a)$, i.e., that $X = \{x \in A : \neg\varphi(x)\} \neq \emptyset$. Then $X$ has an $<$-minimal element, $a$. So $(\forall b < a)\varphi(b)$ but $\neg\varphi(a)$.

This last property should remind you of the principle of strong induction on the naturals, i.e.: if $(\forall n \in \omega)((\forall m < n)\varphi(m) \rightarrow \varphi(n))$, then $(\forall n \in \omega)\varphi(n)$. And this property makes well-ordering into a very robust notion.\(^1\)

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Bibliography

\(^1\)A reminder: all formulas can have parameters (unless explicitly stated otherwise).