

ordinals.1 Von Neumann's Construction of the Ordinals

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?? gives rise to a thought. We could introduce certain objects, called *order types*, to go proxy for the well-orderings. Writing $\text{ord}(A, <)$ for the order type of the well-ordering $\langle A, < \rangle$, we would hope to secure the following two principles:

$$\begin{aligned}\text{ord}(A, <) &= \text{ord}(B, \triangleleft) \text{ iff } \langle A, < \rangle \cong \langle B, \triangleleft \rangle \\ \text{ord}(A, <) < \text{ord}(B, \triangleleft) &\text{ iff } \langle A, < \rangle \cong \langle B_b, \triangleleft_b \rangle \text{ for some } b \in B\end{aligned}$$

Moreover, we might hope to introduce order-types *as certain sets*, just as we can introduce the natural numbers as certain sets.

The most common way to do this—and the approach we will follow—is to define these order-types via certain *canonical* well-ordered sets. These canonical sets were first introduced by von Neumann:

Definition ordinals.1. The set A is *transitive* iff $(\forall x \in A)x \subseteq A$. Then A is an *ordinal* iff A is transitive and well-ordered by \in .

In what follows, we will use Greek letters for ordinals. It follows immediately from the definition that, if α is an ordinal, then $\langle \alpha, \in_\alpha \rangle$ is a well-ordering, where $\in_\alpha = \{\langle x, y \rangle \in \alpha^2 : x \in y\}$. So, abusing notation a little, we can just say that α *itself* is a well-ordering.

Here are our first few ordinals:

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots$$

You will note that these are the first few ordinals that we encountered in our Axiom of Infinity, i.e., in von Neumann's definition of ω (see ??). This is no coincidence. Von Neumann's definition of the ordinals treats natural numbers as ordinals, but allows for transfinite ordinals too.

As always, we can now ask: *are* these the ordinals? Or has von Neumann simply given us some sets that we can *treat* as the ordinals? The kinds of discussions one might have about this question are similar to the discussions we had in ??, ??, ??, and ??, so we will not belabour the point. Instead, in what follows, we will simply use “the ordinals” to speak of “the von Neumann ordinals”.

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Bibliography