In section ordinals.1, we motivated the introduction of ordinals by suggesting that we could treat them as order-types, i.e., canonical proxies for well-orderings. In order for that to work, we would need to prove that every well-ordering is isomorphic to some ordinal. This would allow us to define $\text{ord}(A, <)$ as the ordinal $\alpha$ such that $\langle A, < \rangle \cong \alpha$.

Unfortunately, we cannot prove the desired result only the Axioms we provided introduced so far. (We will see why in ??, but for now: we can’t.) We need a new thought, and here it is:

**Axiom (Scheme of Replacement).** For any formula $\varphi(x, y)$,\(^1\) this is an axiom:

- for any $A$, if $(\forall x \in A) \exists! y \varphi(x, y)$, then $\{y : (\exists x \in A) \varphi(x, y)\}$ exists.

As with Separation, this is a scheme: it yields infinitely many axioms, for each of the infinitely many different $\varphi$’s. And it can equally well be (and normally is) written down thus:

For any formula $\varphi(x, y)$ which does not contain “$B$”,\(^2\) this is an axiom:

$$\forall A[(\forall x \in A) \exists! y \varphi(x, y) \rightarrow \exists B \forall y (y \in B \leftrightarrow (\exists x \in A) \varphi(x, y))]$$

On first encounter, however, this is quite a tangled formula. The following quick consequence of Replacement probably gives a clearer expression to the intuitive idea we are working with:

**Corollary ordinals.1.** For any term $\tau(x)$,\(^3\) and any set $A$, this set exists:

$$\{\tau(x) : x \in A\} = \{y : (\exists x \in A) y = \tau(x)\}.$$ 

**Proof.** Since $\tau$ is a term, $\forall x \exists! y \tau(x) = y$. A fortiori, $(\forall x \in A) \exists! y \tau(x) = y$. So $\{y : (\exists x \in A) \tau(x) = y\}$ exists by Replacement. \(\Box\)

This suggests that “Replacement” is a good name for the Axiom: given a set $A$, you can form a new set, $\{\tau(x) : x \in A\}$, by replacing every member of $A$ with its image under $\tau$. Indeed, following the notation for the image of a set under a function, we might write $\tau[A]$ for $\{\tau(x) : x \in A\}$.

Crucially, however, $\tau$ is a term. It need not be (a name for) a function, in the sense of ??, i.e., a certain set of ordered pairs. After all, if $f$ is a function (in that sense), then the set $f[A] = \{f(x) : x \in A\}$ is just a particular subset of $\text{ran}(f)$, and that is already guaranteed to exist, just using the axioms of $\mathbb{Z}^-$\(^4\).

Replacement, by contrast, is a powerful addition to our axioms, as we will see in ??.

---

\(^1\)Which may have parameters.
\(^2\)Which may have parameters.
\(^3\)Which may have parameters.
\(^4\)Just consider $\{y \in \bigcup f : (\exists x \in A) y = f(x)\}$. 

1
Photo Credits

Bibliography