ordinals.1 Replacement

In ??, we motivated the introduction of ordinals by suggesting that we could treat them as order-types, i.e., canonical proxies for well-orderings. In order for that to work, we would need to prove that every well-ordering is isomorphic to some ordinal. This would allow us to define \( \text{ord}(A, <) \) as the ordinal \( \alpha \) such that \( (A, <) \cong \alpha \).

Unfortunately, we cannot prove the desired result only the Axioms we provided introduced so far. (We will see why in ??, but for now the point is: we can’t.) We need a new thought, and here it is:

**Axiom (Scheme of Replacement).** For any formula \( \varphi(x, y) \), the following is an axiom:

\[
\text{for any } A, \text{ if } (\forall x \in A) \exists! y \varphi(x, y), \text{ then } \{y : (\exists x \in A) \varphi(x, y)\} \text{ exists.}
\]

As with Separation, this is a scheme: it yields infinitely many axioms, for each of the infinitely many different \( \varphi \)'s. And it can equally well be (and normally is) written down thus:

For any formula \( \varphi(x, y) \) which does not contain “\( B \)”, the following is an axiom:

\[
\forall A[(\forall x \in A) \exists! y \varphi(x, y) \rightarrow \exists B \forall y(y \in B \leftrightarrow (\exists x \in A) \varphi(x, y))]
\]

On first encounter, however, this is quite a tangled formula. The following quick consequence of Replacement probably gives a clearer expression to the intuitive idea we are working with:

**Corollary ordinals.1.** For any term \( \tau(x) \), and any set \( A \), this set exists:

\[
\{\tau(x) : x \in A\} = \{y : (\exists x \in A) y = \tau(x)\}.
\]

*Proof.* Since \( \tau \) is a term, \( \forall x \exists! y \tau(x) = y \). A fortiori, \( (\forall x \in A) \exists! y \tau(x) = y \). So \( \{y : (\exists x \in A) \tau(x) = y\} \) exists by Replacement. \( \square \)

This suggests that “Replacement” is a good name for the Axiom: given a set \( A \), you can form a new set, \( \{\tau(x) : x \in A\} \), by replacing every member of \( A \) with its image under \( \tau \). Indeed, following the notation for the image of a set under a function, we might write \( \tau[A] \) for \( \{\tau(x) : x \in A\} \).

Crucially, however, \( \tau \) is a term. It need not be (a name for) a function, in the sense of ??, i.e., a certain set of ordered pairs. After all, if \( f \) is a function (in that sense), then the set \( f[A] = \{f(x) : x \in A\} \) is just a particular subset of \( \text{ran}(f) \), and that is already guaranteed to exist, just using the axioms of \( \mathbb{Z}^1 \).

\(^1\)Just consider \( \{y \in \bigcup f : (\exists x \in A) y = f(x)\} \).
Replacement, by contrast, is a powerful addition to our axioms, as we will see in ??.

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Bibliography