

## ordinals.1 Replacement

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sec

In [section ordinals.1](#), we motivated the introduction of ordinals by suggesting that we could treat them as order-types, i.e., canonical proxies for well-orderings. In order for that to work, we would need to prove that *every well-ordering is isomorphic to some ordinal*. This would allow us to define  $\text{ord}(A, <)$  as the ordinal  $\alpha$  such that  $\langle A, < \rangle \cong \alpha$ .

Unfortunately, we *cannot* prove the desired result only the Axioms we provided introduced so far. (We will see why in ??, but for now: we can't.) We need a new thought, and here it is:

**Axiom** (Scheme of Replacement). For any formula  $\varphi(x, y)$ ,<sup>1</sup> this is an axiom: for any  $A$ , if  $(\forall x \in A)\exists!y \varphi(x, y)$ , then  $\{y : (\exists x \in A)\varphi(x, y)\}$  exists.

As with Separation, this is a scheme: it yields infinitely many axioms, for each of the infinitely many different  $\varphi$ 's. And it can equally well be (and normally is) written down thus:

*For any formula  $\varphi(x, y)$  which does not contain “B”,<sup>2</sup> this is an axiom:*  
 $\forall A[(\forall x \in A)\exists!y \varphi(x, y) \rightarrow \exists B \forall y (y \in B \leftrightarrow (\exists x \in A)\varphi(x, y))]$

On first encounter, however, this is quite a tangled formula. The following quick consequence of Replacement probably gives a *clearer* expression to the intuitive idea we are working with:

**Corollary ordinals.1.** *For any term  $\tau(x)$ ,<sup>3</sup> and any set  $A$ , this set exists:*

$$\{\tau(x) : x \in A\} = \{y : (\exists x \in A)y = \tau(x)\}.$$

*Proof.* Since  $\tau$  is a *term*,  $\forall x \exists!y \tau(x) = y$ . A fortiori,  $(\forall x \in A)\exists!y \tau(x) = y$ . So  $\{y : (\exists x \in A)\tau(x) = y\}$  exists by Replacement.  $\square$

This suggests that “Replacement” is a good name for the Axiom: given a set  $A$ , you can form a new set,  $\{\tau(x) : x \in A\}$ , by replacing every member of  $A$  with its image under  $\tau$ . Indeed, following the notation for the image of a set under a function, we might write  $\tau[A]$  for  $\{\tau(x) : x \in A\}$ .

Crucially, however,  $\tau$  is a *term*. It need not be (a name for) a *function*, in the sense of ??, i.e., a certain set of ordered pairs. After all, if  $f$  is a function (in that sense), then the set  $f[A] = \{f(x) : x \in A\}$  is just a particular subset of  $\text{ran}(f)$ , and that is already guaranteed to exist, just using the axioms of  $\mathbf{Z}^-$ .<sup>4</sup> Replacement, by contrast, is a *powerful* addition to our axioms, as we will see in ??.

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<sup>1</sup>Which may have parameters.

<sup>2</sup>Which may have parameters.

<sup>3</sup>Which may have parameters.

<sup>4</sup>Just consider  $\{y \in \bigcup \bigcup f : (\exists x \in A)y = f(x)\}$ .

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**Bibliography**