ordinals.1 Ordinals as Order-Types

sth:ordinals:ordtype:sec Armed with Replacement, and so now working in $\mathbf{ZF}^-$, we can prove what we wanted:

sth:ordinals:ordtype:thmOrdinalRepresentation **Theorem ordinals.1.** *Every well-ordering is isomorphic to a unique ordinal.*

Proof. Let $\langle A, < \rangle$ be a well-order. By **??**, it is isomorphic to at most one ordinal. So, for reductio, suppose $\langle A, < \rangle$ is not isomorphic to *any* ordinal. We will first "make $\langle A, < \rangle$ as small as possible". In detail: if some proper initial segment $\langle A_a, <_a \rangle$ is not isomorphic to any ordinal, there is a least $a \in A$ with that property; then let $B = A_a$ and $\lessdot = <_a$. Otherwise, let $B = A$ and $\lessdot = <$.

By definition, every proper initial segment of B is isomorphic to some ordinal, which is unique by **??**. So by Replacement, the following set exists, and is a function:

$$f = \{\langle \beta, b \rangle : b \in B \text{ and } \beta \cong \langle B_b, \lessdot_b \rangle\}$$

To complete the reductio, we'll show that f is an isomorphism $\alpha \to B$, for some ordinal α. It is obvious that $\operatorname{ran}(f) = B$. And by **??**, f preserves ordering, i.e., $\gamma \in \beta$ iff $f(\gamma) \lessdot f(\beta)$.

To show that $\operatorname{dom}(f)$ is an ordinal, by **??** it suffices to show that $\operatorname{dom}(f)$ is transitive. So fix $\beta \in \operatorname{dom}(f)$, i.e., $\beta \cong \langle B_b, \lessdot_b \rangle$ for some b. If $\gamma \in \beta$, then $\gamma \in \operatorname{dom}(f)$ by **??**; generalising, $\beta \subseteq \operatorname{dom}(f)$. □

This result licenses the following definition, which we have wanted to offer since **??**:

Definition ordinals.2. If $\langle A, < \rangle$ is a well-ordering, then its order type, $\operatorname{ord}(A, <)$, is the unique ordinal α such that $\langle A, < \rangle \cong \alpha$.

Moreover, this definition licenses two nice principles:

sth:ordinals:ordtype:ordtypesworklikeyouwant **Corollary ordinals.3.** *Where $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ are well-orderings:*

$$\operatorname{ord}(A, <) = \operatorname{ord}(B, \lessdot) \text{ iff } \langle A, < \rangle \cong \langle B, \lessdot \rangle$$
$$\operatorname{ord}(A, <) \in \operatorname{ord}(B, \lessdot) \text{ iff } \langle A, < \rangle \cong \langle B_b, \lessdot_b \rangle \text{ for some } b \in B$$

Proof. The identity holds as isomorphism is an equivalence relation. To prove the second claim, let $\operatorname{ord}(A, <) = \alpha$ and $\operatorname{ord}(B, \lessdot) = \beta$, and let $f \colon \beta \to \langle B, \lessdot \rangle$ be our isomorphism. Then:

$$\begin{aligned} \alpha \in \beta & \text{ iff } f{\restriction}_{\alpha} \colon \alpha \to B_{f(\alpha)} \text{ is an isomorphism} \\ & \text{ iff } \langle A, < \rangle \cong \langle B_{f(\alpha)}, \lessdot_{f(\alpha)} \rangle \\ & \text{ iff } \langle A, < \rangle \cong \langle B_b, \lessdot_b \rangle \text{ for some } b \in B \end{aligned}$$

by **??**, **??**, and **??**. □

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Bibliography