Ordinals as Order-Types

Armed with Replacement, and so now working in $\mathbf{ZF}^-$, we can finally prove the result we have been aiming for:

**Theorem ordinals.1.** Every well-ordering is isomorphic to a unique ordinal.

*Proof.* Let $\langle A, < \rangle$ be a well-order. By $\mathbf{??}$, it is isomorphic to at most one ordinal. So, for reductio, suppose $\langle A, < \rangle$ is not isomorphic to any ordinal. We will first “make $\langle A, < \rangle$ as small as possible”. In detail: if some proper initial segment $\langle A_a, <_a \rangle$ is not isomorphic to any ordinal, there is a least $a \in A$ with that property; then let $B = A_a$ and $<_a = <_a$. Otherwise, let $B = A$ and $<_a = <$.

By definition, every proper initial segment of $B$ is isomorphic to some ordinal, which is unique as above. So by Replacement, the following set exists, and is a function:

$$f = \{ \langle \beta, b \rangle : b \in B \text{ and } \beta \cong \langle B_b, \langle b \rangle \rangle \}$$

To complete the reductio, we’ll show that $f$ is an isomorphism $\alpha \to B$, for some ordinal $\alpha$.

It is obvious that $\text{ran}(f) = B$. And by $\mathbf{??}$, $f$ preserves ordering, i.e., $\gamma \in \beta$ iff $f(\gamma) < f(\beta)$. To show that $\text{dom}(f)$ is an ordinal, by $\mathbf{??}$ it suffices to show that $\text{dom}(f)$ is transitive. So fix $\beta \in \text{dom}(f)$, i.e., $\beta \cong \langle B_b, \langle b \rangle \rangle$ for some $b$. If $\gamma \in \beta$, then $\gamma \in \text{dom}(f)$ by $\mathbf{??}$; generalising, $\beta \subseteq \text{dom}(f)$.

This result licenses the following definition, which we have wanted to offer since $\mathbf{??}$:

**Definition ordinals.2.** If $\langle A, < \rangle$ is a well-ordering, then its order type, $\text{ord}(A, <)$, is the unique ordinal $\alpha$ such that $\langle A, < \rangle \cong \alpha$.

Moreover, this definition licenses two nice principles:

**Corollary ordinals.3.** Where $\langle A, < \rangle$ and $\langle B, \langle \rangle \rangle$ are well-orderings:

$$\text{ord}(A, <) = \text{ord}(B, \langle \rangle) \iff \langle A, < \rangle \cong \langle B, \langle \rangle \rangle$$
$$\text{ord}(A, <) \in \text{ord}(B, \langle \rangle) \iff \langle A, < \rangle \cong \langle B_b, \langle b \rangle \rangle \text{ for some } b \in B$$

*Proof.* The identity holds by $\mathbf{??}$. To prove the second claim, let $\text{ord}(A, <) = \alpha$ and $\text{ord}(B, <) = \beta$, and let $f : \beta \to \langle B, \langle \rangle \rangle$ be our isomorphism. Then:

$$\alpha \in \beta \text{ iff } f|_{\alpha} : \alpha \to B_{f(\alpha)} \text{ is an isomorphism }$$
$$\text{iff } \langle A, < \rangle \cong \langle B_{f(\alpha)}, \langle f(\alpha) \rangle \rangle$$
$$\text{iff } \langle A, < \rangle \cong \langle B_b, \langle b \rangle \rangle \text{ for some } b \in B$$

by $\mathbf{??}$, $\mathbf{??}$, and $\mathbf{??}$.