Chapter udf

Ordinals

ordinals.1 Introduction

sth:ordinals:intro:sec

In **??**, we postulated that there is an infinite-th stage of the hierarchy, in the form of *Stages-hit-infinity* (see also our axiom of Infinity). However, given *Stages-keep-going*, we can't stop at the infinite-th stage; we have to keep going. So: at the next stage after the first infinite stage, we form all possible collections of sets that were available at the first infinite stage; and repeat; and repeat; and repeat; ...

Implicitly what has happened here is that we have started to invoke an "intuitive" notion of number, according to which there can be numbers *after* all the natural numbers. In particular, the notion involved is that of a *transfinite ordinal*. The aim of this chapter is to make this idea more rigorous. We will explore the general notion of an ordinal, and then explicitly define certain sets to be our ordinals.

ordinals.2 The General Idea of an Ordinal

sth:ordinals:idea:sec

Consider the natural numbers, in their usual order:

$$0 \quad < \quad 1 \quad < \quad 2 \quad < \quad 3 \quad < \quad 4 \quad < \quad 5 \quad < \quad \cdots$$

We call this, in the jargon, an ω-sequence. And indeed, this general ordering is mirrored in our initial construction of the stages of the set hierarchy. But, now suppose we move 0 to the end of this sequence, so that it comes after all the other numbers:

$$1 \quad < \quad 2 \quad < \quad 3 \quad < \quad 4 \quad < \quad 5 \quad < \quad \cdots \quad < \quad 0$$

We have the same entities here, but ordered in a fundamentally different way: our first ordering had no last element; our new ordering does. Indeed, our new ordering consists of an ω-sequence of entities $(1, 2, 3, 4, 5, \ldots)$, followed by another entity. It will be an $\omega + 1$-sequence.

We can generate even more types of ordering, using just these entities. For example, consider all the even numbers (in their natural order) followed by all the odd numbers (in their natural order):

$$0 \quad < \quad 2 \quad < \quad 4 \quad < \cdots < \quad 1 \quad < \quad 3 \quad < \quad \cdots$$

This is an ω-sequence followed by another ω-sequence; an $\omega + \omega$-sequence.

Well, we can keep going. But what we would like is a general way to understand this talk about *orderings*.

ordinals.3 Well-Orderings

The fundamental notion is as follows: sth:ordinals:wo:sec

Definition ordinals.1. The relation $<$ *well-orders* A iff it meets these two conditions:

1. $<$ is connected, i.e., for all $a, b \in A$, either $a < b$ or $a = b$ or $b < a$;
2. every non-empty subset of A has a $<$-minimal element, i.e., if $\emptyset \neq X \subseteq A$ then $(\exists m \in X)(\forall z \in X) z \not< m$

It is easy to see that three examples we just considered were indeed well-ordering relations.

Problem ordinals.1. Section ordinals.2 presented three example orderings on the natural numbers. Check that each is a well-ordering.

Here are some elementary but extremely important observations concerning well-ordering.

Proposition ordinals.2. *If $<$ well-orders A, then every non-empty subset of A has a $<$-least member, and $<$ is irreflexive, asymmetric and transitive.* *sth:ordinals:wo:wo:strictorder*

Proof. If X is a non-empty subset of A, it has a $<$-minimal element m, i.e., $(\forall z \in X) z \not< m$. Since $<$ is connected, $(\forall z \in X) m \leq z$. So m is $<$-least.

For irreflexivity, fix $a \in A$; since $\{a\}$ has a $<$-least element, $a \not< a$. For transitivity, if $a < b < c$, then since $\{a, b, c\}$ has a $<$-least element, $a < c$. Asymmetry follows from irreflexivity and transitivity □

Proposition ordinals.3. *If $<$ well-orders A, then for any formula $\varphi(x)$:*[1] *sth:ordinals:wo:propwoinduction*

$$\textit{if } (\forall a \in A)((\forall b < a)\varphi(b) \rightarrow \varphi(a)), \textit{ then } (\forall a \in A)\varphi(a).$$

Proof. We will prove the contrapositive. Suppose $\neg(\forall a \in A)\varphi(a)$, i.e., that $X = \{x \in A : \neg\varphi(x)\} \neq \emptyset$. Then X has an $<$-minimal element, a. So $(\forall b < a)\varphi(b)$ but $\neg\varphi(a)$. □

This last property should remind you of the principle of strong induction on the naturals, i.e.: if $(\forall n \in \omega)((\forall m < n)\varphi(m) \rightarrow \varphi(n))$, then $(\forall n \in \omega)\varphi(n)$. And this property makes well-ordering into a very *robust* notion.

[1] which may have parameters

ordinals.4 Order-Isomorphisms

sth:ordinals:iso: sec

To explain *how* robust well-ordering is, we will start by introducing a method for comparing well-orderings.

Definition ordinals.4. A *well-ordering* is a pair $\langle A, < \rangle$, such that $<$ well-orders A. The well-orderings $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ are *order-isomorphic* iff there is a bijection $f \colon A \to B$ such that: $x < y$ iff $f(x) \lessdot f(y)$. In this case, we write $\langle A, < \rangle \cong \langle B, \lessdot \rangle$, and say that f is an *order-isomorphism*.

In what follows, for brevity, we will speak of "isomorphisms" rather than "order-isomorphisms". Intuitively, isomorphisms are structure-preserving bijections. Here are some simple facts about isomorphisms.

sth:ordinals:iso: isoscompose

Lemma ordinals.5. *Compositions of isomorphisms are isomorphisms, i.e.: if $f \colon A \to B$ and $g \colon B \to C$ are isomorphisms, then $(g \circ f) \colon A \to C$ are isomorphisms. (It follows that $X \cong Y$ is an equivalence relation.)*

Proof. Left as an exercise. □

Problem ordinals.2. Prove Lemma ordinals.5.

sth:ordinals:iso: ordisounique

Proposition ordinals.6. *If $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ are isomorphic well-orderings, then the isomorphism between them is unique.*

Proof. Let f and g be isomorphisms $A \to B$. Fix $a \in A$, and suppose that $(\forall b < a) f(b) = g(b)$, and fix $x \in B$.

If $x \lessdot f(a)$, then $f^{-1}(x) < a$, so $g(f^{-1}(x) \lessdot g(a))$, invoking the fact that f and g are isomorphisms. But since $f^{-1}(x) < a$, by our supposition $x = f(f^{-1}(x)) = g(f^{-1}(x))$. So $x \lessdot g(a)$. Similarly, if $x \lessdot g(a)$ then $x \lessdot f(a)$.

Generalising, $(\forall x \in B)(x \lessdot f(a) \leftrightarrow x \lessdot g(a))$. It follows that $f(a) = g(a)$ by **??**. So $(\forall a \in A) f(a) = g(a)$ by Proposition ordinals.3. □

This gives some sense that well-orderings are robust. But to continue explaining this, it will help to introduce some more notation.

Definition ordinals.7. When $\langle A, < \rangle$ is a well-ordering, let $A_a = \{x \in A : x < a\}$; we say that A_a is a proper *initial segment* of A. (We allow that A itself is an improper initial segment of A.) Let $<_a$ be the restriction of $<$ to the initial segment, i.e., $<\restriction_{A_a^2}$.

Using this notation, we can state and prove that no well-ordering is isomorphic to any of its proper initial segments.

sth:ordinals:iso: wellordnotinitial

Lemma ordinals.8. *If $\langle A, < \rangle$ is a well-ordering with $a \in A$, then $\langle A, a \rangle \ncong \langle A_a, <_a \rangle$*

Proof. For reductio, suppose $f\colon A \to A_a$ is an isomorphism. Since f is a bijection and $A_a \subsetneq A$, let $b \in A$ be the $<$-least element of A such that $b \neq f(b)$. We'll show that $(\forall x \in A)(x < b \leftrightarrow x < f(b))$, from which it will follow by **??** that $b = f(b)$, completing the reductio.

Suppose $x < b$. So $x = f(x)$, by the choice of b. And $f(x) < f(b)$, as f is an isomorphism. So $x < f(b)$.

Suppose $x < f(b)$. So $f^{-1}(x) < b$, since f is an isomorphism, and so $f^{-1}(x) = x$ by the choice of b. So $x < b$. □

Our next result shows, roughly put, that an "initial segment" of an isomorphism is an isomorphism:

Lemma ordinals.9. *Let $\langle A, <\rangle$ and $\langle B, \lessdot\rangle$ be well-orderings. If $f\colon A \to B$ is an isomorphism and $a \in A$, then $f{\restriction}_{A_a} : A_a \to B_{f(a)}$ is an isomorphism.* *sth:ordinals:iso:wellordinitialsegment*

Proof. Since f is an isomorphism:

$$\begin{aligned} f[A_a] &= f[\{x \in A : x < a\}] \\ &= f[\{f^{-1}(y) \in A : f^{-1}(y) < a\}] \\ &= \{y \in B : y \lessdot f(a)\} \\ &= B_{f(a)} \end{aligned}$$

And $f{\restriction}_{A_a}$ preserves order because f does. □

Our next two results establish that well-orderings are always comparable:

Lemma ordinals.10. *Let $\langle A, <\rangle$ and $\langle B, \lessdot\rangle$ be well-orderings. If $\langle A_{a_1}, <_{a_1}\rangle \cong \langle B_{b_1}, \lessdot_{b_1}\rangle$ and $\langle A_{a_2}, <_{a_2}\rangle \cong \langle B_{b_2}, \lessdot_{b_2}\rangle$, then $a_1 < a_2$ iff $b_1 \lessdot b_2$* *sth:ordinals:iso:lemordsegments*

Proof. We will prove *left to right*; the other direction is similar. Suppose both $\langle A_{a_1}, <_{a_1}\rangle \cong \langle B_{b_1}, \lessdot_{b_1}\rangle$ and $\langle A_{a_2}, <_{a_2}\rangle \cong \langle B_{b_2}, \lessdot_{b_2}\rangle$, with $f\colon A_{a_2} \to B_{b_2}$ our isomorphism. Let $a_1 < a_2$; then $\langle A_{a_1}, <_{a_1}\rangle \cong \langle B_{f(a_1)}, \lessdot_{f(a_1)}\rangle$ by Lemma ordinals.9. So $\langle B_{b_1}, \lessdot_{b_1}\rangle \cong \langle B_{f(a_1)}, \lessdot_{f(a_1)}\rangle$, and so $b_1 = f(a_1)$ by Lemma ordinals.8. Now $b_1 \lessdot b_2$ as f's domain is B_{b_2}. □

Theorem ordinals.11. *Given any two well-orderings, one is isomorphic to an initial segment (not necessarily proper) of the other.* *sth:ordinals:iso:thm:woalwayscomparable*

Proof. Let $\langle A, <\rangle$ and $\langle B, \lessdot\rangle$ be well-orderings. Using Separation, let

$$f = \{\langle a, b\rangle \in A \times B : \langle A_a, <_a\rangle \cong \langle B_b, \lessdot_b\rangle\}.$$

By Lemma ordinals.10, $a_1 < a_2$ iff $b_1 \lessdot b_2$ for all $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in f$. So $f\colon \mathrm{dom}(f) \to \mathrm{ran}(f)$ is an isomorphism.

If $a_2 \in \mathrm{dom}(f)$ and $a_1 < a_2$, then $a_1 \in \mathrm{dom}(f)$ by Lemma ordinals.9; so $\mathrm{dom}(f)$ is an initial segment of A. Similarly, $\mathrm{ran}(f)$ is an initial segment of B. For reductio, suppose both are *proper* initial segments. Then let a be the $<$-least element of $A \setminus \mathrm{dom}(f)$, so that $\mathrm{dom}(f) = A_a$, and let b be the $\lessdot$-least element of $B \setminus \mathrm{ran}(f)$, so that $\mathrm{ran}(f) = B_b$. So $f\colon A_a \to B_b$ is an isomorphism, and hence $f(a) = b$, a contradiction. □

ordinals.5 Von Neumann's Construction of the Ordinals

sth:ordinals:vn:sec

Theorem ordinals.11 gives rise to a thought. We could introduce certain objects, called *order types*, to go proxy for the well-orderings. Writing $\mathrm{ord}(A,<)$ for the order type of the well-ordering $\langle A,<\rangle$, we would hope to secure the following two principles:

$$\mathrm{ord}(A,<) = \mathrm{ord}(B,\lessdot) \text{ iff } \langle A,<\rangle \cong \langle B,\lessdot\rangle$$
$$\mathrm{ord}(A,<) < \mathrm{ord}(B,\lessdot) \text{ iff } \langle A,<\rangle \cong \langle B_b,\lessdot_b\rangle \text{ for some } b \in B$$

Moreover, we might hope to introduce order-types *as certain sets*, just as we can introduce the natural numbers as certain sets.

The most common way to do this—and the approach we will follow—is to define these order-types via certain *canonical* well-ordered sets. These canonical sets were first introduced by von Neumann:

Definition ordinals.12. The set A is *transitive* iff $(\forall x \in A)x \subseteq A$. Then A is an *ordinal* iff A is transitive and well-ordered by $\in$.

In what follows, we will use Greek letters for ordinals. It follows immediately from the definition that, if α is an ordinal, then $\langle \alpha, \in_\alpha\rangle$ is a well-ordering, where $\in_\alpha = \{\langle x,y\rangle \in \alpha^2 : x \in y\}$. So, abusing notation a little, we can just say that α *itself* is a well-ordering.

Here are our first few ordinals:

$$\emptyset, \{\emptyset\}, \{\emptyset,\{\emptyset\}\}, \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}, \dots$$

You will note that these are the first few ordinals that we encountered in our Axiom of Infinity, i.e., in von Neumann's definition of ω (see **??**). This is no coincidence. Von Neumann's definition of the ordinals treats natural numbers as ordinals, but allows for transfinite ordinals too.

As always, we can now ask: *are* these the ordinals? Or has von Neumann simply given us some sets that we can *treat* as the ordinals? The kinds of discussions one might have about this question are similar to the discussions we had in **??**, **??**, **??**, and **??**, so we will not belabour the point. Instead, in what follows, we will simply use "the ordinals" to speak of "the von Neumann ordinals".

ordinals.6 Basic Properties of the Ordinals

sth:ordinals:basic:sec

We observed that the first few ordinals are the natural numbers. The main reason for developing a theory of ordinals is to extend the principle of induction which holds on the natural numbers. We will build up to this via a sequence of elementary results.

sth:ordinals:basic:ordmemberord

Lemma ordinals.13. *Every element of an ordinal is an ordinal.*

Proof. Let α be an ordinal with $b \in \alpha$. Since α is transitive, $b \subseteq \alpha$. So $\in$ well-orders b as $\in$ well-orders α.

For transitivity, suppose $x \in c \in b$. So $c \in \alpha$ as $b \subseteq \alpha$. Again, as α is transitive, $c \subseteq \alpha$, so that $x \in \alpha$. So $x, c, b \in \alpha$. But $\in$ well-orders α, so that $\in$ is a transitive relation on α by Proposition ordinals.2. So since $x \in c \in b$, we have $x \in b$. Generalising, $c \subseteq b$ □

Corollary ordinals.14. *$\alpha = \{\beta \in \alpha : \beta$ is an ordinal$\}$, for any ordinal α* *sth:ordinals:basic: ordissetofsmallerord*

Proof. Immediate from Lemma ordinals.13. □

The rough gist of the next two main results, Theorem ordinals.15 and Theorem ordinals.16, is that the ordinals themselves are well-ordered by membership:

Theorem ordinals.15 (Transfinite Induction). *For any formula $\varphi(x)$:*[2] *sth:ordinals:basic: ordinductionschema*

$$\text{if } \exists\alpha\varphi(\alpha), \text{ then } \exists\alpha(\varphi(\alpha) \land (\forall\beta \in \alpha)\neg\varphi(\beta))$$

where the displayed quantifiers are implicitly restricted to ordinals.

Proof. Suppose $\varphi(\alpha)$, for some ordinal α. If $(\forall\beta \in \alpha)\neg\varphi(\beta)$, then we are done. Otherwise, as α is an ordinal, it has some $\in$-least element which is φ, and this is an ordinal by Lemma ordinals.13. □

Note that we can equally express Theorem ordinals.15 as the scheme:

$$\text{if } \forall\alpha((\forall\beta \in \alpha)\varphi(\beta) \rightarrow \varphi(\alpha)), \text{ then } \forall\alpha\varphi(\alpha)$$

just by taking $\neg\varphi(\alpha)$ in Theorem ordinals.15 and reasoning as in Proposition ordinals.3.

Theorem ordinals.16 (Trichotomy). *$\alpha \in \beta \lor \alpha = \beta \lor \beta \in \alpha$, for any ordinals α and β.* *sth:ordinals:basic: ordtrichotomy*

Proof. The proof is by double induction, i.e., using Theorem ordinals.15 twice. Say that x is *comparable* with y iff $x \in y \lor x = y \lor y \in x$.

For induction, suppose that every ordinal in α is comparable with *every* ordinal. For further induction, suppose that α is comparable with every ordinal in β. We will show that α is comparable with β. By induction on β, it will follow that α is comparable with every ordinal; and so by induction on α, *every* ordinal is comparable with *every* ordinal, as required.

It suffices to assume that $\alpha \notin \beta$ and $\beta \notin \alpha$, and show that $\alpha = \beta$.

To show that $\alpha \subseteq \beta$, fix $\gamma \in \alpha$; this is an ordinal by Lemma ordinals.13. So by the first induction hypothesis, γ is comparable with β. But if either $\gamma = \beta$ or $\beta \in \gamma$ then $\beta \in \alpha$ (invoking the fact that α is transitive if necessary), contrary to our assumption; so $\gamma \in \beta$. Generalising, $\alpha \subseteq \beta$.

Exactly similar reasoning, using the second induction hypothesis, shows that $\beta \subseteq \alpha$. So $\alpha = \beta$. □

[2] The formula may have parameters, which need not be ordinals.

As such, we will sometimes write $\alpha < \beta$ rather than $\alpha \in \beta$, since $\in$ is behaving as an ordering relation. There are no deep reasons for this, beyond familiarity, and because it is easier to write $\alpha \leq \beta$ than $\alpha \in \beta \lor \alpha = \beta$.[3]

Here are two quick consequences of our last results, the first of which puts our new notation into action:

sth:ordinals:basic: ordordered

Corollary ordinals.17. *If $\exists\alpha\varphi(\alpha)$, then $\exists\alpha(\varphi(\alpha) \land \forall\beta(\varphi(\beta) \to \alpha \leq \beta))$. Moreover, for any ordinals α, β, γ, both $\alpha \notin \alpha$ and $\alpha \in \beta \in \gamma \to \alpha \in \gamma$.*

Proof. Just like Proposition ordinals.2. □

Problem ordinals.3. Complete the "exactly similar reasoning" in the proof of Theorem ordinals.16.

sth:ordinals:basic: corordtransitiveord

Corollary ordinals.18. *A is an ordinal iff A is a transitive set of ordinals.*

Proof. Left-to-right. By Lemma ordinals.13. *Right-to-left.* If A is a transitive set of ordinals, then $\in$ well-orders A by Theorem ordinals.15 and Theorem ordinals.16. □

But, although we have said that $\in$ well-orders the ordinals, we have to be *very cautious* about all this, thanks to the following:

sth:ordinals:basic: buraliforti

Theorem ordinals.19 (Burali-Forti Paradox)**.** *There is no set of all the ordinals*

Proof. For reductio, suppose O is the set of all ordinals. If $\alpha \in \beta \in O$, then α is an ordinal, by Lemma ordinals.13, so $\alpha \in O$. So O is transitive, and hence O is an ordinal by Corollary ordinals.18. Hence $O \in O$, contradicting Corollary ordinals.17. □

This result is named after Burali-Forti. But, as van Heijenoort explains:

> Burali-Forti himself considered the contradiction as establishing, by *reductio ad absurdum*, the result that the natural ordering of the ordinals is just a partial ordering. (Heijenoort, 1967, p. 105)

It was Cantor in 1899—in a letter to Dedekind—who first saw clearly the *contradiction* in supposing that there is a set of all the ordinals. (For further historical discussion, see Heijenoort 1967, p. 105.)

To summarise, ordinals are sets which are individually well-ordered by membership, and collectively well-ordered by membership.

Rounding this off, here are some more basic properties about the ordinals which follow from Theorem ordinals.15 and Theorem ordinals.16.

Proposition ordinals.20. *Any strictly descending sequence of ordinals is finite.*

[3] We could write $\alpha \underline{\in} \beta$; but that would be wholly non-standard.

Proof. Any infinite strictly descending sequence of ordinals $\ldots \in \alpha_3 \in \alpha_2 \in \alpha_1 \in \alpha_0$ has no $\in$-minimal member, contradicting Theorem ordinals.15. □

Proposition ordinals.21. *$\alpha \subseteq \beta \vee \beta \subseteq \alpha$, for any ordinals α, β.* sth:ordinals:basic: ordinalsaresubsets

Proof. If $\alpha \in \beta$, then $\alpha \subseteq \beta$ as β is transitive. Similarly, if $\beta \in \alpha$, then $\beta \subseteq \alpha$. And if $\alpha = \beta$, then $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$. So by Theorem ordinals.16 we are done. □

Proposition ordinals.22. *$\alpha = \beta$ iff $\alpha \cong \beta$, for any ordinals α, β.* sth:ordinals:basic: ordisoidentity

Proof. The ordinals are well-orders; so this is immediate from Trichotomy (Theorem ordinals.16) and Lemma ordinals.8. □

Problem ordinals.4. Prove that, if every member of X is an ordinal, then $\bigcup X$ is an ordinal. sth:ordinals:basic: probunionordinalsordinal

ordinals.7 Replacement

In section ordinals.7, we motivated the introduction of ordinals by suggesting that we could treat them as order-types, i.e., canonical proxies for well-orderings. In order for that to work, we would need to prove that *every well-ordering is isomorphic to some ordinal*. This would allow us to define $\mathrm{ord}(A, <)$ as the ordinal α such that $\langle A, < \rangle \cong \alpha$. sth:ordinals:replacement: sec

Unfortunately, we *cannot* prove the desired result only the Axioms we provided introduced so far. (We will see why in **??**, but for now: we can't.) We need a new thought, and here it is:

Axiom (Scheme of Replacement)**.** For any formula $\varphi(x, y)$,[4] this is an axiom: for any A, if $(\forall x \in A)\exists! y\, \varphi(x, y)$, then $\{y : (\exists x \in A)\varphi(x, y)\}$ exists.

As with Separation, this is a scheme: it yields infinitely many axioms, for each of the infinitely many different φ's. And it can equally well be (and normally is) written down thus:

For any formula $\varphi(x, y)$ which does not contain "B",[5] this is an axiom:
$\forall A[(\forall x \in A)\exists! y\, \varphi(x, y) \rightarrow \exists B \forall y(y \in B \leftrightarrow (\exists x \in A)\varphi(x, y))]$

On first encounter, however, this is quite a tangled formula. The following quick consequence of Replacement probably gives a *clearer* expression to the intuitive idea we are working with:

Corollary ordinals.23. *For any term $\tau(x)$,[6] and any set A, this set exists:*

$$\{\tau(x) : x \in A\} = \{y : (\exists x \in A) y = \tau(x)\}.$$

[4] Which may have parameters.
[5] Which may have parameters.
[6] Which may have parameters.

Proof. Since τ is a *term*, $\forall x \exists! y\, \tau(x) = y$. A fortiori, $(\forall x \in A)\exists! y\, \tau(x) = y$. So $\{y : (\exists x \in A)\tau(x) = y\}$ exists by Replacement. □

This suggests that "Replacement" is a good name for the Axiom: given a set A, you can form a new set, $\{\tau(x) : x \in A\}$, by replacing every member of A with its image under τ. Indeed, following the notation for the image of a set under a function, we might write $\tau[A]$ for $\{\tau(x) : x \in A\}$.

Crucially, however, τ is a *term*. It need not be (a name for) a *function*, in the sense of **??**, i.e., a certain set of ordered pairs. After all, if f is a function (in that sense), then the set $f[A] = \{f(x) : x \in A\}$ is just a particular subset of $\text{ran}(f)$, and that is already guaranteed to exist, just using the axioms of $\mathbf{Z}^{-}$.[7] Replacement, by contrast, is a *powerful* addition to our axioms, as we will see in **??**.

ordinals.8 ZF^{-}: a milestone

sth:ordinals:zfm: sec

The question of how to justify Replacement (if at all) is not straightforward. As such, we will reserve that for **??**. However, with the addition of Replacement, we have reached another important milestone. We now have all the axioms required for the theory $\mathbf{ZF}^{-}$. In detail:

Definition ordinals.24. The theory $\mathbf{ZF}^{-}$ has these axioms: Extensionality, Union, Pairs, Powersets, Infinity, and all instances of the Separation and Replacement schemes. Otherwise put, $\mathbf{ZF}^{-}$ adds Replacement to $\mathbf{Z}^{-}$.

This stands for *Zermelo–Fraenkel* set theory (*minus* something which we will come to later). Fraenkel gets the honour, since he is credited with the formulation of Replacement in 1922, although the first precise formulation was due to Skolem (1922).

ordinals.9 Ordinals as Order-Types

sth:ordinals:ordtype: sec

Armed with Replacement, and so now working in $\mathbf{ZF}^{-}$, we can prove what we wanted:

sth:ordinals:ordtype: thmOrdinalRepresentation

Theorem ordinals.25. *Every well-ordering is isomorphic to a unique ordinal.*

Proof. Let $\langle A, < \rangle$ be a well-order. By Proposition ordinals.22, it is isomorphic to at most one ordinal. So, for reductio, suppose $\langle A, < \rangle$ is not isomorphic to *any* ordinal. We will first "make $\langle A, < \rangle$ as small as possible". In detail: if some proper initial segment $\langle A_a, <_a \rangle$ is not isomorphic to any ordinal, there is a least $a \in A$ with that property; then let $B = A_a$ and $\lessdot = <_a$. Otherwise, let $B = A$ and $\lessdot = <$.

[7] Just consider $\{y \in \bigcup\bigcup f : (\exists x \in A) y = f(x)\}$.

By definition, every proper initial segment of B is isomorphic to some ordinal, which is unique by Proposition ordinals.22. So by Replacement, the following set exists, and is a function:

$$f = \{\langle \beta, b\rangle : b \in B \text{ and } \beta \cong \langle B_b, \lessdot_b\rangle\}$$

To complete the reductio, we'll show that f is an isomorphism $\alpha \to B$, for some ordinal α. It is obvious that $\operatorname{ran}(f) = B$. And by Lemma ordinals.10, f preserves ordering, i.e., $\gamma \in \beta$ iff $f(\gamma) \lessdot f(\beta)$.

To show that $\operatorname{dom}(f)$ is an ordinal, by Corollary ordinals.18 it suffices to show that $\operatorname{dom}(f)$ is transitive. So fix $\beta \in \operatorname{dom}(f)$, i.e., $\beta \cong \langle B_b, \lessdot_b\rangle$ for some b. If $\gamma \in \beta$, then $\gamma \in \operatorname{dom}(f)$ by Lemma ordinals.9; generalising, $\beta \subseteq \operatorname{dom}(f)$. □

This result licenses the following definition, which we have wanted to offer since section ordinals.5:

Definition ordinals.26. If $\langle A, <\rangle$ is a well-ordering, then its order type, $\operatorname{ord}(A, <)$, is the unique ordinal α such that $\langle A, <\rangle \cong \alpha$.

Moreover, this definition licenses two nice principles:

Corollary ordinals.27. *Where $\langle A, <\rangle$ and $\langle B, \lessdot\rangle$ are well-orderings:* sth:ordinals:ordtype: ordtypesworklikeyouwant

$$\begin{aligned} \operatorname{ord}(A, <) = \operatorname{ord}(B, \lessdot) &\text{ iff } \langle A, <\rangle \cong \langle B, \lessdot\rangle \\ \operatorname{ord}(A, <) \in \operatorname{ord}(B, \lessdot) &\text{ iff } \langle A, <\rangle \cong \langle B_b, \lessdot_b\rangle \text{ for some } b \in B \end{aligned}$$

Proof. The identity holds as isomorphism is an equivalence relation. To prove the second claim, let $\operatorname{ord}(A, <) = \alpha$ and $\operatorname{ord}(B, \lessdot) = \beta$, and let $f\colon \beta \to \langle B, \lessdot\rangle$ be our isomorphism. Then:

$$\begin{aligned} \alpha \in \beta &\text{ iff } f\restriction_{\alpha}\colon \alpha \to B_{f(\alpha)} \text{ is an isomorphism} \\ &\text{ iff } \langle A, <\rangle \cong \langle B_{f(\alpha)}, \lessdot_{f(\alpha)}\rangle \\ &\text{ iff } \langle A, <\rangle \cong \langle B_b, \lessdot_b\rangle \text{ for some } b \in B \end{aligned}$$

by Proposition ordinals.6, Lemma ordinals.9, and Corollary ordinals.14. □

ordinals.10 Successor and Limit Ordinals

In the next few chapters, we will use ordinals a great deal. So it will help if we introduce some simple notions. sth:ordinals:opps: sec

Definition ordinals.28. For any ordinal α, its *successor* is $\alpha^+ = \alpha \cup \{\alpha\}$. We say that α is a *successor* ordinal if $\beta^+ = \alpha$ for some ordinal α. We say that α is a *limit* ordinal iff α is neither empty nor a successor ordinal.

The following result shows that this is the right notion of *successor*:

Proposition ordinals.29. *For any ordinal α:*

1. *$\alpha \in \alpha^+$;*
2. *α^+ is an ordinal;*
3. *there is no ordinal β such that $\alpha \in \beta \in \alpha^+$.*

Proof. Trivially, $\alpha \in \alpha \cup \{\alpha\} = \alpha^+$. Equally, α^+ is a transitive set of ordinals, and hence an ordinal by Corollary ordinals.18. And it is impossible that $\alpha \in \beta \in \alpha^+$, since then either $\beta \in \alpha$ or $\beta = \alpha$, contradicting Corollary ordinals.17. □

This also licenses a variant of proof by transfinite induction:

sth:ordinals:opps:simpletransrecursion

Theorem ordinals.30 (Simple Transfinite Induction)**.** *Let $\varphi(x)$ be a formula such that:*[8]

1. *$\varphi(\emptyset)$; and*
2. *for any ordinal α, if $\varphi(\alpha)$ then $\varphi(\alpha^+)$; and*
3. *if α is a limit ordinal and $(\forall \beta \in \alpha)\varphi(\beta)$, then $\varphi(\alpha)$.*

Then $\forall \alpha \varphi(\alpha)$.

Proof. We prove the contrapositive. So, suppose there is some ordinal which is $\neg\varphi$; let γ be the least such ordinal. Then either $\gamma = \emptyset$, or $\gamma = \alpha^+$ for some α such that $\varphi(\alpha)$; or γ is a limit ordinal and $(\forall \beta \in \gamma)\varphi(\beta)$. □

A final bit of notation will prove helpful.

sth:ordinals:opps:defsupstrict

Definition ordinals.31. If X is a set of ordinals, then $\text{lsub}(X) = \bigcup_{\alpha \in X} \alpha^+$.

Here, "lsub" stands for "least strict upper bound".[9] The following result explains this:

Proposition ordinals.32. *If X is a set of ordinals,* $\text{lsub}(X)$ *is the least ordinal greater than every ordinal in X.*

Proof. Let $Y = \{\alpha^+ : \alpha \in X\}$, so that $\text{lsub}(X) = \bigcup Y$. Since ordinals are transitive and every member of an ordinal is an ordinal, $\text{lsub}(X)$ is a transitive set of ordinals, and so is an ordinal by Corollary ordinals.18.

If $\alpha \in X$, then $\alpha^+ \in Y$, so $\alpha^+ \subseteq \bigcup Y = \text{lsub}(X)$, and hence $\alpha \in \text{lsub}(X)$. So $\text{lsub}(X)$ is strictly greater than every ordinal in X.

Conversely, if $\alpha \in \text{lsub}(X)$, then $\alpha \in \beta^+ \in Y$ for some $\beta \in X$, so that $\alpha \leq \beta \in X$. So $\text{lsub}(X)$ is the *least* strict upper bound on X. □

[8] The formula may have parameters.

[9] Some books use "$\sup(X)$" for this. But other books use "$\sup(X)$" for the least *non-strict* upper bound, i.e., simply $\bigcup X$. If X has a greatest element, α, these notions come apart: the least *strict* upper bound is α^+, whereas the least *non-strict* upper bound is just α.

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Bibliography

Burali-Forti, Cesare. 1897. Una questione sui numeri transfiniti. *Rendiconti del Circolo Matematico di Palermo* 11: 154–64.

Fraenkel, Abraham. 1922. Über den Begriff 'definit' und die Unabhängigkeit des Auswahlaxioms. *Sitzungsberichte der Preussischen Akadademie der Wissenschaften, Physikalisch-mathematische Klasse* 253–257.

Heijenoort, Jean van. 1967. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*. Cambridge, MA: Harvard University Press.

Skolem, Thoralf. 1922. Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre. In *Wissenschaftliche Vorträge gehalten auf dem fünften Kongress der skandanivschen Mathematiker in Helsingfors vom 4. bis zum 7. Juli 1922*, 137–52. Akademiska Bokhandeln.