ordinals.1 Successor and Limit Ordinals

sth:ordinals:opps:sec

In the next few chapters, we will use ordinals a great deal. So it will help if we introduce some simple notions.

Definition ordinals.1. For any ordinal α, its *successor* is $\alpha^+ = \alpha \cup \{\alpha\}$. We say that α is a *successor* ordinal if $\beta^+ = \alpha$ for some ordinal α. We say that α is a *limit* ordinal iff α is neither empty nor a successor ordinal.

The following result shows that this is the right notion of *successor*:

Proposition ordinals.2. *For any ordinal α:*

1. *$\alpha \in \alpha^+$;*
2. *α^+ is an ordinal;*
3. *there is no ordinal β such that $\alpha \in \beta \in \alpha^+$.*

Proof. Trivially, $\alpha \in \alpha \cup \{\alpha\} = \alpha^+$. Equally, α^+ is a transitive set of ordinals, and hence an ordinal by **??**. And it is impossible that $\alpha \in \beta \in \alpha^+$, since then either $\beta \in \alpha$ or $\beta = \alpha$, contradicting **??**. □

This also licenses a variant of proof by transfinite induction:

sth:ordinals:opps:simpletransrecursion

Theorem ordinals.3 (Simple Transfinite Induction)**.** *Let $\varphi(x)$ be a formula such that:*[1]

1. *$\varphi(\emptyset)$; and*
2. *for any ordinal α, if $\varphi(\alpha)$ then $\varphi(\alpha^+)$; and*
3. *if α is a limit ordinal and $(\forall \beta \in \alpha)\varphi(\beta)$, then $\varphi(\alpha)$.*

Then $\forall \alpha \varphi(\alpha)$.

Proof. We prove the contrapositive. So, suppose there is some ordinal which is $\neg\varphi$; let γ be the least such ordinal. Then either $\gamma = \emptyset$, or $\gamma = \alpha^+$ for some α such that $\varphi(\alpha)$; or γ is a limit ordinal and $(\forall \beta \in \gamma)\varphi(\beta)$. □

A final bit of notation will prove helpful.

sth:ordinals:opps:defsupstrict

Definition ordinals.4. If X is a set of ordinals, then $\mathrm{lsub}(X) = \bigcup_{\alpha \in X} \alpha^+$.

Here, "lsub" stands for "least strict upper bound".[2] The following result explains this:

[1] The formula may have parameters.

[2] Some books use "$\sup(X)$" for this. But other books use "$\sup(X)$" for the least *non-strict* upper bound, i.e., simply $\bigcup X$. If X has a greatest element, α, these notions come apart: the least *strict* upper bound is α^+, whereas the least *non-strict* upper bound is just α.

Proposition ordinals.5. *If X is a set of ordinals,* $\mathrm{lsub}(X)$ *is the least ordinal greater than every ordinal in X.*

Proof. Let $Y = \{\alpha^+ : \alpha \in X\}$, so that $\mathrm{lsub}(X) = \bigcup Y$. Since ordinals are transitive and every member of an ordinal is an ordinal, $\mathrm{lsub}(X)$ is a transitive set of ordinals, and so is an ordinal by **??**.

If $\alpha \in X$, then $\alpha^+ \in Y$, so $\alpha^+ \subseteq \bigcup Y = \mathrm{lsub}(X)$, and hence $\alpha \in \mathrm{lsub}(X)$. So $\mathrm{lsub}(X)$ is strictly greater than every ordinal in X.

Conversely, if $\alpha \in \mathrm{lsub}(X)$, then $\alpha \in \beta^+ \in Y$ for some $\beta \in X$, so that $\alpha \leq \beta \in X$. So $\mathrm{lsub}(X)$ is the *least* strict upper bound on X. □

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Bibliography