In the next few chapters, we will use ordinals a great deal. So it will help if we introduce some simple notions.

**Definition ordinals.1.** For any ordinal $\alpha$, its **successor** is $\alpha^+ = \alpha \cup \{\alpha\}$. We say that $\alpha$ is a successor ordinal if $\beta^+ = \alpha$ for some ordinal $\alpha$. We say that $\alpha$ is a limit ordinal iff $\alpha$ is neither empty nor a successor ordinal.

The following result shows that this is the right notion of successor:

**Proposition ordinals.2.** For any ordinal $\alpha$:

1. $\alpha \in \alpha^+$;
2. $\alpha^+$ is an ordinal;
3. there is no ordinal $\beta$ such that $\alpha \in \beta \in \alpha^+$.

*Proof.* Trivially, $\alpha \in \alpha \cup \{\alpha\} = \alpha^+$. Equally, $\alpha^+$ is a transitive set of ordinals, and hence an ordinal by ???. And it is impossible that $\alpha \in \beta \in \alpha^+$, since then either $\beta \in \alpha$ or $\beta = \alpha$, contradicting ???.

This also licenses a variant of proof by transfinite induction:

**Theorem ordinals.3 (Simple Transfinite Induction).** Let $\varphi(x)$ be a formula such that,$^1$

1. $\varphi(\emptyset)$; and
2. for any ordinal $\alpha$, if $\varphi(\alpha)$ then $\varphi(\alpha^+)$; and
3. if $\alpha$ is a limit ordinal and $(\forall \beta \in \alpha) \varphi(\beta)$, then $\varphi(\alpha)$.

Then $\forall \alpha \varphi(\alpha)$.

*Proof.* We prove the contrapositive. So, suppose there is some ordinal which is $\neg \varphi$; let $\gamma$ be the least such ordinal. Then either $\gamma = \emptyset$, or $\gamma = \alpha^+$ for some $\alpha$ such that $\varphi(\alpha)$; or $\gamma$ is a limit ordinal and $(\forall \beta \in \gamma) \varphi(\beta)$.

A final bit of notation will prove helpful.

**Definition ordinals.4.** If $X$ is a set of ordinals, then $\text{lsub}(X) = \bigcup_{\alpha \in X} \alpha^+$.

Here, “lsub” stands for “least strict upper bound”.$^2$ The following result explains this:

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$^1$The formula may have parameters.

$^2$Some books use “$\sup(X)$” for this. But other books use “$\sup(X)$” for the least non-strict upper bound, i.e., simply $\bigcup X$. If $X$ has a greatest element, $\alpha$, these notions come apart: the least strict upper bound is $\alpha^+$, whereas the least non-strict upper bound is just $\alpha$. 

Proposition ordinals.5. If $X$ is a set of ordinals, $\text{lsub}(X)$ is the least ordinal greater than every ordinal in $X$.

Proof. Let $Y = \{\alpha^+ : \alpha \in X\}$, so that $\text{lsub}(X) = \bigcup Y$. Since ordinals are transitive and every member of an ordinal is an ordinal, $\text{lsub}(X)$ is a transitive set of ordinals, and so is an ordinal by ??.

If $\alpha \in X$, then $\alpha^+ \in Y$, so $\alpha^+ \subseteq \bigcup Y = \text{lsub}(X)$, and hence $\alpha \in \text{lsub}(X)$. So $\text{lsub}(X)$ is strictly greater than every ordinal in $X$.

Conversely, if $\alpha \in \text{lsub}(X)$, then $\alpha \in \beta^+ \in Y$ for some $\beta \in X$, so that $\alpha \leq \beta \in X$. So $\text{lsub}(X)$ is the least strict upper bound on $X$. \qed

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Bibliography