ordinals.1  Order-Isomorphisms

To explain how robust well-ordering is, we will start by introducing a method for comparing well-orderings.

Definition ordinals.1. A well-ordering is a pair $\langle A, < \rangle$, such that $<$ well-orders $A$. The well-orderings $\langle A, < \rangle$ and $\langle B, \preceq \rangle$ are order-isomorphic iff there is a bijection $f : A \to B$ such that: $x < y$ iff $f(x) \preceq f(y)$. In this case, we write $\langle A, < \rangle \cong \langle B, \preceq \rangle$, and say that $f$ is an order-isomorphism.

In what follows, for brevity, we will speak of “isomorphisms” rather than “order-isomorphisms”. Intuitively, isomorphisms are structure-preserving bijections. Here are some simple facts about isomorphisms.

Lemma ordinals.2. Compositions of isomorphisms are isomorphisms, i.e.: if $f : A \to B$ and $g : B \to C$ are isomorphisms, then $(g \circ f) : A \to C$ is an isomorphism.

Problem ordinals.1. Prove Lemma ordinals.2.

Proof. Left as an exercise.

Corollary ordinals.3. $X \cong Y$ is an equivalence relation.

Proposition ordinals.4. If $\langle A, < \rangle$ and $\langle B, \preceq \rangle$ are isomorphic well-orderings, then the isomorphism between them is unique.

Proof. Let $f$ and $g$ be isomorphisms $A \to B$. We will prove the result by induction, i.e. using $\textbf{??}$. Fix $a \in A$, and suppose (for induction) that ($\forall b < a) f(b) = g(b)$. Fix $x \in B$.

If $x \prec f(a)$, then $f^{-1}(x) < a$, so $g(f^{-1}(x)) \prec g(a)$, invoking the fact that $f$ and $g$ are isomorphisms. But since $f^{-1}(x) < a$, by our supposition $x = f(f^{-1}(x)) = g(f^{-1}(x))$. So $x \prec g(a)$. Similarly, if $x \preceq g(a)$ then $x \preceq f(a)$.

Generalising, ($\forall x \in B)(x \prec f(a) \leftrightarrow x \prec g(a))$. It follows that $f(a) = g(a)$ by $\textbf{??}$. So ($\forall a \in A)f(a) = g(a)$ by $\textbf{??}$.

This gives some sense that well-orderings are robust. But to continue explaining this, it will help to introduce some more notation.

Definition ordinals.5. When $\langle A, < \rangle$ is a well-ordering with $a \in A$, let $A_a = \{x \in A : x < a\}$. We say that $A_a$ is a proper initial segment of $A$ (and allow that $A$ itself is an improper initial segment of $A$). Let $<_a$ be the restriction of $<$ to the initial segment, i.e., $<_a |_{A_a}$.

Using this notation, we can state and prove that no well-ordering is isomorphic to any of its proper initial segments.

Lemma ordinals.6. If $\langle A, < \rangle$ is a well-ordering with $a \in A$, then $\langle A, < \rangle \not\cong \langle A_a, <_a \rangle$.
Proof. For reductio, suppose $f : A \to A_a$ is an isomorphism. Since $f$ is a bijection and $A_a \subseteq A$, using ?? let $b \in A$ be the $<$-least element of $A$ such that $b \neq f(b)$. We'll show that $(\forall x \in A)(x < b \iff x < f(b))$, from which it will follow by ?? that $b = f(b)$, completing the reductio.

Suppose $x < b$. So $x = f(x)$, by the choice of $b$. And $f(x) < f(b)$, as $f$ is an isomorphism. So $x < f(b)$.

Suppose $x < f(b)$. So $f^{-1}(x) < b$, since $f$ is an isomorphism, and so $f^{-1}(x) = x$ by the choice of $b$. So $x < b$.

Our next result shows, roughly put, that an “initial segment” of an isomorphism is an isomorphism:

**Lemma ordinals.7.** Let $\langle A, \langle \rangle \rangle$ and $\langle B, \langle \rangle \rangle$ be well-orderings. If $f : A \to B$ is an isomorphism and $a \in A$, then $f\restriction_{A_a} : A_a \to B_{f(a)}$ is an isomorphism.

Proof. Since $f$ is an isomorphism:

$$f[A_a] = f[\{x \in A : x < a\}]$$

$$= f[\{f^{-1}(y) \in A : f^{-1}(y) < a\}]$$

$$= \{y \in B : y < f(a)\}$$

$$= B_{f(a)}$$

And $f\restriction_{A_a}$ preserves order because $f$ does. \hfill \Box

Our next two results establish that well-orderings are always comparable:

**Lemma ordinals.8.** Let $\langle A, \langle \rangle \rangle$ and $\langle B, \langle \rangle \rangle$ be well-orderings. If $\langle A_{a_1}, \langle a_1 \rangle \rangle \cong \langle B_{b_1}, \langle b_1 \rangle \rangle$ and $\langle A_{a_2}, \langle a_2 \rangle \rangle \cong \langle B_{b_2}, \langle b_2 \rangle \rangle$, then $a_1 < a_2$ iff $b_1 < b_2$

Proof. We will prove left to right; the other direction is similar. Suppose both $\langle A_{a_1}, \langle a_1 \rangle \rangle \cong \langle B_{b_1}, \langle b_1 \rangle \rangle$ and $\langle A_{a_2}, \langle a_2 \rangle \rangle \cong \langle B_{b_2}, \langle b_2 \rangle \rangle$, with $f : A_{a_2} \to B_{b_2}$ our isomorphism. Let $a_1 < a_2$; then $\langle A_{a_1}, \langle a_1 \rangle \rangle \cong \langle B_{f(a_1)}, \langle f(a_1) \rangle \rangle$ by Lemma ordinals.7. So $\langle B_{b_1}, \langle b_1 \rangle \rangle \cong \langle B_{f(a_1)}, \langle f(a_1) \rangle \rangle$, and so $b_1 = f(a_1)$ by Lemma ordinals.6. Now $b_1 < b_2$ as $f$’s domain is $B_{b_2}$. \hfill \Box

**Theorem ordinals.9.** Given any two well-orderings, one is isomorphic to an initial segment (not necessarily proper) of the other.

Proof. Let $\langle A, \langle \rangle \rangle$ and $\langle B, \langle \rangle \rangle$ be well-orderings. Using Separation, let

$$f = \{\langle a, b \rangle \in A \times B : \langle a, \langle a \rangle \rangle \cong \langle B_{b}, \langle b \rangle \rangle\}.$$ 

By Lemma ordinals.8, $a_1 < a_2$ iff $b_1 < b_2$ for all $\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in f$. So $f : \text{dom}(f) \to \text{ran}(f)$ is an isomorphism.

If $a_2 \in \text{dom}(f)$ and $a_1 < a_2$, then $a_1 \in \text{dom}(f)$ by Lemma ordinals.7; so $\text{dom}(f)$ is an initial segment of $A$. Similarly, $\text{ran}(f)$ is an initial segment of $B$. For reductio, suppose both are proper initial segments. Then let $a$ be the $<$-least element of $A \setminus \text{dom}(f)$, so that $\text{dom}(f) = A_a$, and let $b$ be the $<$-least element of $B \setminus \text{ran}(f)$, so that $\text{ran}(f) = B_b$. So $f : A_a \to B_b$ is an isomorphism, and hence $\langle a, b \rangle \in f$, a contradiction. \hfill \Box