ordinals.1 Order-Isomorphisms

To explain how robust well-ordering is, we will start by introducing a method for comparing well-orderings.

Definition ordinals.1. A well-ordering is a pair \( \langle A, < \rangle \), such that \(< \) well-orders \( A \). The well-orderings \( \langle A, < \rangle \) and \( \langle B, \not< \rangle \) are order-isomorphic iff there is a bijection \( f : A \to B \) such that: \( x < y \iff f(x) \not< f(y) \). In this case, we write \( \langle A, < \rangle \cong \langle B, \not< \rangle \), and say that \( f \) is an order-isomorphism.

In what follows, for brevity, we will speak of “isomorphisms” rather than “order-isomorphisms”. Intuitively, isomorphisms are structure-preserving bijections. Here are some simple facts about isomorphisms.

Lemma ordinals.2. Compositions of isomorphisms are isomorphisms, i.e.: if \( f : A \to B \) and \( g : B \to C \) are isomorphisms, then \( (g \circ f) : A \to C \) is an isomorphism.

Problem ordinals.1. Prove Lemma ordinals.2.

Proof. Left as an exercise.

Corollary ordinals.3. \( X \cong Y \) is an equivalence relation.

Proposition ordinals.4. If \( \langle A, < \rangle \) and \( \langle B, \not< \rangle \) are isomorphic well-orderings, then the isomorphism between them is unique.

Proof. Let \( f \) and \( g \) be isomorphisms \( A \to B \). We will prove the result by induction, i.e. using ???. Fix \( a \in A \), and suppose (for induction) that \( (\forall b < a) f(b) = g(b) \). Fix \( x \in B \).

If \( x < f(a) \), then \( f^{-1}(x) < a \), so \( g(f^{-1}(x)) \not< g(a) \), invoking the fact that \( f \) and \( g \) are isomorphisms. But since \( f^{-1}(x) < a \), by our supposition \( x = f(f^{-1}(x)) = g(f^{-1}(x)) \). So \( x < g(a) \). Similarly, if \( x < g(a) \) then \( x < f(a) \).

Generalising, \( (\forall x \in B)(x < f(a) \iff x < g(a)) \). It follows that \( f(a) = g(a) \) by ???. So \( (\forall a \in A) f(a) = g(a) \) by ??.

This gives some sense that well-orderings are robust. But to continue explaining this, it will help to introduce some more notation.

Definition ordinals.5. When \( \langle A, < \rangle \) is a well-ordering with \( a \in A \), let \( A_a = \{ x \in A : x < a \} \). We say that \( A_a \) is a proper initial segment of \( A \) (and allow that \( A \) itself is an improper initial segment of \( A \)). Let \( <_a \) be the restriction of \(< \) to the initial segment, i.e., \( < \mid_{A_a} \).

Using this notation, we can state and prove that no well-ordering is isomorphic to any of its proper initial segments.

Lemma ordinals.6. If \( \langle A, < \rangle \) is a well-ordering with \( a \in A \), then \( \langle A, < \rangle \not\cong \langle A_a, <_a \rangle \).
Proof. For reductio, suppose \( f : A \to A_a \) is an isomorphism. Since \( f \) is a bijection and \( A_a \subseteq A \), using ?? let \( b \in A \) be the \(<\)-least element of \( A \) such that \( b \neq f(b) \). We'll show that \( (\forall x \in A)(x < b \iff x < f(b)) \), from which it will follow by ?? that \( b = f(b) \), completing the reductio.

Suppose \( x < b \). So \( x = f(x) \), by the choice of \( b \). And \( f(x) < f(b) \), as \( f \) is an isomorphism. So \( x < f(b) \).

Suppose \( x < f(b) \). So \( f^{-1}(x) < b \), since \( f \) is an isomorphism, and so \( f^{-1}(x) = x \) by the choice of \( b \). So \( x < b \).

Our next result shows, roughly put, that an “initial segment” of an isomorphism is an isomorphism:

**Lemma ordinals.7.** Let \( \langle A, < \rangle \) and \( \langle B, \preceq \rangle \) be well-orderings. If \( f : A \to B \) is an isomorphism and \( a \in A \), then \( f\mid_{A_a} : A_a \to B_{f(a)} \) is an isomorphism.

Proof. Since \( f \) is an isomorphism:

\[
\begin{align*}
  f[A_a] &= f\{x \in A : x < a\} \\
  &= f\{f^{-1}(y) \in A : f^{-1}(y) < a\} \\
  &= \{y \in B : y < f(a)\} \\
  &= B_{f(a)}
\end{align*}
\]

And \( f\mid_{A_a} \) preserves order because \( f \) does.

Our next two results establish that well-orderings are always comparable:

**Lemma ordinals.8.** Let \( \langle A, < \rangle \) and \( \langle B, \preceq \rangle \) be well-orderings. If \( \langle A_{a_1}, <_{a_1} \rangle \cong \langle B_{b_1}, \preceq_{b_1} \rangle \) and \( \langle A_{a_2}, <_{a_2} \rangle \cong \langle B_{b_2}, \preceq_{b_2} \rangle \), then \( a_1 < a_2 \) iff \( b_1 < b_2 \).

Proof. We will prove left to right; the other direction is similar. Suppose both \( \langle A_{a_1}, <_{a_1} \rangle \cong \langle B_{b_1}, \preceq_{b_1} \rangle \) and \( \langle A_{a_2}, <_{a_2} \rangle \cong \langle B_{b_2}, \preceq_{b_2} \rangle \), with \( f : A_{a_2} \to B_{b_2} \) our isomorphism. Let \( a_1 < a_2 \); then \( \langle A_{a_1}, <_{a_1} \rangle \cong \langle B_{f(a_1)}, \preceq_{f(a_1)} \rangle \) by Lemma ordinals.7. So \( \langle B_{b_1}, \preceq_{b_1} \rangle \cong \langle B_{f(a_1)}, \preceq_{f(a_1)} \rangle \), and so \( b_1 = f(a_1) \) by Lemma ordinals.6. Now \( b_1 < b_2 \) as \( f \)’s domain is \( B_{b_2} \).

**Theorem ordinals.9.** Given any two well-orderings, one is isomorphic to an initial segment (not necessarily proper) of the other.

Proof. Let \( \langle A, < \rangle \) and \( \langle B, \preceq \rangle \) be well-orderings. Using Separation, let

\[
  f = \{ \langle a, b \rangle \in A \times B : \langle A_{a_b}, <_{a_b} \rangle \cong \langle B_{b_b}, \preceq_{b_b} \rangle \}.
\]

By Lemma ordinals.8, \( a_1 < a_2 \) iff \( b_1 < b_2 \) for all \( \langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in f \). So \( f : \text{dom}(f) \to \text{ran}(f) \) is an isomorphism.

If \( a_2 \in \text{dom}(f) \) and \( a_1 < a_2 \), then \( a_1 \in \text{dom}(f) \) by Lemma ordinals.7; so \( \text{dom}(f) \) is an initial segment of \( A \). Similarly, \( \text{ran}(f) \) is an initial segment of \( B \). For reductio, suppose both are proper initial segments. Then let \( a \) be the \(<\)-least element of \( A \setminus \text{dom}(f) \), so that \( \text{dom}(f) = A_a \), and let \( b \) be the \(<\)-least element of \( B \setminus \text{ran}(f) \), so that \( \text{ran}(f) = B_b \). So \( f : A_a \to B_b \) is an isomorphism, and hence \( \langle a, b \rangle \in f \), a contradiction.

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