ordinals.1 Order-Isomorphisms

sth:ordinals:iso:sec To explain *how* robust well-ordering is, we will start by introducing a method for comparing well-orderings.

Definition ordinals.1. A *well-ordering* is a pair $\langle A, < \rangle$, such that $<$ well-orders A. The well-orderings $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ are *order-isomorphic* iff there is a bijection $f \colon A \to B$ such that: $x < y$ iff $f(x) \lessdot f(y)$. In this case, we write $\langle A, < \rangle \cong \langle B, \lessdot \rangle$, and say that f is an *order-isomorphism*.

In what follows, for brevity, we will speak of "isomorphisms" rather than "order-isomorphisms". Intuitively, isomorphisms are structure-preserving bijections. Here are some simple facts about isomorphisms.

sth:ordinals:iso:isoscompose **Lemma ordinals.2.** *Compositions of isomorphisms are isomorphisms, i.e.: if $f \colon A \to B$ and $g \colon B \to C$ are isomorphisms, then $(g \circ f) \colon A \to C$ are isomorphisms. (It follows that $X \cong Y$ is an equivalence relation.)*

Proof. Left as an exercise. □

Problem ordinals.1. Prove Lemma ordinals.2.

sth:ordinals:iso:ordisounique **Proposition ordinals.3.** *If $\langle A, < \rangle$ and $\langle B, \lessdot \rangle$ are isomorphic well-orderings, then the isomorphism between them is unique.*

Proof. Let f and g be isomorphisms $A \to B$. Fix $a \in A$, and suppose that $(\forall b < a) f(b) = g(b)$, and fix $x \in B$.

If $x \lessdot f(a)$, then $f^{-1}(x) < a$, so $g(f^{-1}(x) \lessdot g(a))$, invoking the fact that f and g are isomorphisms. But since $f^{-1}(x) < a$, by our supposition $x = f(f^{-1}(x)) = g(f^{-1}(x))$. So $x \lessdot g(a)$. Similarly, if $x \lessdot g(a)$ then $x \lessdot f(a)$.

Generalising, $(\forall x \in B)(x \lessdot f(a) \leftrightarrow x \lessdot g(a))$. It follows that $f(a) = g(a)$ by **??**. So $(\forall a \in A) f(a) = g(a)$ by **??**. □

This gives some sense that well-orderings are robust. But to continue explaining this, it will help to introduce some more notation.

Definition ordinals.4. When $\langle A, < \rangle$ is a well-ordering, let $A_a = \{x \in A : x < a\}$; we say that A_a is a proper *initial segment* of A. (We allow that A itself is an improper initial segment of A.) Let $<_a$ be the restriction of $<$ to the initial segment, i.e., $<\restriction_{A_a^2}$.

Using this notation, we can state and prove that no well-ordering is isomorphic to any of its proper initial segments.

sth:ordinals:iso:wellordnotinitial **Lemma ordinals.5.** *If $\langle A, < \rangle$ is a well-ordering with $a \in A$, then $\langle A, a \rangle \ncong \langle A_a, <_a \rangle$*

Proof. For reductio, suppose $f\colon A \to A_a$ is an isomorphism. Since f is a bijection and $A_a \subsetneq A$, let $b \in A$ be the $<$-least element of A such that $b \neq f(b)$. We'll show that $(\forall x \in A)(x < b \leftrightarrow x < f(b))$, from which it will follow by **??** that $b = f(b)$, completing the reductio.

Suppose $x < b$. So $x = f(x)$, by the choice of b. And $f(x) < f(b)$, as f is an isomorphism. So $x < f(b)$.

Suppose $x < f(b)$. So $f^{-1}(x) < b$, since f is an isomorphism, and so $f^{-1}(x) = x$ by the choice of b. So $x < b$. $\square$

Our next result shows, roughly put, that an "initial segment" of an isomorphism is an isomorphism:

Lemma ordinals.6. *Let $\langle A, <\rangle$ and $\langle B, \lessdot\rangle$ be well-orderings. If $f\colon A \to B$ is an isomorphism and $a \in A$, then $f{\restriction}_{A_a} : A_a \to B_{f(a)}$ is an isomorphism.* sth:ordinals:iso:wellordinitialsegment

Proof. Since f is an isomorphism:

$$\begin{aligned} f[A_a] &= f[\{x \in A : x < a\}] \\ &= f[\{f^{-1}(y) \in A : f^{-1}(y) < a\}] \\ &= \{y \in B : y \lessdot f(a)\} \\ &= B_{f(a)} \end{aligned}$$

And $f{\restriction}_{A_a}$ preserves order because f does. $\square$

Our next two results establish that well-orderings are always comparable:

Lemma ordinals.7. *Let $\langle A, <\rangle$ and $\langle B, \lessdot\rangle$ be well-orderings. If $\langle A_{a_1}, <_{a_1}\rangle \cong \langle B_{b_1}, \lessdot_{b_1}\rangle$ and $\langle A_{a_2}, <_{a_2}\rangle \cong \langle B_{b_2}, \lessdot_{b_2}\rangle$, then $a_1 < a_2$ iff $b_1 \lessdot b_2$* sth:ordinals:iso:lemordsegments

Proof. We will prove *left to right*; the other direction is similar. Suppose both $\langle A_{a_1}, <_{a_1}\rangle \cong \langle B_{b_1}, \lessdot_{b_1}\rangle$ and $\langle A_{a_2}, <_{a_2}\rangle \cong \langle B_{b_2}, \lessdot_{b_2}\rangle$, with $f\colon A_{a_2} \to B_{b_2}$ our isomorphism. Let $a_1 < a_2$; then $\langle A_{a_1}, <_{a_1}\rangle \cong \langle B_{f(a_1)}, \lessdot_{f(a_1)}\rangle$ by Lemma ordinals.6. So $\langle B_{b_1}, \lessdot_{b_1}\rangle \cong \langle B_{f(a_1)}, \lessdot_{f(a_1)}\rangle$, and so $b_1 = f(a_1)$ by Lemma ordinals.5. Now $b_1 \lessdot b_2$ as f's domain is B_{b_2}. $\square$

Theorem ordinals.8. *Given any two well-orderings, one is isomorphic to an initial segment (not necessarily proper) of the other.* sth:ordinals:iso:thm:woalwayscomparable

Proof. Let $\langle A, <\rangle$ and $\langle B, \lessdot\rangle$ be well-orderings. Using Separation, let

$$f = \{\langle a, b\rangle \in A \times B : \langle A_a, <_a\rangle \cong \langle B_b, \lessdot_b\rangle\}.$$

By Lemma ordinals.7, $a_1 < a_2$ iff $b_1 \lessdot b_2$ for all $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in f$. So $f\colon \mathrm{dom}(f) \to \mathrm{ran}(f)$ is an isomorphism.

If $a_2 \in \mathrm{dom}(f)$ and $a_1 < a_2$, then $a_1 \in \mathrm{dom}(f)$ by Lemma ordinals.6; so $\mathrm{dom}(f)$ is an initial segment of A. Similarly, $\mathrm{ran}(f)$ is an initial segment of B. For reductio, suppose both are *proper* initial segments. Then let a be the $<$-least element of $A \setminus \mathrm{dom}(f)$, so that $\mathrm{dom}(f) = A_a$, and let b be the $\lessdot$-least element of $B \setminus \mathrm{ran}(f)$, so that $\mathrm{ran}(f) = B_b$. So $f\colon A_a \to B_b$ is an isomorphism, and hence $f(a) = b$, a contradiction. $\square$

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Bibliography