ordinals.1 Basic Properties of the Ordinals

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We observed that the first few ordinals are the natural numbers. The main reason for developing a theory of ordinals is to extend the principle of induction which holds on the natural numbers. We will build up to this via a sequence of elementary results.

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Lemma ordinals.1. *Every element of an ordinal is an ordinal.*

Proof. Let α be an ordinal with $b \in \alpha$. Since α is transitive, $b \subseteq \alpha$. So $\in$ well-orders b as $\in$ well-orders α.

For transitivity, suppose $x \in c \in b$. So $c \in \alpha$ as $b \subseteq \alpha$. Again, as α is transitive, $c \subseteq \alpha$, so that $x \in \alpha$. So $x, c, b \in \alpha$. But $\in$ well-orders α, so that $\in$ is a transitive relation on α by **??**. So since $x \in c \in b$, we have $x \in b$. Generalising, $c \subseteq b$ □

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Corollary ordinals.2. $\alpha = \{\beta \in \alpha : \beta \text{ is an ordinal}\}$*, for any ordinal* α

Proof. Immediate from Lemma ordinals.1. □

The rough gist of the next two main results, Theorem ordinals.3 and Theorem ordinals.4, is that the ordinals themselves are well-ordered by membership:

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Theorem ordinals.3 (Transfinite Induction)**.** *For any formula* $\varphi(x)$*:*[1]

$$\text{if } \exists\alpha\varphi(\alpha), \text{ then } \exists\alpha(\varphi(\alpha) \land (\forall\beta \in \alpha)\neg\varphi(\beta))$$

where the displayed quantifiers are implicitly restricted to ordinals.

Proof. Suppose $\varphi(\alpha)$, for some ordinal α. If $(\forall\beta \in \alpha)\neg\varphi(\beta)$, then we are done. Otherwise, as α is an ordinal, it has some $\in$-least element which is φ, and this is an ordinal by Lemma ordinals.1. □

Note that we can equally express Theorem ordinals.3 as the scheme:

$$\text{if } \forall\alpha((\forall\beta \in \alpha)\varphi(\beta) \to \varphi(\alpha)), \text{ then } \forall\alpha\varphi(\alpha)$$

just by taking $\neg\varphi(\alpha)$ in Theorem ordinals.3 and reasoning as in **??**.

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Theorem ordinals.4 (Trichotomy)**.** $\alpha \in \beta \lor \alpha = \beta \lor \beta \in \alpha$*, for any ordinals* α *and* β*.*

Proof. The proof is by double induction, i.e., using Theorem ordinals.3 twice. Say that x is *comparable* with y iff $x \in y \lor x = y \lor y \in x$.

For induction, suppose that every ordinal in α is comparable with *every* ordinal. For further induction, suppose that α is comparable with every ordinal in β. We will show that α is comparable with β. By induction on β, it will

[1] The formula may have parameters, which need not be ordinals.

follow that α is comparable with every ordinal; and so by induction on α, *every* ordinal is comparable with *every* ordinal, as required.

It suffices to assume that $\alpha \notin \beta$ and $\beta \notin \alpha$, and show that $\alpha = \beta$.

To show that $\alpha \subseteq \beta$, fix $\gamma \in \alpha$; this is an ordinal by Lemma ordinals.1. So by the first induction hypothesis, γ is comparable with β. But if either $\gamma = \beta$ or $\beta \in \gamma$ then $\beta \in \alpha$ (invoking the fact that α is transitive if necessary), contrary to our assumption; so $\gamma \in \beta$. Generalising, $\alpha \subseteq \beta$.

Exactly similar reasoning, using the second induction hypothesis, shows that $\beta \subseteq \alpha$. So $\alpha = \beta$. □

As such, we will sometimes write $\alpha < \beta$ rather than $\alpha \in \beta$, since $\in$ is behaving as an ordering relation. There are no deep reasons for this, beyond familiarity, and because it is easier to write $\alpha \leq \beta$ than $\alpha \in \beta \lor \alpha = \beta$.[2]

Here are two quick consequences of our last results, the first of which puts our new notation into action:

Corollary ordinals.5. *If $\exists \alpha \varphi(\alpha)$, then $\exists \alpha(\varphi(\alpha) \land \forall \beta(\varphi(\beta) \rightarrow \alpha \leq \beta))$. Moreover, for any ordinals α, β, γ, both $\alpha \notin \alpha$ and $\alpha \in \beta \in \gamma \rightarrow \alpha \in \gamma$.* *sth:ordinals:basic: ordordered*

Proof. Just like **??**. □

Problem ordinals.1. Complete the "exactly similar reasoning" in the proof of Theorem ordinals.4.

Corollary ordinals.6. *A is an ordinal iff A is a transitive set of ordinals.* *sth:ordinals:basic: corordtransitiveord*

Proof. Left-to-right. By Lemma ordinals.1. *Right-to-left.* If A is a transitive set of ordinals, then $\in$ well-orders A by Theorem ordinals.3 and Theorem ordinals.4. □

But, although we have said that $\in$ well-orders the ordinals, we have to be *very cautious* about all this, thanks to the following:

Theorem ordinals.7 (Burali-Forti Paradox)**.** *There is no set of all the ordinals* *sth:ordinals:basic: buraliforti*

Proof. For reductio, suppose O is the set of all ordinals. If $\alpha \in \beta \in O$, then α is an ordinal, by Lemma ordinals.1, so $\alpha \in O$. So O is transitive, and hence O is an ordinal by Corollary ordinals.6. Hence $O \in O$, contradicting Corollary ordinals.5. □

This result is named after Burali-Forti. But, as van Heijenoort explains:

> Burali-Forti himself considered the contradiction as establishing, by *reductio ad absurdum*, the result that the natural ordering of the ordinals is just a partial ordering. (Heijenoort, 1967, p. 105)

[2] We could write $\alpha \in\!\!\!\!_{=} \beta$; but that would be wholly non-standard.

It was Cantor in 1899—in a letter to Dedekind—who first saw clearly the *contradiction* in supposing that there is a set of all the ordinals. (For further historical discussion, see Heijenoort 1967, p. 105.)

To summarise, ordinals are sets which are individually well-ordered by membership, and collectively well-ordered by membership.

Rounding this off, here are some more basic properties about the ordinals which follow from Theorem ordinals.3 and Theorem ordinals.4.

Proposition ordinals.8. *Any strictly descending sequence of ordinals is finite.*

Proof. Any infinite strictly descending sequence of ordinals $\ldots \in \alpha_3 \in \alpha_2 \in \alpha_1 \in \alpha_0$ has no $\in$-minimal member, contradicting Theorem ordinals.3. □

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Proposition ordinals.9. $\alpha \subseteq \beta \vee \beta \subseteq \alpha$, *for any ordinals* α, β.

Proof. If $\alpha \in \beta$, then $\alpha \subseteq \beta$ as β is transitive. Similarly, if $\beta \in \alpha$, then $\beta \subseteq \alpha$. And if $\alpha = \beta$, then $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$. So by Theorem ordinals.4 we are done. □

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Proposition ordinals.10. $\alpha = \beta$ *iff* $\alpha \cong \beta$, *for any ordinals* α, β.

Proof. The ordinals are well-orders; so this is immediate from Trichotomy (Theorem ordinals.4) and **??**. □

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Problem ordinals.2. Prove that, if every member of X is an ordinal, then $\bigcup X$ is an ordinal.

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Bibliography

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