ordinals.1  Basic Properties of the Ordinals

We observed that the first few ordinals are the natural numbers. The main reason for developing a theory of ordinals is to extend the principle of induction which holds on the natural numbers. We will build up to this via a sequence of elementary results.

Lemma ordinals.1. Every element of an ordinal is an ordinal.

Proof. Let $\alpha$ be an ordinal with $b \in \alpha$. Since $\alpha$ is transitive, $b \subseteq \alpha$. So $\in$ well-orders $b$ as $\in$ well-orders $\alpha$.

For transitivity, suppose $x \in c \in b$. So $c \subseteq \alpha$, so that $x \in \alpha$. So $x,c,b \in \alpha$. But $\in$ well-orders $\alpha$, so that $\in$ is a transitive relation on $\alpha$ by ??.

Generalising, $c \subseteq b$

Corollary ordinals.2. $\alpha = \{ \beta \in \alpha : \beta \text{ is an ordinal} \}$, for any ordinal $\alpha$

Proof. Immediate from Lemma ordinals.1.

The rough gist of the next two main results, Theorem ordinals.3 and Theorem ordinals.4, is that the ordinals themselves are well-ordered by membership:

Theorem ordinals.3 (Transfinite Induction). For any formula $\varphi(x)$:

if $\exists \alpha \varphi(\alpha)$, then $\exists \alpha (\varphi(\alpha) \land (\forall \beta \in \alpha) \neg \varphi(\beta))$

where the displayed quantifiers are implicitly restricted to ordinals.

Proof. Suppose $\varphi(\alpha)$, for some ordinal $\alpha$. If $(\forall \beta \in \alpha) \neg \varphi(\beta)$, then we are done. Otherwise, as $\alpha$ is an ordinal, it has some $\in$-least element which is $\varphi$, and this is an ordinal by Lemma ordinals.1.

Note that we can equally express Theorem ordinals.3 as the scheme:

if $\forall \alpha ((\forall \beta \in \alpha) \varphi(\beta) \rightarrow \varphi(\alpha))$, then $\forall \alpha \varphi(\alpha)$

just by taking $\neg \varphi(\alpha)$ in Theorem ordinals.3 and reasoning as in ??.

Theorem ordinals.4 (Trichotomy). $\alpha \in \beta \lor \alpha = \beta \lor \beta \in \alpha$, for any ordinals $\alpha$ and $\beta$.

Proof. The proof is by double induction, i.e., using Theorem ordinals.3 twice. Say that $x$ is comparable with $y$ iff $x \in y \lor x = y \lor y \in x$.

For induction, suppose that every ordinal in $\alpha$ is comparable with every ordinal. For further induction, suppose that $\alpha$ is comparable with every ordinal in $\beta$. We will show that $\alpha$ is comparable with $\beta$. By induction on $\beta$, it will

\[\text{The formula may have parameters, which need not be ordinals.}\]
follow that \( \alpha \) is comparable with every ordinal; and so by induction on \( \alpha \), every ordinal is comparable with every ordinal, as required.

It suffices to assume that \( \alpha \notin \beta \) and \( \beta \notin \alpha \), and show that \( \alpha = \beta \).

To show that \( \alpha \subseteq \beta \), fix \( \gamma \in \alpha \); this is an ordinal by Lemma ordinals.1. So by the first induction hypothesis, \( \gamma \) is comparable with \( \beta \). But if either \( \gamma = \beta \) or \( \beta \in \gamma \) then \( \beta \in \alpha \) (invoking the fact that \( \alpha \) is transitive if necessary), contrary to our assumption; so \( \gamma \in \beta \). Generalising, \( \alpha \subseteq \beta \).

Exactly similar reasoning, using the second induction hypothesis, shows that \( \beta \subseteq \alpha \). So \( \alpha = \beta \).

As such, we will sometimes write \( \alpha < \beta \) rather than \( \alpha \in \beta \), since \( \in \) is behaving as an ordering relation. There are no deep reasons for this, beyond familiarity, and because it is easier to write \( \alpha \leq \beta \) than \( \alpha \in \beta \lor \alpha = \beta \).

Here are two quick consequences of our last results, the first of which puts our new notation into action:

**Corollary ordinals.5.** If \( \exists \alpha \varphi(\alpha) \), then \( \exists \alpha (\varphi(\alpha) \land \forall \beta (\varphi(\beta) \rightarrow \alpha \leq \beta)) \). Moreover, for any ordinals \( \alpha, \beta, \gamma \), both \( \alpha \notin \alpha \) and \( \alpha \in \beta \in \gamma \rightarrow \alpha \in \gamma \).

*Proof.* Just like ??.

**Problem ordinals.1.** Complete the “exactly similar reasoning” in the proof of Theorem ordinals.4.

**Corollary ordinals.6.** \( A \) is an ordinal iff \( A \) is a transitive set of ordinals.

*Proof.* Left-to-right. By Lemma ordinals.1. Right-to-left. If \( A \) is a transitive set of ordinals, then \( \in \) well-orders \( A \) by Theorem ordinals.3 and Theorem ordinals.4.

But, although we have said that \( \in \) well-orders the ordinals, we have to be very cautious about all this, thanks to the following:

**Theorem ordinals.7 (Burali-Forti Paradox).** There is no set of all the ordinals

*Proof.* For reductio, suppose \( O \) is the set of all ordinals. If \( \alpha \in \beta \in O \), then \( \alpha \) is an ordinal, by Lemma ordinals.1, so \( \alpha \in O \). So \( O \) is transitive, and hence \( O \) is an ordinal by Corollary ordinals.6. Hence \( O \in O \), contradicting Corollary ordinals.5.

This result is named after Burali-Forti. But, as van Heijenoort explains:

Burali-Forti himself considered the contradiction as establishing, by *reductio ad absurdum*, the result that the natural ordering of the ordinals is just a partial ordering. (Heijenoort, 1967, p. 105)

\[ \text{We could write } \alpha \subseteq \beta; \text{ but that would be wholly non-standard.} \]
It was Cantor in 1899—in a letter to Dedekind—who first saw clearly the contradiction in supposing that there is a set of all the ordinals. (For further historical discussion, see Heijenoort 1967, p. 105.)

To summarise, ordinals are sets which are individually well-ordered by membership, and collectively well-ordered by membership.

Rounding this off, here are some more basic properties about the ordinals which follow from Theorem ordinals.3 and Theorem ordinals.4.

**Proposition ordinals.8.** Any strictly descending sequence of ordinals is finite.

*Proof.* Any infinite strictly descending sequence of ordinals \( \ldots \in \alpha_3 \in \alpha_2 \in \alpha_1 \in \alpha_0 \) has no \( \in \)-minimal member, contradicting Theorem ordinals.3.

**Proposition ordinals.9.** \( \alpha \subseteq \beta \lor \beta \subseteq \alpha \), for any ordinals \( \alpha, \beta \).

*Proof.* If \( \alpha \in \beta \), then \( \alpha \subseteq \beta \) as \( \beta \) is transitive. Similarly, if \( \beta \in \alpha \), then \( \beta \subseteq \alpha \). And if \( \alpha = \beta \), then \( \alpha \subseteq \beta \) and \( \beta \subseteq \alpha \). So by Theorem ordinals.4 we are done.

**Proposition ordinals.10.** \( \alpha = \beta \iff \alpha \cong \beta \), for any ordinals \( \alpha, \beta \).

*Proof.* The ordinals are well-orders; so this is immediate from Trichotomy (Theorem ordinals.4) and ??.

**Problem ordinals.2.** Prove that, if every member of \( X \) is an ordinal, then \( \bigcup X \) is an ordinal.

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**Bibliography**
