ord-arithmetic.1 Using Ordinal Addition

h:ord-arithmetic:using-addition: sec

Using addition on the ordinals, we can explicitly calculate the ranks of various sets, in the sense of **??**:

sth:ord-arithmetic:using-addition: *rankcomputation*

Lemma ord-arithmetic.1. *If* $\text{rank}(A) = \alpha$ *and* $\text{rank}(B) = \beta$, *then:*

sth:ord-arithmetic:using-addition: *exrankpow*

1. $\text{rank}(\wp(A)) = \alpha + 1$

sth:ord-arithmetic:using-addition: *exrankpair*

2. $\text{rank}(\{A, B\}) = \max(\alpha, \beta) + 1$

sth:ord-arithmetic:using-addition: *exrankcup*

3. $\text{rank}(A \cup B) = \max(\alpha, \beta)$

sth:ord-arithmetic:using-addition: *exranktuple*

4. $\text{rank}(\langle A, B\rangle) = \max(\alpha, \beta) + 2$

sth:ord-arithmetic:using-addition: *exranktimes*

5. $\text{rank}(A \times B) \leq \max(\alpha, \beta) + 2$

sth:ord-arithmetic:using-addition: *exrankunion*

6. $\text{rank}(\bigcup A) = \alpha$ *when* α *is empty or a limit;* $\text{rank}(\bigcup A) = \gamma$ *when* $\alpha = \gamma + 1$

Proof. Throughout, we invoke **??** repeatedly.

(1). If $x \subseteq A$ then $\text{rank}(x) \leq \text{rank}(A)$. So $\text{rank}(\wp(A)) \leq \alpha + 1$. Since $A \in \wp(A)$ in particular, $\text{rank}(\wp(A)) = \alpha + 1$.

(2). By **??**

(3). By **??**.

(4). By (2), twice.

(5). Note that $A \times B \subseteq \wp(\wp(A \cup B))$, and invoke (4).

(6). If $\alpha = \gamma + 1$, there is some $c \in A$ with $\text{rank}(c) = \gamma$, and no element of A has higher rank; so $\text{rank}(\bigcup A) = \gamma$. If α is a limit ordinal, then A has elements with rank arbitrarily close to (but strictly less than) α, so that $\bigcup A$ also has elements with rank arbitrarily close to (but strictly less than) α, so that $\text{rank}(\bigcup A) = \alpha$. □

We leave it as an exercise to show why (5) involves an *in*equality.

Problem ord-arithmetic.1. Produce sets A and B such that $\text{rank}(A \times B) = \max(\text{rank}(A), \text{rank}(B))$. Produce sets A and B such that $\text{rank}(A \times B) \max(\text{rank}(A), \text{rank}(B)) + 2$. Are any other ranks possible?

We are also now in a position to show that several reasonable notions of what it might mean to describe an ordinal as "finite" or "infinite" coincide:

sth:ord-arithmetic:using-addition: *ordinfinitycharacter*

Lemma ord-arithmetic.2. *For any ordinal* α, *the following are equivalent:*

sth:ord-arithmetic:using-addition: *ord:notinomega*

1. $\alpha \notin \omega$, *i.e.,* α *is not a natural number*

sth:ord-arithmetic:using-addition: *ord:omegaplus*

2. $\omega \leq \alpha$

sth:ord-arithmetic:using-addition: *ord:oneplus*

3. $1 + \alpha = \alpha$

sth:ord-arithmetic:using-addition: *ord:plusone*

4. $\alpha \approx \alpha + 1$, *i.e.,* α *and* $\alpha + 1$ *are equinumerous*

5. *α is Dedekind infinite* sth:ord-arithmetic:using-addition:ord:infinite

So we have five provably equivalent ways to understand what it takes for an ordinal to be (in)finite.

Proof. *(1)* $\Rightarrow$ *(2)*. By Trichotomy.

(2) $\Rightarrow$ *(3)*. Fix $\alpha \geq \omega$. By Transfinite Induction, there is some least ordinal γ (possibly 0) such that there is a limit ordinal β with $\alpha = \beta + \gamma$. Now:

$$1 + \alpha = 1 + (\beta + \gamma) = (1 + \beta) + \gamma = \operatorname{lsub}_{\delta < \beta}(1 + \delta) + \gamma = \beta + \gamma = \alpha.$$

(3) $\Rightarrow$ *(4)*. There is clearly a bijection $f\colon (\alpha \sqcup 1) \to (1 \sqcup \alpha)$. If $1 + \alpha = \alpha$, there is an isomorphism $g\colon (1 \sqcup \alpha) \to \alpha$. Now consider $g \circ f$.

(4) $\Rightarrow$ *(5)*. If $\alpha \approx \alpha + 1$, there is a bijection $f\colon (\alpha \sqcup 1) \to \alpha$. Define $g(\gamma) = f(\gamma, 0)$ for each $\gamma < \alpha$; this injection witnesses that α is Dedekind infinite, since $f(0, 1) \in \alpha \setminus \operatorname{ran}(g)$.

(5) $\Rightarrow$ *(1)*. This is **??**. $\square$

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Bibliography