Using Ordinal Addition

Using addition on the ordinals, we can explicitly calculate the ranks of various sets, in the sense of ??:

**Lemma ord-arithmetic.1.** If \( \text{rank}(A) = \alpha \) and \( \text{rank}(B) = \beta \), then:

1. \( \text{rank}(\wp(A)) = \alpha + 1 \)
2. \( \text{rank}(\{A, B\}) = \max(\alpha, \beta) + 1 \)
3. \( \text{rank}(A \cup B) = \max(\alpha, \beta) \)
4. \( \text{rank}(\langle A, B \rangle) = \max(\alpha, \beta) + 2 \)
5. \( \text{rank}(A \times B) \leq \max(\alpha, \beta) + 2 \)
6. \( \text{rank}(\bigcup A) = \alpha \) when \( \alpha \) is empty or a limit; \( \text{rank}(\bigcup A) = \gamma \) when \( \alpha = \gamma + 1 \)

**Proof.** Throughout, we invoke ?? repeatedly.

(1). If \( x \subseteq A \) then \( \text{rank}(x) \leq \text{rank}(A) \). So \( \text{rank}(\wp(A)) \leq \alpha + 1 \). Since \( A \in \wp(A) \) in particular, \( \text{rank}(\wp(A)) = \alpha + 1 \).

(2). By ??.

(3). By ??.

(4). By (2), twice.

(5). Note that \( A \times B \subseteq \wp(\wp(A \cup B)) \), and invoke (4).

(6). If \( \alpha = \gamma + 1 \), there is some \( c \in A \) with \( \text{rank}(c) = \gamma \), and no element of \( A \) has higher rank; so \( \text{rank}(\bigcup A) = \gamma \). If \( \alpha \) is a limit ordinal, then \( A \) has elements with rank arbitrarily close to (but strictly less than) \( \alpha \), so that \( \bigcup A \) also has elements with rank arbitrarily close to (but strictly less than) \( \alpha \), so that \( \text{rank}(\bigcup A) = \alpha \).

We leave it as an exercise to show why (5) involves an inequality.

**Problem ord-arithmetic.1.** Produce sets \( A \) and \( B \) such that \( \text{rank}(A \times B) = \max(\text{rank}(A), \text{rank}(B)) \). Produce sets \( A \) and \( B \) such that \( \text{rank}(A \times B) \max(\text{rank}(A), \text{rank}(B)) + 2 \). Are any other ranks possible?

We are also now in a position to show that several reasonable notions of what it might mean to describe an ordinal as “finite” or “infinite” coincide:

**Lemma ord-arithmetic.2.** For any ordinal \( \alpha \), the following are equivalent:

1. \( \alpha \notin \omega \), i.e., \( \alpha \) is not a natural number
2. \( \omega \leq \alpha \)
3. \( 1 + \alpha = \alpha \)
4. \( \alpha \approx \alpha + 1 \), i.e., \( \alpha \) and \( \alpha + 1 \) are equinumerous
5. \( \alpha \) is Dedekind infinite

So we have five provably equivalent ways to understand what it takes for an ordinal to be (in)finite.

Proof. \((1) \Rightarrow (2)\). By Trichotomy.

\((2) \Rightarrow (3)\). Fix \( \alpha \geq \omega \). By Transfinite Induction, there is some least ordinal \( \gamma \) (possibly 0) such that there is a limit ordinal \( \beta \) with \( \alpha = \beta + \gamma \). Now:

\[
1 + \alpha = 1 + (\beta + \gamma) = (1 + \beta) + \gamma = \sum_{\delta < \beta} (1 + \delta) + \gamma = \beta + \gamma = \alpha.
\]

\((3) \Rightarrow (4)\). There is clearly a bijection \( f : (\alpha \sqcup 1) \rightarrow (1 \sqcup \alpha) \). If \( 1 + \alpha = \alpha \), there is an isomorphism \( g : (1 \sqcup \alpha) \rightarrow \alpha \). Now consider \( g \circ f \).

\((4) \Rightarrow (5)\). If \( \alpha \approx \alpha + 1 \), there is a bijection \( f : (\alpha \sqcup 1) \rightarrow \alpha \). Define \( g(\gamma) = f(\gamma, 0) \) for each \( \gamma < \alpha \); this injection witnesses that \( \alpha \) is Dedekind infinite, since \( f(0, 1) \in \alpha \setminus \text{ran}(g) \).

\((5) \Rightarrow (1)\). This is ??.

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Bibliography