ord-arithmetic.1 Using Ordinal Addition

Using addition on the ordinals, we can explicitly calculate the ranks of various sets, in the sense of ??:

**Lemma ord-arithmetic.1.** If \( \text{rank}(A) = \alpha \) and \( \text{rank}(B) = \beta \), then:

1. \( \text{rank}(\wp(A)) = \alpha + 1 \)
2. \( \text{rank}([A, B]) = \max(\alpha, \beta) + 1 \)
3. \( \text{rank}(A \cup B) = \max(\alpha, \beta) \)
4. \( \text{rank}([A, B]) = \max(\alpha, \beta) + 2 \)
5. \( \text{rank}(A \times B) \leq \max(\alpha, \beta) + 2 \)
6. \( \text{rank}([A]) = \alpha \) when \( \alpha \) is empty or a limit; \( \text{rank}([A]) = \gamma \) when \( \alpha = \gamma + 1 \)

**Proof.** Throughout, we invoke ?? repeatedly.

(1). If \( x \subseteq A \) then \( \text{rank}(x) \leq \text{rank}(A) \). So \( \text{rank}(\wp(A)) \leq \alpha + 1 \). Since \( A \in \wp(A) \) in particular, \( \text{rank}(\wp(A)) = \alpha + 1 \).

(2). By ??.

(3). By ??.

(4). By (2), twice.

(5). Note that \( A \times B \subseteq \wp(\wp(A)) \), and invoke (4).

(6). If \( \alpha = \gamma + 1 \), there is some \( c \in A \) with \( \text{rank}(c) = \gamma \), and no element of \( A \) has higher rank; so \( \text{rank}(\bigcup A) = \gamma \). If \( \alpha \) is a limit ordinal, then \( A \) has elements with rank arbitrarily close to (but strictly less than) \( \alpha \), so that \( \bigcup A \) also has elements with rank arbitrarily close to (but strictly less than) \( \alpha \), so that \( \text{rank}(\bigcup A) = \alpha \).

We leave it as an exercise to show why (5) involves an inequality.

**Problem ord-arithmetic.1.** Produce sets \( A \) and \( B \) such that \( \text{rank}(A \times B) = \max(\text{rank}(A), \text{rank}(B)) \). Produce sets \( A \) and \( B \) such that \( \text{rank}(A \times B) \max(\text{rank}(A), \text{rank}(B)) + 2 \). Are any other ranks possible?

We are also now in a position to show that several reasonable notions of what it might mean to describe an ordinal as “finite” or “infinite” coincide:

**Lemma ord-arithmetic.2.** For any ordinal \( \alpha \), the following are equivalent:

1. \( \alpha \notin \omega \), i.e., \( \alpha \) is not a natural number
2. \( \omega \leq \alpha \)
3. \( 1 + \alpha = \alpha \)
4. \( \alpha \approx \alpha + 1 \), i.e., \( \alpha \) and \( \alpha + 1 \) are equinumerous

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5. \( \alpha \) is Dedekind infinite

So we have five provably equivalent ways to understand what it takes for an ordinal to be (in)finite.

**Proof.** \((1) \Rightarrow (2)\). By Trichotomy.

\((2) \Rightarrow (3)\). Fix \( \alpha \geq \omega \). By Transfinite Induction, there is some least ordinal \( \gamma \) (possibly 0) such that there is a limit ordinal \( \beta \) with \( \alpha = \beta + \gamma \). Now:

\[
1 + \alpha = 1 + (\beta + \gamma) = (1 + \beta) + \gamma = \sub_{\delta \prec \beta'} (1 + \delta) + \gamma = \beta + \gamma = \alpha.
\]

\((3) \Rightarrow (4)\). There is clearly a bijection \( f : (\alpha \sqcup 1) \to (1 \sqcup \alpha) \). If \( 1 + \alpha = \alpha \), there is an isomorphism \( g : (1 \sqcup \alpha) \to \alpha \). Now consider \( g \circ f \).

\((4) \Rightarrow (5)\). If \( \alpha \approx \alpha + 1 \), there is a bijection \( f : (\alpha \sqcup 1) \to \alpha \). Define \( g(\gamma) = f(\gamma, 0) \) for each \( \gamma < \alpha \); this injection witnesses that \( \alpha \) is Dedekind infinite, since \( f(0, 1) \in \alpha \setminus \text{ran}(g) \).

\((5) \Rightarrow (1)\). This is ??.

\[\square\]

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**Bibliography**