Chapter udf

Ordinal Arithmetic

ord-arithmetic.1 Introduction

sth:ord-arithmetic:intro:sec

In **??**, we developed a theory of ordinal numbers. We saw in **??** that we can think of the ordinals as a spine around which the remainder of the hierarchy is constructed. But that is not the only role for the ordinals. There is also the task of performing ordinal arithmetic.

We already gestured at this, back in **??**, when we spoke of ω, $\omega + 1$ and $\omega + \omega$. At the time, we spoke informally; the time has come to spell it out properly. However, we should mention that there is not much philosophy in this chapter; just technical developments, coupled with a (mildly) interesting observation that we can do the same thing in two different ways.

ord-arithmetic.2 Ordinal Addition

sth:ord-arithmetic:add:sec

Suppose we want to add α and β. We can simply put a copy of β immediately after a copy of α. (We need to take *copies*, since we know from **??** that either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.) The intuitive effect of this is to run through an α-sequence of stages, *and then* to run through a β-sequence. The resulting sequence will be well-ordered; so by **??** it is isomorphic to a (unique) ordinal. That ordinal can be regarded as the *sum* of α and β.

That is the intuitive idea behind ordinal addition. To define it rigorously, we start with the idea of taking *copies* of sets. The idea here is to use arbitrary tags, 0 and 1, to keep track of which object came from where:

sth:ord-arithmetic:add:defdissum

Definition ord-arithmetic.1. The *disjoint sum* of A and B is $A \sqcup B = (A \times \{0\}) \cup (B \times \{1\})$.

We next define an ordering on pairs of ordinals:

Definition ord-arithmetic.2. For any ordinals $\alpha_1, \alpha_2, \beta_1, \beta_2$, say that:

$$\begin{aligned} \langle \alpha_1, \alpha_2 \rangle \lessdot \langle \beta_1, \beta_2 \rangle \text{ iff either } & \alpha_2 \in \beta_2 \\ \text{or both } & \alpha_2 = \beta_2 \text{ and } \alpha_1 \in \beta_1 \end{aligned}$$

This is a *reverse lexicographic* ordering, since you order by the second element, then by the first. Now recall that we wanted to define $\alpha + \beta$ as the order type of a copy of α followed by a copy of β. To achieve that, we say:

Definition ord-arithmetic.3. For any ordinals α, β, their sum is $\alpha + \beta = \text{ord}(\alpha \sqcup \beta, \lhd)$.[1] sth:ord-arithmetic:add:defordplus

The following result, together with **??**, confirms that our definition is well-formed:

Lemma ord-arithmetic.4. *$\langle \alpha \sqcup \beta, \lhd \rangle$ is a well-order, for any ordinals α and β.* *sth:ord-arithmetic:add:ordsumlessiswo*

Proof. Obviously $\lhd$ is connected on $\alpha \sqcup \beta$. To show it is well-founded, fix a non-empty $X \subseteq \alpha \sqcup \beta$, and let

$$X_0 = \{\langle a, b\rangle \in X : (\forall \langle x, y\rangle \in X) b \leq y\}.$$

Now choose the element of X_0 with smallest first coordinate. □

So we have a lovely, explicit definition of ordinal addition. Here is an unsurprising fact (recall that $1 = \{0\}$, by **??**):

Proposition ord-arithmetic.5. *$\alpha + 1 = \alpha^+$, for any ordinal α.*

Proof. Consider the isomorphism f from $\alpha^+ = \alpha \cup \{\alpha\}$ to $\alpha \sqcup 1 = (\alpha \times \{0\}) \sqcup (\{0\} \times \{1\})$ given by $f(\gamma) = \langle \gamma, 0\rangle$ for $\gamma \in \alpha$, and $f(\alpha) = \langle 0, 1\rangle$. □

Moreover, it is easy to show that addition obeys certain recursive conditions:

Lemma ord-arithmetic.6. *For any ordinals α, β, we have:* *sth:ord-arithmetic:add:ordadditionrecursion*

$$\begin{aligned} \alpha + 0 &= \alpha \\ \alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\ \alpha + \beta &= \operatorname*{lsub}_{\delta < \beta}(\alpha + \delta) && \textit{if } \beta \textit{ is a limit ordinal} \end{aligned}$$

Proof. We check case-by-case; first:

$$\begin{aligned} \alpha + 0 &= \text{ord}((\alpha \times \{0\}) \sqcup (0 \times \{1\}), \lhd) \\ &= \text{ord}((\alpha \times \{0\}) \times \{0\}, \lhd) \\ &= \alpha \\ \alpha + (\beta + 1) &= \text{ord}((\alpha \times \{0\}) \cup (\beta^+ \times \{1\}), \lhd) \\ &= \text{ord}((\alpha \times \{0\}) \cup (\beta \times \{1\}), \lhd) + 1 \\ &= (\alpha + \beta) + 1 \end{aligned}$$

[1] This is a slight abuse of notation; strictly we should write "$\{\langle x, y\rangle \in \alpha \sqcup \beta : x \lhd y\}$" in place of "$\lhd$".

Now let $\beta \neq \emptyset$ be a limit. If $\delta < \beta$ then also $\delta + 1 < \beta$, so $\alpha + \delta$ is a proper initial segment of $\alpha + \beta$. So $\alpha + \beta$ is a strict upper bound on $X = \{\alpha + \delta : \delta < \beta\}$. Moreover, if $\alpha \leq \gamma < \alpha + \beta$, then clearly $\gamma = \alpha + \delta$ for some $\delta < \beta$. So $\alpha + \beta = \text{lsub}_{\delta < \beta}(\alpha + \delta)$. □

But here is a striking fact. To define ordinal addition, we could *instead* have simply used the Transfinite Recursion Theorem, and laid down the recursion equations, exactly as given in Lemma ord-arithmetic.6 (though using "β^+" rather than "$\beta + 1$").

There are, then, two different ways to define operations on the ordinals. We can define them *synthetically*, by explicitly constructing a well-ordered set and considering its order type. Or we can define them *recursively*, just by laying down the recursion equations. Done correctly, though, the outcome is identical. For **??** guarantees that these recursion equations pin down *unique* ordinals.

In many ways, ordinal arithmetic behaves just like addition of the natural numbers. For example, we can prove the following:

sth:ord-arithmetic:add:ordinaladditionisnice

Lemma ord-arithmetic.7. *If α, β, γ are ordinals, then:*

sth:ord-arithmetic:add:ordaddition1
1. *if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$*

sth:ord-arithmetic:add:ordaddition2
2. *if $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$*

sth:ord-arithmetic:add:ordaddition3
3. *$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$, i.e., addition is associative*

sth:ord-arithmetic:add:ordaddition4
4. *If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$*

Proof. We prove (3), leaving the rest as an exercise. The proof is by Simple Transfinite Induction on γ, using Lemma ord-arithmetic.6. When $\gamma = 0$:

$$(\alpha + \beta) + 0 = \alpha + \beta = \alpha + (\beta + 0)$$

When $\gamma = \delta + 1$, suppose for induction that $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$; now:

$$\begin{aligned}
(\alpha + \beta) + (\delta + 1) &= ((\alpha + \beta) + \delta) + 1 \\
&= (\alpha + (\beta + \delta)) + 1 \\
&= \alpha + ((\beta + \delta) + 1) \\
&= \alpha + (\beta + (\delta + 1))
\end{aligned}$$

When γ is a limit ordinal, suppose for induction that if $\delta \in \gamma$ then $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$; now:

$$\begin{aligned}
(\alpha + \beta) + \gamma &= \operatorname*{lsub}_{\delta < \gamma}((\alpha + \beta) + \delta) \\
&= \operatorname*{lsub}_{\delta < \gamma}(\alpha + (\beta + \delta)) \\
&= \alpha + \operatorname*{lsub}_{\delta < \gamma}(\beta + \delta) \\
&= \alpha + (\beta + \gamma)
\end{aligned}$$

□

In these ways, ordinal addition should be very familiar.

Problem ord-arithmetic.1. Prove the remainder of Lemma ord-arithmetic.7.

But, there is a crucial way in which ordinal addition is *not* like addition on the natural numbers.

Proposition ord-arithmetic.8. *Ordinal addition is not commutative;* $1 + \omega = \omega < \omega + 1$. sth:ord-arithmetic:add:ordsumnotcommute

Proof. Note that $1 + \omega = \text{lsub}_{n<\omega}(1 + n) = \omega \in \omega \cup \{\omega\} = \omega^+ = \omega + 1$. □

Whilst this may initially come as a surprise, *it shouldn't*. On the one hand, when you consider $1 + \omega$, you are thinking about the order type you get by putting an extra element *before* all the natural numbers. Reasoning as we did with Hilbert's Hotel in **??**, intuitively, this extra first element shouldn't make any difference to the overall order type. On the other hand, when you consider $\omega + 1$, you are thinking about the order type you get by putting an extra element *after* all the natural numbers. And that's a radically different beast!

ord-arithmetic.3 Using Ordinal Addition

Using addition on the ordinals, we can explicitly calculate the ranks of various sets, in the sense of **??**: sth:ord-arithmetic:using-addition:sec

Lemma ord-arithmetic.9. *If* $\text{rank}(A) = \alpha$ *and* $\text{rank}(B) = \beta$*, then:* sth:ord-arithmetic:using-addition:rankcomputation

1. $\text{rank}(\wp(A)) = \alpha + 1$ sth:ord-arithmetic:using-addition:exrankpow
2. $\text{rank}(\{A, B\}) = \max(\alpha, \beta) + 1$ sth:ord-arithmetic:using-addition:exrankpair
3. $\text{rank}(A \cup B) = \max(\alpha, \beta)$ sth:ord-arithmetic:using-addition:exrankcup
4. $\text{rank}(\langle A, B\rangle) = \max(\alpha, \beta) + 2$ sth:ord-arithmetic:using-addition:exranktuple
5. $\text{rank}(A \times B) \leq \max(\alpha, \beta) + 2$ sth:ord-arithmetic:using-addition:exranktimes
6. $\text{rank}(\bigcup A) = \alpha$ *when* α *is empty or a limit;* $\text{rank}(\bigcup A) = \gamma$ *when* $\alpha = \gamma + 1$ sth:ord-arithmetic:using-addition:exrankunion

Proof. Throughout, we invoke **??** repeatedly.

(1). If $x \subseteq A$ then $\text{rank}(x) \leq \text{rank}(A)$. So $\text{rank}(\wp(A)) \leq \alpha + 1$. Since $A \in \wp(A)$ in particular, $\text{rank}(\wp(A)) = \alpha + 1$.

(2). By **??**

(3). By **??**.

(4). By (2), twice.

(5). Note that $A \times B \subseteq \wp(\wp(A \cup B))$, and invoke (4).

(6). If $\alpha = \gamma + 1$, there is some $c \in A$ with $\text{rank}(c) = \gamma$, and no element of A has higher rank; so $\text{rank}(\bigcup A) = \gamma$. If α is a limit ordinal, then A has elements with rank arbitrarily close to (but strictly less than) α, so that $\bigcup A$

also has elements with rank arbitrarily close to (but strictly less than) α, so that $\text{rank}(\bigcup A) = \alpha$. □

We leave it as an exercise to show why (5) involves an *in*equality.

Problem ord-arithmetic.2. Produce sets A and B such that $\text{rank}(A \times B) = \max(\text{rank}(()A), \text{rank}(B))$. Produce sets A and B such that $\text{rank}(A \times B) \max(\text{rank}(()A), \text{rank}(B)) + 2$. Are any other ranks possible?

We are also now in a position to prove that several reasonable notions of "finite" coincide, when considering ordinals:

sth:ord-arithmetic:using-addition: ordinfinitycharacter

Lemma ord-arithmetic.10. *For any ordinal α, the following are equivalent:*

sth:ord-arithmetic:using-addition: ord:notinomega

1. $\alpha \notin \omega$, *i.e., α is not a natural number*

sth:ord-arithmetic:using-addition: ord:omegaplus

2. $\omega \leq \alpha$

sth:ord-arithmetic:using-addition: ord:oneplus

3. $1 + \alpha = \alpha$

sth:ord-arithmetic:using-addition: ord:plusone

4. $\alpha \approx \alpha + 1$, *i.e., α and $\alpha + 1$ are equinumerous*

sth:ord-arithmetic:using-addition: ord:infinite

5. *α is Dedekind infinite*

Proof. *(1)* ⇒ *(2)*. By Trichotomy.

(2) ⇒ *(3)*. Fix $\alpha \geq \omega$. By Transfinite Induction, there is some least ordinal γ (possibly 0) such that there is a limit ordinal β with $\alpha = \beta + \gamma$. Now:

$$1 + \alpha = 1 + (\beta + \gamma) = (1 + \beta) + \gamma = \underset{1+\delta}{\text{lsub}}(\delta < \beta) + \gamma = \beta + \gamma = \alpha.$$

(3) ⇒ *(4)*. There is clearly a bijection $f \colon (\alpha \sqcup 1) \to (1 \sqcup \alpha)$. If $1 + \alpha = \alpha$, there is an isomorphism $g \colon (1 \sqcup \alpha) \to \alpha$. Now consider $g \circ f$.

(4) ⇒ *(5)*. If $\alpha \approx \alpha + 1$, there is a bijection $f \colon (\alpha \sqcup 1) \to \alpha$. Define $g(\gamma) = f(\gamma, 0)$ for each $\gamma < \alpha$; this injection witnesses that α is Dedekind infinite, since $f(0, 1) \in \alpha \setminus \text{ran}(g)$.

(5) ⇒ *(1)*. This is **??**. □

ord-arithmetic.4 Ordinal Multiplication

sth:ord-arithmetic:mult: sec

We now turn to ordinal multiplication, and we approach this much like ordinal addition. So, suppose we want to multiply α by β. To do this, you might imagine a rectangular grid, with width α and height β; the product of α and β is now the result of moving along each row, then moving through the next row... until you have moved through the entire grid. Otherwise put, the product of α and β arises by replacing *each* element in β with a copy of α.

To make this formal, we simply use the reverse lexicographic ordering on the Cartesian product of α and β:

Definition ord-arithmetic.11. For any ordinals α, β, their product $\alpha \cdot \beta = \text{ord}(\alpha \times \beta, \lhd)$.

We must again confirm that this is a well-formed definition:

Lemma ord-arithmetic.12. *$\langle \alpha \times \beta, \lhd \rangle$ is a well-order, for any ordinals α and β.* sth:ord-arithmetic:mult:ordtimeslessiswo

Proof. Exactly as for Lemma ord-arithmetic.4. □

And it is also not hard to prove that multiplication behaves thus:

Lemma ord-arithmetic.13. *For any ordinals α, β:* sth:ord-arithmetic:mult:ordtimesrecursion

$$\begin{aligned} \alpha \cdot 0 &= 0 \\ \alpha \cdot (\beta + 1) &= (\alpha \cdot \beta) + \alpha \\ \alpha \cdot \beta &= \operatorname*{lsub}_{\delta < \beta} (\alpha \cdot \delta) && \text{when } \beta \text{ is a limit ordinal.} \end{aligned}$$

Proof. Left as an exercise. □

Indeed, just as in the case of addition, we could have defined ordinal multiplication via these recursion equations, rather than offering a direct definition. Equally, as with addition, certain behaviour is familiar:

Lemma ord-arithmetic.14. *If α, β, γ are ordinals, then:* sth:ord-arithmetic:mult:ordinalmultiplicationisnice

1. *if $\alpha \neq 0$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$;* sth:ord-arithmetic:mult:ordtimes1
2. *if $\alpha \neq 0$ and $\alpha \cdot \beta = \alpha \cdot \gamma$, then $\beta = \gamma$;* sth:ord-arithmetic:mult:ordtimes2
3. *$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$;* sth:ord-arithmetic:mult:ordtimes3
4. *If $\alpha \leq \beta$, then $\alpha \cdot \gamma \leq \beta \cdot \gamma$;* sth:ord-arithmetic:mult:ordtimes4
5. *$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.* sth:ord-arithmetic:mult:ordtimes5

Proof. Left as an exercise. □

You can prove (or look up) other results, to your heart's content. But, given Proposition ord-arithmetic.8, the following should not come as a surprise:

Proposition ord-arithmetic.15. *Ordinal multiplication is not commutative: $2 \cdot \omega = \omega < \omega \cdot 2$*

Proof. $2 \cdot \omega = \operatorname{lsub}_{n<\omega}(2 \cdot n) = \omega \in \operatorname{lsub}_{n<\omega}(\omega + n) = \omega + \omega = \omega \cdot 2$. □

Again, the intuitive rationale is quite straightforward. To compute $2 \cdot \omega$, you replace each natural number with two entities. You would get the same order type if you simply inserted all the "half" numbers into the natural numbers, i.e., you considered the natural ordering on $\{n/2 : n \in \omega\}$. And, put like that, the order type is plainly the same as that of ω itself. But, to compute $\omega \cdot 2$, you place down two copies of ω, one after the other.

Problem ord-arithmetic.3. Prove Lemma ord-arithmetic.12, Lemma ord-arithmetic.13, and Lemma ord-arithmetic.14

ord-arithmetic.5 Ordinal Exponentiation

sth:ord-arithmetic:expo:sec We now move to ordinal exponentiation. Sadly, there is no *nice* synthetic definition for ordinal exponentiation.

Sure, there *are* explicit synthetic definitions. Here is one. Let $\mathrm{finfun}(\alpha, \beta)$ be the set of all functions $f\colon \alpha \to \beta$ such that $\{\gamma \in \alpha : f(\gamma) \neq 0\}$ is equinumerous with some natural number. Define a well-ordering on $\mathrm{finfun}(\alpha, \beta)$ by $f \sqsubset g$ iff $f \neq g$ and $f(\gamma_0) < g(\gamma_0)$, where $\gamma_0 = \max\{\gamma \in \alpha : f(\gamma) \neq g(\gamma)\}$. Then we can define $\alpha^{(\beta)}$ as $\mathrm{ord}(\mathrm{finfun}(\alpha, \beta), \sqsubset)$. Potter employs this explicit definition, and then immediately explains:

> The choice of this ordering is determined purely by our desire to obtain a definition of ordinal exponentiation which obeys the appropriate recursive condition…, and it is much harder to picture than either the ordered sum or the ordered product. (Potter, 2004, p. 199)

Quite. We explained addition as "a copy of α followed by a copy of β", and multiplication as "a β-sequence of copies of α". But we have nothing pithy to say about $\mathrm{finfun}(\alpha, \gamma)$. So instead, we'll offer the definition of ordinal exponentiation just *by* transfinite recursion, i.e.:

sth:ord-arithmetic:expo:ordexporecursion **Definition ord-arithmetic.16.**

$$\begin{aligned} \alpha^{(0)} &= 1 \\ \alpha^{(\beta+1)} &= \alpha^{(\beta)} \cdot \alpha \\ \alpha^{(\beta)} &= \bigcup_{\delta<\beta} \alpha^{(\delta)} && \text{when } \beta \text{ is a limit ordinal} \end{aligned}$$

If we were working *as* set theorists, we might want to explore some of the properties of ordinal exponentiation. But we have nothing much more to add, except to note the unsurprising fact that ordinal exponentiation does not commute. Thus $2^{(\omega)} = \bigcup_{\delta<\omega} 2^{(\delta)} = \omega$, whereas $\omega^{(2)} = \omega \cdot \omega$. But then, we should not *expect* exponentiation to commute, since it does not commute with natural numbers: $2^{(3)} = 8 < 9 = 3^{(2)}$.

Problem ord-arithmetic.4. Using Transfinite Induction, prove that, if we define $\alpha^{(\beta)} = \mathrm{ord}(\mathrm{finfun}(\alpha, \beta), \sqsubset)$, we obtain the recursion equations of Definition ord-arithmetic.16.

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Bibliography

Potter, Michael. 2004. *Set Theory and its Philosophy*. Oxford: Oxford University Press.