

ord-arithmetic.1 Ordinal Multiplication

sth:ord-arithmetic:mult:
sec

We now turn to ordinal multiplication, and we approach this much like ordinal addition. So, suppose we want to multiply α by β . To do this, you might imagine a rectangular grid, with width α and height β ; the product of α and β is now the result of moving along each row, then moving through the next row... until you have moved through the entire grid. Otherwise put, the product of α and β arises by replacing *each* element in β with a copy of α .

To make this formal, we simply use the reverse lexicographic ordering on the Cartesian product of α and β :

Definition ord-arithmetic.1. For any ordinals α, β , their product $\alpha \cdot \beta = \text{ord}(\alpha \times \beta, \triangleleft)$.

We must again confirm that this is a well-formed definition:

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ordtimeslessiswo

Lemma ord-arithmetic.2. $\langle \alpha \times \beta, \triangleleft \rangle$ is a well-order, for any ordinals α and β .

Proof. Exactly as for ??.

□

And it is also not hard to prove that multiplication behaves thus:

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ordtimesrecursion

Lemma ord-arithmetic.3. For any ordinals α, β :

$$\begin{aligned} \alpha \cdot 0 &= 0 \\ \alpha \cdot (\beta + 1) &= (\alpha \cdot \beta) + \alpha \\ \alpha \cdot \beta &= \text{lsub}_{\delta < \beta}(\alpha \cdot \delta) \quad \text{when } \beta \text{ is a limit ordinal.} \end{aligned}$$

Proof. Left as an exercise.

□

Indeed, just as in the case of addition, we could have defined ordinal multiplication via these recursion equations, rather than offering a direct definition. Equally, as with addition, certain behaviour is familiar:

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ordinalmultiplicationisnice

Lemma ord-arithmetic.4. If α, β, γ are ordinals, then:

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ordtimes1

1. if $\alpha \neq 0$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$;

sth:ord-arithmetic:mult:
ordtimes2

2. if $\alpha \neq 0$ and $\alpha \cdot \beta = \alpha \cdot \gamma$, then $\beta = \gamma$;

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ordtimes3

3. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$;

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ordtimes4

4. If $\alpha \leq \beta$, then $\alpha \cdot \gamma \leq \beta \cdot \gamma$;

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ordtimes5

5. $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.

Proof. Left as an exercise.

□

You can prove (or look up) other results, to your heart's content. But, given ??, the following should not come as a surprise:

Proposition ord-arithmetic.5. *Ordinal multiplication is not commutative:*
 $2 \cdot \omega = \omega < \omega \cdot 2$

Proof. $2 \cdot \omega = \text{lsub}_{n < \omega}(2 \cdot n) = \omega \in \text{lsub}_{n < \omega}(\omega + n) = \omega + \omega = \omega \cdot 2.$ \square

Again, the intuitive rationale is quite straightforward. To compute $2 \cdot \omega$, you replace each natural number with two entities. You would get the same order type if you simply inserted all the “half” numbers into the natural numbers, i.e., you considered the natural ordering on $\{n/2 : n \in \omega\}$. And, put like that, the order type is plainly the same as that of ω itself. But, to compute $\omega \cdot 2$, you place down two copies of ω , one after the other.

Problem ord-arithmetic.1. Prove [Lemma ord-arithmetic.2](#), [Lemma ord-arithmetic.3](#), and [Lemma ord-arithmetic.4](#)

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Bibliography