choice.1 The Well-Ordering Problem

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Evidently rather a lot hangs on whether we accept Well-Ordering. But the discussion of this principle has tended to focus on an equivalent principle, the Axiom of Choice. So we will now turn our attention to that (and prove the equivalence).

In 1883, Cantor expressed his support for the Axiom of Well-Ordering, calling it "a law of thought which appears to me to be fundamental, rich in its consequences, and particularly remarkable for its general validity" (cited in Potter 2004, p. 243). But Cantor ultimately became convinced that the "Axiom" was in need of proof. So did the mathematical community.

The problem was "solved" by Zermelo in 1904. To explain his solution, we need some definitions.

Definition choice.1. A function f is a *choice function* iff $f(x) \in x$ for all $x \in \text{dom}(f)$. We say that f is a *choice function for A* iff f is a choice function with $\text{dom}(f) = A \setminus \{\emptyset\}$.

Intuitively, for every (non-empty) set $x \in A$, a choice function for A *chooses* a particular element, $f(x)$, from x. The Axiom of Choice is then:

Axiom (Choice)**.** Every set has a choice function.

Zermelo showed that Choice entails well-ordering, and vice versa:

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Theorem choice.2 (in **ZF**)**.** *Well-Ordering and Choice are equivalent.*

Proof. Left-to-right. Let A be a set of sets. Then $\bigcup A$ exists by the Axiom of Union, and so by Well-Ordering there is some $<$ which well-orders $\bigcup A$. Now let $f(x) =$ the $<$-least member of x. This is a choice function for A.

Right-to-left. Fix A. By Choice, there is a choice function, f, for $\wp(A) \setminus \{\emptyset\}$. Using Transfinite Recursion, define:

$$g(0) = f(A)$$
$$g(\alpha) = \begin{cases} \text{stop!} & \text{if } A = g[\alpha] \\ f(A \setminus g[\alpha]) & \text{otherwise} \end{cases}$$

(The indication to "stop!" is just a shorthand for what would otherwise be a more long-winded definition. That is, when $A = g[\alpha]$ for the first time, let $g(\delta) = A$ for all $\delta \leq \alpha$.) Note that we do stop, since otherwise we would have that $\alpha \prec \wp(A) \setminus \{\emptyset\}$ for every ordinal α, contradicting **??**.

We do not stop until we have exhausted A. Since we stop, $\text{ran}(g) = A$.

Since f is a choice function, for each α we have $g(\alpha) = f(A \setminus g[\alpha]) \in A \setminus g[\alpha]$; i.e., $g(\alpha) \notin g[\alpha]$. So if $g(\alpha) = g(\beta)$ then $g(\beta) \notin g[\alpha]$, i.e., $\beta \notin \alpha$, and similarly $\alpha \notin \beta$. So $\alpha = \beta$, by Trichotomy. So g is an injection.

Assembling these two facts, g is a bijection from some ordinal to A. Now g can be used to well-order A. □

So Well-Ordering and Choice stand or fall together. But the question remains: do they stand or fall?

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