

## choice.1 Intrinsic Considerations about Choice

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The broader question, then, is whether Well-Ordering, or Choice, or indeed the comparability of all sets as regards their size—it doesn't matter which—can be justified.

Here is an attempted *intrinsic* justification. Back in ??, we introduced several principles about the hierarchy. One of these is worth restating:

*Stages-accumulate.* For any stage  $S$ , and for any sets which were formed before stage  $S$ : a set is formed at stage  $S$  whose members are exactly those sets. Nothing else is formed at stage  $S$ .

In fact, many authors have suggested that the Axiom of Choice can be justified via (something like) this principle. We will briefly provide a gloss on that approach.

We will start with a simple little result, which offers *yet another* equivalent for Choice:

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**Theorem choice.1** (in ZF). *Choice is equivalent to the following principle. If the elements of  $A$  are disjoint and non-empty, then there is some  $C$  such that  $C \cap x$  is a singleton for every  $x \in A$ . (We call such a  $C$  a choice set for  $A$ .)*

The proof of this result is straightforward, and we leave it as an exercise for the reader.

**Problem choice.1.** Prove **Theorem choice.1**. If you struggle, you can find a proof in (Potter, 2004, pp. 242–3).

The essential point is that a choice set for  $A$  is just the range of a choice function for  $A$ . So, to justify Choice, we can simply try to justify its equivalent formulation, in terms of the existence of choice sets. And we will now try to do exactly that.

Let  $A$ 's elements be disjoint and non-empty. By *Stages-are-key* (see ??),  $A$  is formed at some stage  $S$ . Note that all the elements of  $\bigcup A$  are available before stage  $S$ . Now, by *Stages-accumulate*, for *any* sets which were formed before  $S$ , a set is formed whose members are exactly those sets. Otherwise put: every *possible* collection of earlier-available sets will exist at  $S$ . But it is certainly *possible* to select objects which could be formed into a choice set for  $A$ ; that is just some very specific subset of  $\bigcup A$ . So: some such choice set exists, as required.

Well, that's a *very* quick attempt to offer a justification of Choice on intrinsic grounds. But, to pursue this idea further, you should read Potter's (2004, §14.8) neat development of it.

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## Bibliography

Potter, Michael. 2004. *Set Theory and its Philosophy*. Oxford: Oxford University Press.