choice.1 Intrinsic Considerations about Choice

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The broader question, then, is whether Well-Ordering, or Choice, or indeed the comparability of all sets as regards their size—it doesn't matter which—can be justified.

Here is an attempted *intrinsic* justification. Back in **??**, we introduced several principles about the hierarchy. One of these is worth restating:

> *Stages-accumulate.* For any stage S, and for any sets which were formed *before* stage S: a set is formed at stage S whose members are exactly those sets. Nothing else is formed at stage S.

In fact, many authors have suggested that the Axiom of Choice can be justified via (something like) this principle. We will briefly provide a gloss on that approach.

We will start with a simple little result, which offers *yet another* equivalent for Choice:

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Theorem choice.1 (in ZF). *Choice is equivalent to the following principle. If the elements of A are disjoint and non-empty, then there is some C such that $C \cap x$ is a singleton for every $x \in A$. (We call such a C a choice set for A.)*

The proof of this result is straightforward, and we leave it as an exercise for the reader.

Problem choice.1. Prove Theorem choice.1. If you struggle, you can find a proof in (Potter, 2004, pp. 242–3).

The essential point is that a choice set for A is just the range of a choice function for A. So, to justify Choice, we can simply try to justify its equivalent formulation, in terms of the existence of choice sets. And we will now try to do exactly that.

Let A's elements be disjoint and non-empty. By *Stages-are-key* (see **??**), A is formed at some stage S. Note that all the elements of $\bigcup A$ are available before stage S. Now, by *Stages-accumulate*, for *any* sets which were formed before S, a set is formed whose members are exactly those sets. Otherwise put: every *possible* collections of earlier-available sets will exist at S. But it is certainly *possible* to select objects which could be formed into a choice set for A; that is just some very specific subset of $\bigcup A$. So: some such choice set exists, as required.

Well, that's a *very* quick attempt to offer a justification of Choice on intrinsic grounds. But, to pursue this idea further, you should read Potter's (2004, §14.8) neat development of it.

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Bibliography

Potter, Michael. 2004. *Set Theory and its Philosophy*. Oxford: Oxford University Press.