choice.1 Comparability and Hartogs' Lemma

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That's the plus side. Here's the minus side. Without Choice, things get *messy*. To see why, here is a nice result due to Hartogs (1915):

sth:choice:hartogs:HartogsLemma

Lemma choice.1 (*in* ZF). *For any set A, there is an ordinal α such that $\alpha \not\preceq A$*

Proof. If $B \subseteq A$ and $R \subseteq B^2$, then $\langle B, R\rangle \subseteq V_{\text{rank}(A)+4}$ by **??**. So, using Separation, consider:

$$C = \{\langle B, R\rangle \in V_{\text{rank}(A)+5} : B \subseteq A \text{ and } \langle B, R\rangle \text{ is a well-ordering}\}$$

Using Replacement and **??**, form the set:

$$\alpha = \{\text{ord}(B, R) : \langle B, R\rangle \in C\}.$$

By **??**, α is an ordinal, since it is a transitive set of ordinals. After all, if $\gamma \in \beta \in \alpha$, then $\beta = \text{ord}(B, R)$ for some $B \subseteq R$, whereupon $\gamma = \text{ord}(B_b, R_b)$ for some $b \in B$ by **??**, so that $\gamma \in \alpha$.

For reductio, suppose there is an injection $f\colon \alpha \to A$. Then, where:

$$\begin{aligned} B &= \text{ran}(f) \\ R &= \{\langle f(\alpha), f(\beta)\rangle \in A \times A : \alpha \in \beta\}. \end{aligned}$$

Clearly $\alpha = \text{ord}(B, R)$ and $\langle B, R\rangle \in C$. So $\alpha \in \alpha$, which is a contradiction. □

This entails a deep result:

Theorem choice.2 (*in* ZF). *The following claims are equivalent:*

sth:choice:hartogs:equivwo

1. *The Axiom of Well-Ordering*

sth:choice:hartogs:equivcompare

2. *Either $A \preceq B$ or $B \preceq A$, for any sets A and B*

Proof. *(1) ⇒ (2).* Fix A and B. Invoking (1), there are well-orderings $\langle A, R\rangle$ and $\langle B, S\rangle$. Invoking **??**, let $f\colon \alpha \to \langle A, R\rangle$ and $g\colon \beta \to \langle B, S\rangle$ be isomorphisms. By **??**, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. If $\alpha \subseteq \beta$, then $g \circ f^{-1}\colon A \to B$ is an injection, and hence $A \preceq B$; similarly, if $\beta \subseteq \alpha$ then $B \preceq A$.

(2) ⇒ (1). Fix A; by Lemma choice.1 there is some ordinal β such that $\beta \not\preceq A$. Invoking (2), we have $A \preceq \beta$. So there is some injection $f\colon A \to \beta$, and we can use this injection to well-order the elements of A, by defining an order $\{\langle a, b\rangle \in A \times A : f(a) \in f(b)\}$. □

As an immediate consequence: if Well-Ordering fails, then some sets are *literally incomparable* with regard to their size. So, if Well-Ordering fails, then transfinite cardinal arithmetic will be messy. For example, we will have to abandon the idea that if A and B are infinite then $A \sqcup B \approx A \times B \approx M$, where

M is the larger of A and B (see **??**). The problem is simple: if we cannot *compare* the size of A and B, then it is nonsensical to ask which is larger.

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Bibliography

Hartogs, Friedrich. 1915. Über das Problem der Wohlordnung. *Mathematische Annalen* 76: 438–43.