That’s the plus side. Here’s the minus side. Without Choice, things get messy. To see why, here is a nice result due to Hartogs (1915):

**Lemma choice.1 (in ZF).** For any set $A$, there is an ordinal $\alpha$ such that $\alpha \not\leq A$

*Proof.* If $B \subseteq A$ and $R \subseteq B^2$, then $\langle B, R \rangle \subseteq V_{\text{rank}(A)+4}$ by ???. So, using Separation, consider:

$$C = \{ \langle B, R \rangle \in V_{\text{rank}(A)+5} : B \subseteq A \text{ and } \langle B, R \rangle \text{ is a well-ordering} \}$$

Using Replacement and ??, form the set:

$$\alpha = \{ \text{ord}(B, R) : \langle B, R \rangle \in C \}.$$

By ??, $\alpha$ is an ordinal, since it is a transitive set of ordinals. After all, if $\gamma \in \beta \in \alpha$, then $\beta = \text{ord}(B, R)$ for some $B \subseteq R$, whereupon $\gamma = \text{ord}(B_0, R_0)$ for some $b \in B$ by ??, so that $\gamma \in \alpha$.

For reductio, suppose there is an injection $f : \alpha \to A$. Then, where:

$$B = \text{ran}(f)$$

$$R = \{ \langle f(\alpha), f(\beta) \rangle \in A \times A : \alpha \in \beta \}.$$

Clearly $\alpha = \text{ord}(B, R)$ and $\langle B, R \rangle \in C$. So $\alpha \in \alpha$, which is a contradiction. \(\square\)

This entails a deep result:

**Theorem choice.2 (in ZF).** The following claims are equivalent:

1. The Axiom of Well-Ordering
2. Either $A \preceq B$ or $B \preceq A$, for any sets $A$ and $B$

*Proof. (1) \(\Rightarrow\) (2).* Fix $A$ and $B$. Invoking (1), there are well-orderings $\langle A, R \rangle$ and $\langle B, S \rangle$. Invoking ??, let $f : \alpha \to \langle A, R \rangle$ and $g : \beta \to \langle B, S \rangle$ be isomorphisms. By Trichotomy, either $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$. In the first two cases $\alpha \subseteq \beta$, so $g \circ f^{-1} : A \to B$ is an injection, and hence $A \preceq B$. Similarly if $\beta \in \alpha$ then $B \preceq A$.

(1) \(\Rightarrow\) (2). Fix $A$; by Lemma choice.1 there is some ordinal $\beta$ such that $\beta \not\preceq A$. Invoking (2), we have $A \preceq \beta$. So there is some injection $f : A \to \beta$, and we can use this injection to well-order the elements of $A$, by defining an order $\{ (a, b) \in A \times A : f(a) \in f(b) \}$.

As an immediate consequence: if Well-Ordering fails, then some sets are literally incomparable with regard to their size. So, if Well-Ordering fails, then transfinite cardinal arithmetic will be messy. For example, we will have to abandon the idea that if $A$ and $B$ are infinite then $A \cup B \approx A \times B \approx M$, where
$M$ is the larger of $A$ and $B$ (see ??). The problem is simple: if we cannot compare the size of $A$ and $B$, then it is nonsensical to ask which is larger.

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**Bibliography**