

cardinals.1 Appendix: Hume’s Principle

st:cardinals:hp: In ??, we described Cantor’s Principle. This was:
sec

$$|A| = |B| \text{ iff } A \approx B.$$

This is very similar to what is now called *Hume’s Principle*, which says:

$$\#x F(x) = \#x G(x) \text{ iff } F \sim G$$

where ‘ $F \sim G$ ’ abbreviates that there are exactly as many F s as G s, i.e., the F s can be put into a bijection with the G s, i.e.:

$$\begin{aligned} \exists R(\forall v\forall y(Rvy \rightarrow (Fv \wedge Gy)) \wedge \\ \forall v(Fv \rightarrow \exists!y Rvy) \wedge \\ \forall y(Gy \rightarrow \exists!v Rvy)) \end{aligned}$$

But there is a type-difference between Hume’s Principle and Cantor’s Principle. In the statement of Cantor’s Principle, the variables “ A ” and “ B ” are first-order terms which stand for *sets*. In the statement of Hume’s Principle, “ F ”, “ G ” and “ R ” are *not* first-order terms; rather, they are in *predicate position*. (Maybe they stand for *properties*.) So we might gloss Hume’s Principle in English as: the number of F s is the number of G s iff the F s are bijective with the G s. This is called *Hume’s Principle*, because Hume once wrote this:

When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal. (Hume, 1740, Pt.III Bk.1 §1)

And Hume’s Principle was brought to contemporary mathematico-logical prominence by Frege (1884, §63), who quoted this passage from Hume, before (in effect) sketching (what we have called) Hume’s Principle.

You should note the structural similarity between Hume’s Principle and Basic Law V. We formulated this in ?? as follows:

$$\epsilon x F(x) = \epsilon x G(x) \text{ iff } \forall x (F(x) \leftrightarrow G(x)).$$

And, at this point, some commentary and comparison might help.

There are two ways to take a principle like Hume’s Principle or Basic Law V: *predicatively* or *impredicatively* (recall ??). On the impredicative reading of Basic Law V, for each F , the object $\epsilon x F(x)$ falls within the domain of quantification that we used in formulating Basic Law V itself. Similarly, on the impredicative reading of Hume’s Principle, for each F , the object $\#x F(x)$ falls within the domain of quantification that we used in formulating Hume’s Principle. By contrast, on the *predicative* understanding, the objects $\epsilon x F(x)$ and $\#x F(x)$ would be entities from some *different* domain.

Now, if we read Basic Law V impredicatively, it leads to inconsistency, via Naïve Comprehension (for the details, see ??). Much like Naïve Comprehension, it can be rendered consistent by reading it *predicatively*. But it probably will not do everything that we wanted it to.

Hume's Principle, however, *can* consistently be read impredicatively. And, read thus, it is quite powerful.

To illustrate: consider the predicate " $x \neq x$ ", which obviously nothing satisfies. Hume's Principle now yields an object $\#x(x \neq x)$. We might treat this as the number 0. Now, on the *impredicative* understanding—but *only* on the impredicative understanding—this entity 0 falls within our original domain of quantification. So we can sensibly apply Hume's Principle with the predicate " $x = 0$ " to obtain an object $\#x(x = 0)$. We might treat this as the number 1. Moreover, Hume's Principle entails that $0 \neq 1$, since there cannot be a bijection from the non-self-identical objects to the objects identical with 0 (there are none of the former, but one of the latter). Now, working impredicatively again, 1 falls within our original domain of quantification. So we can sensibly apply Hume's Principle with the predicate " $(x = 0 \vee x = 1)$ " to obtain an object $\#x(x = 0 \vee x = 1)$. We might treat this as the number 2, and we can show that $0 \neq 2$ and $1 \neq 2$ and so on.

In short, taken impredicatively, Hume's Principle entails that there are *infinitely many objects*. And this has encouraged *neo-Fregean logicians* to take Hume's Principle as the foundation for arithmetic.

Frege *himself*, though, did not take Hume's Principle as his foundation for arithmetic. Instead, Frege proved Hume's Principle from an explicit definition: $\#x F(x)$ is defined as the extension of the concept $F \sim \Phi$. In modern terms, we might attempt to render this as $\#x F(x) = \{G : F \sim G\}$; but this will pull us back into the problems of Naïve Comprehension.

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Bibliography

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