cardinals.1 Cantor’s Principle

Cast your mind back to ???. We were discussing well-ordered sets, and suggested that it would be nice to have objects which go proxy for well-orders. With this in mind, we introduced ordinals, and then showed in ?? that these behave as we would want them to, i.e.:

\[
\text{ord}(A, <) = \text{ord}(B, <) \text{ iff } \langle A, < \rangle \cong \langle B, < \rangle.
\]

Cast your mind back even further, to ???. There, working naïvely, we introduced the notion of the “size” of a set. Specifically, we said that two sets are equinumerous, \( A \approx B \), just in case there is a bijection \( f: A \to B \). This is an intrinsically simpler notion than that of a well-ordering: we are only interested in bijections, and not (as with order-isomorphisms) whether the bijections “preserve any structure”.

This all gives rise to an obvious thought. Just as we introduced certain objects, ordinals, to calibrate well-orders, we can introduce certain objects, cardinals, to calibrate size. That is the aim of this chapter.

Before we say what these cardinals will be, we should lay down a principle which they ought to satisfy. Writing \(|X|\) for the cardinality of the set \( X \), we would hope to secure the following principle:

\[|A| = |B| \text{ iff } A \approx B.\]

We’ll call this Cantor’s Principle, since Cantor was probably the first to have it very clearly in mind. (We’ll say more about its relationship to Hume’s Principle in ??.) So our aim is to define \(|X|\), for each \( X \), in such a way that it delivers Cantor’s Principle.

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Bibliography