cardinals.1 Cantor's Principle

sth:cardinals:cp:sec

Cast your mind back to **??**. We were discussing well-ordered sets, and suggested that it would be nice to have objects which go proxy for well-orders. With this is mind, we introduced ordinals, and then showed in **??** that these behave as we would want them to, i.e.:

$$\text{ord}(A, <) = \text{ord}(B, \lessdot) \text{ iff } \langle A, < \rangle \cong \langle B, \lessdot \rangle.$$

Cast your mind back even further, to **??**. There, working naïvely, we introduced the notion of the "size" of a set. Specifically, we said that two sets are equinumerous, $A \approx B$, just in case there is a bijection $f \colon A \to B$. This is an intrinsically simpler notion than that of a well-ordering: we are only interested in bijections, and not (as with order-isomorphisms) whether the bijections "preserve any structure".

This all gives rise to an obvious thought. Just as we introduced certain objects, *ordinals*, to calibrate well-orders, we can introduce certain objects, *cardinals*, to calibrate size. That is the aim of this chapter.

Before we say what these cardinals will be, we should lay down a principle which they ought to satisfy. Writing $|X|$ for the cardinality of the set X, we would want them to obey:

$$|A| = |B| \text{ iff } A \approx B.$$

We'll call this *Cantor's* Principle, since Cantor was probably the first to have it very clearly in mind. (We'll say more about its relationship to *Hume's* Principle in **??**.) So our aim is to define $|X|$, for each X, in such a way that it delivers Cantor's Principle.

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Bibliography