Now that we have been introduced to cardinals, it is worth spending a little time talking about different varieties of cardinals; specifically, finite, enumerable, and non-enumerable cardinals.

Our first two results entail that the finite cardinals will be exactly the finite ordinals, which we defined as our natural numbers back in ??:

**Proposition cardinals.1.** Let \( n, m \in \omega \). Then \( n = m \) iff \( n \approx m \).

**Proof.** Left-to-right is trivial. To prove right-to-left, suppose \( n \approx m \) although \( n \neq m \). By Trichotomy, either \( n \in m \) or \( m \in n \); suppose \( n \in m \) without loss of generality. Then \( n \subseteq m \) and there is a bijection \( f: m \to n \), so that \( m \) is Dedekind infinite, contradicting ??.

**Corollary cardinals.2.** If \( n \in \omega \), then \( n \) is a cardinal.

**Proof.** Immediate.

It also follows that several reasonable notions of what it might mean to describe a cardinal as “finite” or “infinite” coincide:

**Theorem cardinals.3.** For any set \( A \), the following are equivalent:

1. \( |A| \notin \omega \), i.e., \( A \) is not a natural number;
2. \( \omega \leq |A| \);
3. \( A \) is Dedekind infinite.

**Proof.** From ??, ??, and Corollary cardinals.2.

This licenses the following *definition* of some notions which we used rather informally in ??:

**Definition cardinals.4.** We say that \( A \) is *finite* iff \( |A| \) is a natural number, i.e., \( |A| \in \omega \). Otherwise, we say that \( A \) is *infinite*.

But note that this definition is presented against the background of ZFC. After all, we needed Well-Ordering to guarantee that every set has a cardinality. And indeed, without Well-Ordering, there can be a set which is neither finite nor Dedekind infinite. We will return to this sort of issue in ?? For now, we continue to rely upon Well-Ordering.

Let us now turn from the finite cardinals to the infinite cardinals. Here are two elementary points:

**Corollary cardinals.5.** \( \omega \) is the least infinite cardinal.

**Proof.** \( \omega \) is a cardinal, since \( \omega \) is Dedekind infinite and if \( \omega \approx n \) for any \( n \in \omega \) then \( n \) would be Dedekind infinite, contradicting ??.

Now \( \omega \) is the least infinite cardinal by definition.
Corollary cardinals.6. Every infinite cardinal is a limit ordinal.

Proof. Let \( \alpha \) be an infinite successor ordinal, so \( \alpha = \beta + 1 \) for some \( \beta \). By Proposition cardinals.1, \( \beta \) is also infinite, so \( \beta \approx \beta + 1 \) by \( \text{??} \). Now \( |\beta| = |\beta + 1| = |\alpha| \) by \( \text{??} \), so that \( \alpha \neq |\alpha| \).

Now, as early as \( \text{??} \), we flagged we can distinguish between enumerable and non-enumerable infinite sets. That definition naturally leads to the following:

Proposition cardinals.7. A is enumerable iff \( |A| \leq \omega \), and A is non-enumerable iff \( \omega < |A| \).

Proof. By Trichotomy, the two claims are equivalent, so it suffices to prove that A is enumerable iff \( |A| \leq \omega \). For right-to-left: if \( |A| \leq \omega \), then \( A \leq \omega \) by \( \text{??} \) and Corollary cardinals.5. For left-to-right: suppose A is enumerable; then by \( \text{??} \) there are three possible cases:

1. if \( A = \emptyset \), then \( |A| = 0 \in \omega \), by Corollary cardinals.2 and \( \text{??} \).
2. if \( n \approx A \), then \( |A| = n \in \omega \), by Corollary cardinals.2 and \( \text{??} \).
3. if \( \omega \approx A \), then \( |A| = \omega \), by Corollary cardinals.5.

So in all cases, \( |A| \leq \omega \).

Indeed, \( \omega \) has a special place. Whilst there are many countable ordinals:

Corollary cardinals.8. \( \omega \) is the only enumerable infinite cardinal.

Proof. Let \( a \) be a enumerable infinite cardinal. Since \( a \) is infinite, \( \omega \leq a \). Since \( a \) is a enumerable cardinal, \( a = |a| \leq \omega \). So \( a = \omega \) by Trichotomy.

Evidently there are infinitely many cardinals. So we might ask: How many cardinals are there? The following results show that we might want to reconsider that question.

Proposition cardinals.9. If every member of \( X \) is a cardinal, then \( \bigcup X \) is a cardinal.

Proof. It is easy to check that \( \bigcup X \) is an ordinal. Let \( \alpha \in \bigcup X \) be an ordinal; then \( \alpha \in b \in X \) for some cardinal \( b \). Since \( b \) is a cardinal, \( \alpha \prec b \). Since \( b \subseteq \bigcup X \), we have \( b \leq \bigcup X \), and so \( \alpha \approx \bigcup X \). Generalising, \( \bigcup X \) is a cardinal.

Theorem cardinals.10. There is no largest cardinal.

Proof. For any cardinal \( a \), Cantor’s Theorem (\( \text{??} \)) and \( \text{??} \) entail that \( a < |\varphi(a)| \).

Theorem cardinals.11. The set of all cardinals does not exist.
Proof. For reductio, suppose $C = \{a : a \text{ is a cardinal}\}$. Now $\bigcup C$ is a cardinal by Proposition cardinals.9, so by Theorem cardinals.10 there is a cardinal $b > \bigcup C$. By definition $b \in C$, so $b \subseteq \bigcup C$, so that $b \leq \bigcup C$, a contradiction. □

You should compare this with both Russell’s Paradox and Burali-Forti.

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Bibliography