Now that we have been introduced to cardinals, it is worth spending a little time talking about different varieties of cardinals; specifically, finite, enumerable, and non-enumerable cardinals.

Our first two results entail that the finite cardinals will be exactly the finite ordinals, which we defined as our natural numbers back in ??:

**Proposition cardinals.1.** Let \( n, m \in \omega \). Then \( n = m \) iff \( n \approx m \).

**Proof.** Left-to-right is trivial. To prove right-to-left, suppose \( n \approx m \) although \( n \neq m \). By Trichotomy, either \( n \in m \) or \( m \in n \); suppose \( n \in m \) without loss of generality. Then \( n \subset m \) and there is a bijection \( f : m \to n \), so that \( m \) is Dedekind infinite, contradicting ??.

**Corollary cardinals.2.** If \( n \in \omega \), then \( n \) is a cardinal.

**Proof.** Immediate.

It also follows that several reasonable notions of what it might mean to describe a cardinal as “finite” or “infinite” coincide:

**Theorem cardinals.3.** For any set \( A \), the following are equivalent:

1. \( |A| \notin \omega \), i.e., \( A \) is not a natural number;
2. \( \omega \leq |A| \);
3. \( A \) is Dedekind infinite.

**Proof.** From ??, ??, and Corollary cardinals.2.

This licenses the following definition of some notions which we used rather informally in ??:

**Definition cardinals.4.** We say that \( A \) is finite iff \( |A| \) is a natural number, i.e., \( |A| \in \omega \). Otherwise, we say that \( A \) is infinite.

But note that this definition is presented against the background of ZFC. After all, we needed Well-Ordering to guarantee that every set has a cardinality. And indeed, without Well-Ordering, there can be a set which is neither finite nor Dedekind infinite. We will return to this sort of issue in ??.

Let us now turn from the finite cardinals to the infinite cardinals. Here are two elementary points:

**Corollary cardinals.5.** \( \omega \) is the least infinite cardinal.

**Proof.** \( \omega \) is a cardinal, since \( \omega \) is Dedekind infinite and if \( \omega \approx n \) for any \( n \in \omega \) then \( n \) would be Dedekind infinite, contradicting ??.

\( \omega \) is the least infinite cardinal by definition.

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Corollary cardinals.6. Every infinite cardinal is a limit ordinal.

Proof. Let \( \alpha \) be an infinite successor ordinal, so \( \alpha = \beta + 1 \) for some \( \beta \). By Proposition cardinals.1, \( \beta \) is also infinite, so \( \beta \approx \beta + 1 \) by \ref{prop:infinite-ordinals}. Now \( |\beta| = |\beta + 1| = |\alpha| \) by \ref{cor:cardinals}, so that \( \alpha \neq |\alpha| \).

Now, as early as \ref{prop:cardinals}, we flagged we can distinguish between enumerable and non-enumerable infinite sets. That definition naturally leads to the following:

Proposition cardinals.7. A is enumerable iff \( |A| \leq \omega \), and A is non-enumerable iff \( \omega < |A| \).

Proof. By Trichotomy, the two claims are equivalent, so it suffices to prove that \( A \) is enumerable iff \( |A| \leq \omega \). For right-to-left: if \( |A| \leq \omega \), then \( A \preceq \omega \) by \ref{prop:cardinals} and Corollary cardinals.5. For left-to-right: suppose \( A \) is enumerable; then by \ref{prop:cardinals} there are three possible cases:

1. if \( A = \emptyset \), then \( |A| = 0 \in \omega \), by Corollary cardinals.2 and \ref{cor:cardinals}.
2. if \( n \approx A \), then \( |A| = n \in \omega \), by Corollary cardinals.2 and \ref{cor:cardinals}.
3. if \( \omega \approx A \), then \( |A| = \omega \), by Corollary cardinals.5.

So in all cases, \( |A| \leq \omega \).

Indeed, \( \omega \) has a special place. Whilst there are many countable ordinals:

Corollary cardinals.8. \( \omega \) is the only enumerable infinite cardinal.

Proof. Let \( a \) be an enumerable infinite cardinal. Since \( a \) is infinite, \( \omega \leq a \). Since \( a \) is an enumerable cardinal, \( a = |a| \leq \omega \). So \( a = \omega \) by Trichotomy.

Of course, there are infinitely many cardinals. So we might ask: How many cardinals are there? The following results show that we might want to reconsider that question.

Proposition cardinals.9. If every member of \( X \) is a cardinal, then \( \bigcup X \) is a cardinal.

Proof. It is easy to check that \( \bigcup X \) is an ordinal. Let \( \alpha \in \bigcup X \) be an ordinal; then \( \alpha \in b \in X \) for some cardinal \( b \). Since \( b \) is a cardinal, \( \alpha \prec b \). Since \( b \subseteq \bigcup X \), we have \( b \preceq \bigcup X \), and so \( \alpha \preceq \bigcup X \). Generalising, \( \bigcup X \) is a cardinal.

Theorem cardinals.10. There is no largest cardinal.

Proof. For any cardinal \( a \), Cantor’s Theorem (\ref{thm:cantor}) and \ref{thm:cardinals} entail that \( a < |\wp(a)| \).

Theorem cardinals.11. The set of all cardinals does not exist.
Proof. For reductio, suppose $C = \{a : a \text{ is a cardinal}\}$. Now $\bigcup C$ is a cardinal by Proposition cardinals.9, so by Theorem cardinals.10 there is a cardinal $b > \bigcup C$. By definition $b \in C$, so $b \subseteq \bigcup C$, so that $b \leq \bigcup C$, a contradiction. \qed

You should compare this with both Russell’s Paradox and Burali-Forti.

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Bibliography