Now that we have been introduced to cardinals, it is worth spending a little time talking about different varieties of cardinals; specifically, finite, enumerable, and non-enumerable cardinals.

Our first two results entail that the finite cardinals will be exactly the finite ordinals, which we defined as our natural numbers back in ??:

**Proposition cardinals.1.** Let \( n, m \in \omega \). Then \( n = m \) iff \( n \approx m \).

**Proof.** Left-to-right is trivial. To prove right-to-left, suppose \( n \approx m \) although \( n \neq m \). By Trichotomy, either \( n \in m \) or \( m \in n \); suppose \( n \in m \) without loss of generality. Then \( n \subseteq m \) and there is a bijection \( f: m \to n \), so that \( m \) is Dedekind infinite, contradicting ??.

**Corollary cardinals.2.** If \( n \in \omega \), then \( n \) is a cardinal.

**Proof.** Immediate.

It also follows that several reasonable notions of what it might mean to describe a cardinal as “finite” or “infinite” coincide:

**Theorem cardinals.3.** For any set \( A \), the following are equivalent:

1. \( |A| \notin \omega \), i.e., \( A \) is not a natural number;
2. \( \omega \leq |A| \);
3. \( A \) is Dedekind infinite.

**Proof.** From ??, ??, and Corollary cardinals.2.

This licenses the following definition of some notions which we used rather informally in ??:

**Definition cardinals.4.** We say that \( A \) is finite iff \( |A| \) is a natural number, i.e., \( |A| \in \omega \). Otherwise, we say that \( A \) is infinite.

But note that this definition is presented against the background of ZFC. After all, we needed Well-Ordering to guarantee that every set has a cardinality. And indeed, without Well-Ordering, there can be a set which is neither finite nor Dedekind infinite. We will return to this sort of issue in ?? . For now, we continue to rely upon Well-Ordering.

Let us now turn from the finite cardinals to the infinite cardinals. Here are two elementary points:

**Corollary cardinals.5.** \( \omega \) is the least infinite cardinal.

**Proof.** \( \omega \) is a cardinal, since \( \omega \) is Dedekind infinite and if \( \omega \approx n \) for any \( n \in \omega \) then \( n \) would be Dedekind infinite, contradicting ?? . Now \( \omega \) is the least infinite cardinal by definition.
Corollary cardinals.6. Every infinite cardinal is a limit ordinal.

Proof. Let $\alpha$ be an infinite successor ordinal, so $\alpha = \beta + 1$ for some $\beta$. By Proposition cardinals.1, $\beta$ is also infinite, so $\beta \approx \beta + 1$ by ???. Now $|\beta| = |\beta + 1| = |\alpha|$ by ??, so that $\alpha \neq |\alpha|$.

Now, as early as ??, we flagged we can distinguish between enumerable and non-enumerable infinite sets. That definition naturally leads to the following:

Proposition cardinals.7. $A$ is enumerable iff $|A| \leq \omega$, and $A$ is non-enumerable iff $\omega < |A|$.

Proof. By Trichotomy, the two claims are equivalent, so it suffices to prove that $A$ is enumerable iff $|A| \leq \omega$. For right-to-left: if $|A| \leq \omega$, then $A \leq \omega$ by ?? and Corollary cardinals.5. For left-to-right: suppose $A$ is enumerable; then by ?? there are three possible cases:

1. if $A = \emptyset$, then $|A| = 0 \leq \omega$, by Corollary cardinals.2 and ??.
2. if $n \approx A$, then $|A| = n \leq \omega$, by Corollary cardinals.2 and ??.
3. if $\omega \approx A$, then $|A| = \omega$, by Corollary cardinals.5.

So in all cases, $|A| \leq \omega$.

Indeed, $\omega$ has a special place. Whilst there are many countable ordinals:

Corollary cardinals.8. $\omega$ is the only enumerable infinite cardinal.

Proof. Let $a$ be an enumerable infinite cardinal. Since $a$ is infinite, $\omega \leq a$. Since $a$ is an enumerable cardinal, $a = |a| \leq \omega$. So $a = \omega$ by Trichotomy.

Of course, there are infinitely many cardinals. So we might ask: How many cardinals are there? The following results show that we might want to reconsider that question.

Proposition cardinals.9. If every member of $X$ is a cardinal, then $\bigcup X$ is a cardinal.

Proof. It is easy to check that $\bigcup X$ is an ordinal. Let $\alpha \in \bigcup X$ be an ordinal; then $\alpha \in b \in X$ for some cardinal $b$. Since $b$ is a cardinal, $\alpha \prec b$. Since $b \subseteq \bigcup X$, we have $b \preceq \bigcup X$, and so $\alpha \preceq \bigcup X$. Generalising, $\bigcup X$ is a cardinal.

Theorem cardinals.10. There is no largest cardinal.

Proof. For any cardinal $a$, Cantor’s Theorem (??) and ?? entail that $a < |\wp(a)|$.

Theorem cardinals.11. The set of all cardinals does not exist.
Proof. For reductio, suppose $C = \{a : a \text{ is a cardinal}\}$. Now $\bigcup C$ is a cardinal by Proposition cardinals.9, so by Theorem cardinals.10 there is a cardinal $b > \bigcup C$. By definition $b \in C$, so $b \subseteq \bigcup C$, so that $b \leq \bigcup C$, a contradiction. \qed

You should compare this with both Russell’s Paradox and Burali-Forti.

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Bibliography