cardinals.1 Finite, Enumerable, Non-enumerable

sth:cardinals:classing:sec

Now that we have been introduced to cardinals, it is worth spending a little time talking about different varieties of cardinals; specifically, finite, enumerable, and non-enumerable cardinals.

Our first two results entail that the finite cardinals will be exactly the finite ordinals, which we defined as our *natural numbers* back in **??**:

sth:cardinals:classing:finitecardisoequal

Proposition cardinals.1. *Let $n, m \in \omega$. Then $n = m$ iff $n \approx m$.*

Proof. *Left-to-right* is trivial. To prove *right-to-left*, suppose $n \approx m$ although $n \neq m$. By Trichotomy, either $n \in m$ or $m \in n$; suppose $n \in m$ without loss of generality. Then $n \subsetneq m$ and there is a bijection $f\colon m \to n$, so that m is Dedekind infinite, contradicting **??**. □

sth:cardinals:classing:naturalsarecardinals

Corollary cardinals.2. *If $n \in \omega$, then n is a cardinal.*

Proof. Immediate. □

It also follows that several reasonable notions of what it might mean to describe a cardinal as "finite" or "infinite" coincide:

sth:cardinals:classing:generalinfinitycharacter

Theorem cardinals.3. *For any set A, the following are equivalent:*

sth:cardinals:classing:card:notinomega

1. *$|A| \notin \omega$, i.e., A is not a natural number;*

sth:cardinals:classing:card:omegaplus

2. *$\omega \leq |A|$;*

sth:cardinals:classing:card:infinite

3. *A is Dedekind infinite.*

Proof. From **??**, **??**, and Corollary cardinals.2. □

This licenses the following *definition* of some notions which we used rather informally in **??**:

sth:cardinals:classing:defnfinite

Definition cardinals.4. We say that A is *finite* iff $|A|$ is a natural number, i.e., $|A| \in \omega$. Otherwise, we say that A is *infinite*.

But note that this definition is presented against the background of **ZFC**. After all, we needed Well-Ordering to guarantee that every set has a cardinality. And indeed, without Well-Ordering, there can be a set which is neither finite nor Dedekind infinite. We will return to this sort of issue in **??**. For now, we continue to rely upon Well-Ordering.

Let us now turn from the finite cardinals to the infinite cardinals. Here are two elementary points:

sth:cardinals:classing:omegaisacardinal

Corollary cardinals.5. *ω is the least infinite cardinal.*

Proof. ω is a cardinal, since ω is Dedekind infinite and if $\omega \approx n$ for any $n \in \omega$ then n would be Dedekind infinite, contradicting **??**. Now ω is the least infinite cardinal by definition. □

Corollary cardinals.6. *Every infinite cardinal is a limit ordinal.*

Proof. Let α be an infinite successor ordinal, so $\alpha = \beta + 1$ for some β. By Proposition cardinals.1, β is also infinite, so $\beta \approx \beta + 1$ by **??**. Now $|\beta| = |\beta + 1| = |\alpha|$ by **??**, so that $\alpha \neq |\alpha|$. □

Now, as early as **??**, we flagged we can distinguish between enumerable and non-enumerable infinite sets. That definition naturally leads to the following:

Proposition cardinals.7. *A is enumerable iff $|A| \leq \omega$, and A is non-enumerable iff $\omega < |A|$.*

Proof. By Trichotomy, the two claims are equivalent, so it suffices to prove that A is enumerable iff $|A| \leq \omega$. For *right-to-left*: if $|A| \leq \omega$, then $A \preceq \omega$ by **??** and Corollary cardinals.5. For *left-to-right*: suppose A is enumerable; then by **??** there are three possible cases:

1. if $A = \emptyset$, then $|A| = 0 \in \omega$, by Corollary cardinals.2 and **??**.
2. if $n \approx A$, then $|A| = n \in \omega$, by Corollary cardinals.2 and **??**.
3. if $\omega \approx A$, then $|A| = \omega$, by Corollary cardinals.5.

So in all cases, $|A| \leq \omega$. □

Indeed, ω has a special place. Whilst there are many countable ordinals:

Corollary cardinals.8. *ω is the only enumerable infinite cardinal.*

Proof. Let $\mathfrak{a}$ be an enumerable infinite cardinal. Since $\mathfrak{a}$ is infinite, $\omega \leq \mathfrak{a}$. Since $\mathfrak{a}$ is an enumerable cardinal, $\mathfrak{a} = |\mathfrak{a}| \leq \omega$. So $\mathfrak{a} = \omega$ by Trichotomy. □

Of course, there are infinitely many cardinals. So we might ask: *How many cardinals are there?* The following results show that we might want to reconsider that question.

Proposition cardinals.9. *If every member of X is a cardinal, then $\bigcup X$ is a cardinal.* *sth:cardinals:classing:unioncardinalscardinal*

Proof. It is easy to check that $\bigcup X$ is an ordinal. Let $\alpha \in \bigcup X$ be an ordinal; then $\alpha \in \mathfrak{b} \in X$ for some cardinal $\mathfrak{b}$. Since $\mathfrak{b}$ is a cardinal, $\alpha \prec \mathfrak{b}$. Since $\mathfrak{b} \subseteq \bigcup X$, we have $\mathfrak{b} \preceq \bigcup X$, and so $\alpha \not\approx \bigcup X$. Generalising, $\bigcup X$ is a cardinal. □

Theorem cardinals.10. *There is no largest cardinal.* *sth:cardinals:classing:lem:NoLargestCardinal*

Proof. For any cardinal $\mathfrak{a}$, Cantor's Theorem (**??**) and **??** entail that $\mathfrak{a} < |\wp(\mathfrak{a})|$. □

Theorem cardinals.11. *The set of all cardinals does not exist.*

Proof. For reductio, suppose $C = \{\mathfrak{a} : \mathfrak{a} \text{ is a cardinal}\}$. Now $\bigcup C$ is a cardinal by Proposition cardinals.9, so by Theorem cardinals.10 there is a cardinal $\mathfrak{b} > \bigcup C$. By definition $\mathfrak{b} \in C$, so $\mathfrak{b} \subseteq \bigcup C$, so that $\mathfrak{b} \leq \bigcup C$, a contradiction. □

You should compare this with both Russell's Paradox and Burali-Forti.

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Bibliography