Chapter udf

Cardinals

cardinals.1 Cantor's Principle

sth:cardinals:cp:sec Cast your mind back to **??**. We were discussing well-ordered sets, and suggested that it would be nice to have objects which go proxy for well-orders. With this is mind, we introduced ordinals, and then showed in **??** that these behave as we would want them to, i.e.:

$$\text{ord}(A, <) = \text{ord}(B, \lessdot) \text{ iff } \langle A, < \rangle \cong \langle B, \lessdot \rangle.$$

Cast your mind back even further, to **??**. There, working naïvely, we introduced the notion of the "size" of a set. Specifically, we said that two sets are equinumerous, $A \approx B$, just in case there is a bijection $f \colon A \to B$. This is an intrinsically simpler notion than that of a well-ordering: we are only interested in bijections, and not (as with order-isomorphisms) whether the bijections "preserve any structure".

This all gives rise to an obvious thought. Just as we introduced certain objects, *ordinals*, to calibrate well-orders, we can introduce certain objects, *cardinals*, to calibrate size. That is the aim of this chapter.

Before we say what these cardinals will be, we should lay down a principle which they ought to satisfy. Writing $|X|$ for the cardinality of the set X, we would want them to obey:

$$|A| = |B| \text{ iff } A \approx B.$$

We'll call this *Cantor's* Principle, since Cantor was probably the first to have it very clearly in mind. (We'll say more about its relationship to *Hume's* Principle in section cardinals.5.) So our aim is to define $|X|$, for each X, in such a way that it delivers Cantor's Principle.

cardinals.2 Cardinals as Ordinals

sth:cardinals:cardsasords:sec

In fact, our theory of cardinals will just make (shameless) use of our theory of ordinals. That is: we will just define cardinals as certain specific ordinals. In particular, we will offer the following:

Definition cardinals.1. If A can be well-ordered, then $|A|$ is the least ordinal γ such that $A \approx \gamma$. For any ordinal γ, we say that γ is a *cardinal* iff $\gamma = |\gamma|$. sth:cardinals:cardsasords: defcardinalasordinal

We just used the phrase "A can be well-ordered". As is almost always the case in mathematics, the modal locution here is just a hand-waving gloss on an existential claim: to say "A can be well-ordered" is just to say "there is a relation which well-orders A".

But there is a snag with Definition cardinals.1. We would like it to be the case that *every* set has a size, i.e., that $|A|$ exists for every A. The definition we just gave, though, begins with a conditional: "*If* A can be well-ordered…". If there is some set A which cannot be well-ordered, then our definition will simply fail to define an object $|A|$.

So, to use Definition cardinals.1, we need a guarantee that every set can be well-ordered. Sadly, though, this guarantee is unavailable in **ZF**. So, if we want to use Definition cardinals.1, there is no alternative but to add a new axiom, such as:

Axiom (Well-Ordering). Every set can be well-ordered.

We will discuss whether the Well-Ordering Axiom is acceptable in **??**. From now on, though, we will simply help ourselves to it. And, using it, it is quite straightforward to prove that cardinals (as defined in Definition cardinals.1) exist and behave nicely:

Lemma cardinals.2. *For every set A:* *sth:cardinals:cardsasords: lem:CardinalsExist*

1. $|A|$ *exists and is unique;* *sth:cardinals:cardsasords: cardaexists*
2. $|A| \approx A$*;* *sth:cardinals:cardsasords: cardaapprox*
3. $|A|$ *is a cardinal, i.e.,* $|A| = ||A||$*;* *sth:cardinals:cardsasords: cardaidem*

Proof. Fix A. By Well-Ordering, there is a well-ordering $\langle A, R\rangle$. By **??**, $\langle A, R\rangle$ is isomorphic to a unique ordinal, β. So $A \approx \beta$. By Transfinite Induction, there is a uniquely least ordinal, γ, such that $A \approx \gamma$. So $|A| = \gamma$, establishing (1) and (2). To establish (3), note that if $\delta \in \gamma$ then $\delta \prec A$, by our choice of γ, so that also $\delta \prec \gamma$ since equinumerosity is an equivalence relation (**??**). So $\gamma = |\gamma|$. □

The next result guarantees Cantor's Principle, and more besides. (Note that cardinals inherit their ordering from the ordinals, i.e., $\mathfrak{a} < \mathfrak{b}$ iff $\mathfrak{a} \in \mathfrak{b}$. In formulating this, we will use Fraktur letters for objects we know to be cardinals. This is fairly standard. A common alternative is to use Greek letters, since cardinals are ordinals, but to choose them from the middle of the alphabet, e.g.: κ, λ.):

sth:cardinals:cardsasords: lem:CardinalsBehaveRight

Lemma cardinals.3. *For any sets A and B:*

$$\begin{aligned} A \approx B &\text{ iff } |A| = |B| \\ A \preceq B &\text{ iff } |A| \leq |B| \\ A \prec B &\text{ iff } |A| < |B| \end{aligned}$$

Proof. We will prove the left-to-right direction of the second claim (the other cases are similar, and left as an exercise). So, consider the following diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, leftrightarrow] & B \arrow[d, leftrightarrow] \\ {|A|} \arrow[r, dashed] & {|B|} \end{tikzcd}$$

The double-headed arrows indicate bijections, whose existence is guaranteed by Lemma cardinals.2. In assuming that $A \preceq B$, there is an injection $A \to B$. Now, chasing the arrows around from $|A|$ to A to B to $|B|$, we obtain an injection $|A| \to |B|$ (the dashed arrow). □

We can also use Lemma cardinals.3 to re-prove Schröder–Bernstein. This is the claim that if $A \preceq B$ and $B \preceq A$ then $A \approx B$. We stated this as **??**, but first proved it—with some effort—in **??**. Now consider:

Re-proof of Schröder-Bernstein. If $A \preceq B$ and $B \preceq A$, then $|A| \leq |B|$ and $|B| \leq |A|$ by Lemma cardinals.3. So $|A| = |B|$ and $A \approx B$ by Trichotomy and Lemma cardinals.3. □

Whilst this is a very simple proof, it implicitly relies on both Replacement (to secure **??**) and on Well-Ordering (to guarantee Lemma cardinals.3). By contrast, the proof of **??** was much more self-standing (indeed, it can be carried out in $\mathbf{Z}^-$).

cardinals.3 ZFC: A Milestone

sth:cardinals:zfc: sec

With the addition of Well-Ordering, we have reached the final theoretical milestone. We now have all the axioms required for **ZFC**. In detail:

Definition cardinals.4. The theory **ZFC** has these axioms: Extensionality, Union, Pairs, Powersets, Infinity, Foundation, Well-Ordering and all instances of the Separation and Replacement schemes. Otherwise put, **ZFC** adds Well-Ordering to **ZF**.

ZFC stands for *Zermelo-Fraenkel* set theory with *Choice*. Now this might seem slightly odd, since the axiom we added was called "Well-Ordering", not "Choice". But, when we later formulate Choice, it will turn out that Well-Ordering is equivalent (modulo **ZF**) to Choice (see **??**). So which to take as our "basic" axiom is a matter of indifference. And the name "**ZFC**" is entirely standard in the literature.

cardinals.4 Finite, Enumerable, Non-enumerable

Now that we have been introduced to cardinals, it is worth spending a little time talking about different varieties of cardinals; specifically, finite, enumerable, and non-enumerable cardinals. sth:cardinals:classing:sec

Our first two results entail that the finite cardinals will be exactly the finite ordinals, which we defined as our *natural numbers* back in **??**:

Proposition cardinals.5. *Let $n, m \in \omega$. Then $n = m$ iff $n \approx m$.* *sth:cardinals:classing:finitecardisoequal*

Proof. *Left-to-right* is trivial. To prove *right-to-left*, suppose $n \approx m$ although $n \neq m$. By Trichotomy, either $n \in m$ or $m \in n$; suppose $n \in m$ without loss of generality. Then $n \subsetneq m$ and there is a bijection $f : m \to n$, so that m is Dedekind infinite, contradicting **??**. □

Corollary cardinals.6. *If $n \in \omega$, then n is a cardinal.* *sth:cardinals:classing:naturalsarecardinals*

Proof. Immediate. □

It also follows that several reasonable notions of what it might mean to describe a cardinal as "finite" or "infinite" coincide:

Theorem cardinals.7. *For any set A, the following are equivalent:* *sth:cardinals:classing:generalinfinitycharacter*

1. *$|A| \notin \omega$, i.e., A is not a natural number;* *sth:cardinals:classing:card:notinomega*
2. *$\omega \leq |A|$;* *sth:cardinals:classing:card:omegaplus*
3. *A is Dedekind infinite.* *sth:cardinals:classing:card:infinite*

Proof. From **??**, Lemma cardinals.3, and Corollary cardinals.6. □

This licenses the following *definition* of some notions which we used rather informally in **??**:

Definition cardinals.8. We say that A is *finite* iff $|A|$ is a natural number, i.e., $|A| \in \omega$. Otherwise, we say that A is *infinite*. sth:cardinals:classing:defnfinite

But note that this definition is presented against the background of **ZFC**. After all, we needed Well-Ordering to guarantee that every set has a cardinality. And indeed, without Well-Ordering, there can be a set which is neither finite nor Dedekind infinite. We will return to this sort of issue in **??**. For now, we continue to rely upon Well-Ordering.

Let us now turn from the finite cardinals to the infinite cardinals. Here are two elementary points:

Corollary cardinals.9. *ω is the least infinite cardinal.* *sth:cardinals:classing:omegaisacardinal*

Proof. ω is a cardinal, since ω is Dedekind infinite and if $\omega \approx n$ for any $n \in \omega$ then n would be Dedekind infinite, contradicting **??**. Now ω is the least infinite cardinal by definition. □

Corollary cardinals.10. *Every infinite cardinal is a limit ordinal.*

Proof. Let α be an infinite successor ordinal, so $\alpha = \beta + 1$ for some β. By Proposition cardinals.5, β is also infinite, so $\beta \approx \beta + 1$ by **??**. Now $|\beta| = |\beta + 1| = |\alpha|$ by Lemma cardinals.3, so that $\alpha \neq |\alpha|$. □

Now, as early as **??**, we flagged we can distinguish between enumerable and non-enumerable infinite sets. That definition naturally leads to the following:

Proposition cardinals.11. *A is enumerable iff $|A| \leq \omega$, and A is non-enumerable iff $\omega < |A|$.*

Proof. By Trichotomy, the two claims are equivalent, so it suffices to prove that A is enumerable iff $|A| \leq \omega$. For *right-to-left*: if $|A| \leq \omega$, then $A \preceq \omega$ by Lemma cardinals.3 and Corollary cardinals.9. For *left-to-right*: suppose A is enumerable; then by **??** there are three possible cases:

1. if $A = \emptyset$, then $|A| = 0 \in \omega$, by Corollary cardinals.6 and Lemma cardinals.3.
2. if $n \approx A$, then $|A| = n \in \omega$, by Corollary cardinals.6 and Lemma cardinals.3.
3. if $\omega \approx A$, then $|A| = \omega$, by Corollary cardinals.9.

So in all cases, $|A| \leq \omega$. □

Indeed, ω has a special place. Whilst there are many countable ordinals:

Corollary cardinals.12. *ω is the only enumerable infinite cardinal.*

Proof. Let $\mathfrak{a}$ be an enumerable infinite cardinal. Since $\mathfrak{a}$ is infinite, $\omega \leq \mathfrak{a}$. Since $\mathfrak{a}$ is an enumerable cardinal, $\mathfrak{a} = |\mathfrak{a}| \leq \omega$. So $\mathfrak{a} = \omega$ by Trichotomy. □

Of course, there are infinitely many cardinals. So we might ask: *How many cardinals are there?* The following results show that we might want to reconsider that question.

sth:cardinals:classing:unioncardinalscardinal **Proposition cardinals.13.** *If every member of X is a cardinal, then $\bigcup X$ is a cardinal.*

Proof. It is easy to check that $\bigcup X$ is an ordinal. Let $\alpha \in \bigcup X$ be an ordinal; then $\alpha \in \mathfrak{b} \in X$ for some cardinal $\mathfrak{b}$. Since $\mathfrak{b}$ is a cardinal, $\alpha \prec \mathfrak{b}$. Since $\mathfrak{b} \subseteq \bigcup X$, we have $\mathfrak{b} \preceq \bigcup X$, and so $\alpha \not\approx \bigcup X$. Generalising, $\bigcup X$ is a cardinal. □

sth:cardinals:classing:lem:NoLargestCardinal **Theorem cardinals.14.** *There is no largest cardinal.*

Proof. For any cardinal $\mathfrak{a}$, Cantor's Theorem (**??**) and Lemma cardinals.2 entail that $\mathfrak{a} < |\wp(\mathfrak{a})|$. □

Theorem cardinals.15. *The set of all cardinals does not exist.*

Proof. For reductio, suppose $C = \{\mathfrak{a} : \mathfrak{a} \text{ is a cardinal}\}$. Now $\bigcup C$ is a cardinal by Proposition cardinals.13, so by Theorem cardinals.14 there is a cardinal $\mathfrak{b} > \bigcup C$. By definition $\mathfrak{b} \in C$, so $\mathfrak{b} \subseteq \bigcup C$, so that $\mathfrak{b} \leq \bigcup C$, a contradiction.□

You should compare this with both Russell's Paradox and Burali-Forti.

cardinals.5 Appendix: Hume's Principle

In section cardinals.1, we described Cantor's Principle. This was: sth:cardinals:hp:sec

$$|A| = |B| \text{ iff } A \approx B.$$

This is very similar to what is now called *Hume's Principle*, which says:

$$\#x\, F(x) = \#x\, G(x) \text{ iff } F \sim G$$

where '$F \sim G$' abbreviates that there are exactly as many Fs as Gs, i.e., the Fs can be put into a bijection with the Gs, i.e.:

$$\begin{aligned}\exists R(&\forall v \forall y(Rvy \to (Fv \land Gy)) \land\\ &\forall v(Fv \to \exists! y\, Rvy) \land\\ &\forall y(Gy \to \exists! v\, Rvy))\end{aligned}$$

But there is a type-difference between Hume's Principle and Cantor's Principle. In the statement of Cantor's Principle, the variables "A" and "B" are first-order terms which stand for *sets*. In the statement of Hume's Principle, "F", "G" and "R" are *not* first-order terms; rather, they are in *predicate position*. (Maybe they stand for *properties*.) So we might gloss Hume's Principle in English as: the number of Fs is the number of Gs iff the Fs are bijective with the Gs. This is called *Hume's Principle*, because Hume once wrote this:

> When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal. (Hume, 1740, Pt.III Bk.1 §1)

And Hume's Principle was brought to contemporary mathematico-logical prominence by Frege (1884, §63), who quoted this passage from Hume, before (in effect) sketching (what we have called) Hume's Principle.

You should note the structural similarity between Hume's Principle and Basic Law V. We formulated this in **??** as follows:

$$\epsilon x\, F(x) = \epsilon x\, G(x) \text{iff } \forall x\, (F(x) \leftrightarrow G(x)).$$

And, at this point, some commentary and comparison might help.

There are two ways to take a principle like Hume's Principle or Basic Law V: *predicatively* or *impredicatively* (recall **??**). On the impredicative reading of

Basic Law V, for each F, the object $\epsilon x\, F(x)$ falls within the domain of quantification that we used in formulating Basic Law V itself. Similarly, on the impredicative reading of Hume's Principle, for each F, the object $\#x\, F(x)$ falls within the domain of quantification that we used in formulating Hume's Principle. By contrast, on the *predicative* understanding, the objects $\epsilon x\, F(x)$ and $\#x\, F(x)$ would be entities from some *different* domain.

Now, if we read Basic Law V impredicatively, it leads to inconsistency, via Naïve Comprehension (for the details, see **??**). Much like Naïve Comprehension, it can be rendered consistent by reading it *predicatively.* But it probably will not do everything that we wanted it to.

Hume's Principle, however, *can* consistently be read impredicatively. And, read thus, it is quite powerful.

To illustrate: consider the predicate "$x \neq x$", which obviously nothing satisfies. Hume's Principle now yields an object $\#x(x \neq x)$. We might treat this as the number 0. Now, on the *impredicative* understanding—but *only* on the impredicative understanding—this entity 0 falls within our original domain of quantification. So we can sensibly apply Hume's Principle with the predicate "$x = 0$" to obtain an object $\#x(x = 0)$. We might treat this as the number 1. Moreover, Hume's Principle entails that $0 \neq 1$, since there cannot be a bijection from the non-self-identical objects to the objects identical with 0 (there are none of the former, but one of the latter). Now, working impredicatively again, 1 falls within our original domain of quantification. So we can sensibly apply Hume's Principle with the predicate "$(x = 0 \lor x = 1)$" to obtain an object $\#x(x = 0 \lor x = 1)$. We might treat this as the number 2, and we can show that $0 \neq 2$ and $1 \neq 2$ and so on.

In short, taken impredicatively, Hume's Principle entails that there are *infinitely many objects*. And this has encouraged *neo-Fregean logicists* to take Hume's Principle as the foundation for arithmetic.

Frege *himself*, though, did not take Hume's Principle as his foundation for arithmetic. Instead, Frege proved Hume's Principle from an explicit definition: $\#x\, F(x)$ is defined as the extension of the concept $F \sim \Phi$. In modern terms, we might attempt to render this as $\#x\, F(x) = \{G : F \sim G\}$; but this will pull us back into the problems of Naïve Comprehension.

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