card-arithmetic.1 Simplifying Addition and Multiplication

sth:card-arithmetic:simp:sec

It turns out that transfinite cardinal addition and multiplication is *extremely* easy. This follows from the fact that cardinals are (certain) ordinals, and so well-ordered, and so can be manipulated in a certain way. Showing this, though, is *not* so easy. To start, we need a tricksy definition:

Definition card-arithmetic.1. We define a *canonical ordering*, $\lhd$, on pairs of ordinals, by stipulating that $\langle \alpha_1, \alpha_2 \rangle \lhd \langle \beta_1, \beta_2 \rangle$ iff either:

1. $\max(\alpha_1, \alpha_2) < \max(\beta_1, \beta_2)$; or

2. $\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)$ and $\alpha_1 < \beta_1$; or

3. $\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)$ and $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$

Lemma card-arithmetic.2. *$\langle \alpha \times \alpha, \lhd \rangle$ is a well-order, for any ordinal α.*[1]

Proof. Evidently $\lhd$ is connected on $\alpha \times \alpha$. For suppose that neither $\langle \alpha_1, \alpha_2 \rangle$ nor $\langle \beta_1, \beta_2 \rangle$ is $\lhd$-less than the other. Then $\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)$ and $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, so that $\langle \alpha_1, \alpha_2 \rangle = \langle \beta_1, \beta_2 \rangle$.

To show well-ordering, let $X \subseteq \alpha \times \alpha$ be non-empty. Since α is an ordinal, some δ is the least member of $\{\max(\gamma_1, \gamma_2) : \langle \gamma_1, \gamma_2 \rangle \in X\}$. Now discard all pairs from $\{\langle \gamma_1, \gamma_2 \rangle \in X : \max(\gamma_1, \gamma_2) = \delta\}$ except those with least first coordinate; from among these, the pair with least second coordinate is the $\lhd$-least element of X. □

Now for a teensy, simple observation:

sth:card-arithmetic:simp:simplecardproduct

Proposition card-arithmetic.3. *If $\alpha \approx \beta$, then $\alpha \times \alpha \approx \beta \times \beta$.*

Proof. Just let $f : \alpha \to \beta$ induce $\langle \gamma_1, \gamma_2 \rangle \mapsto \langle f(\gamma_1), f(\gamma_2) \rangle$. □

And now we will put all this to work, in proving a crucial lemma:

sth:card-arithmetic:simp:alphatimesalpha

Lemma card-arithmetic.4. *$\alpha \approx \alpha \times \alpha$, for any infinite ordinal α*

Proof. For reductio, let α be the least infinite ordinal for which this is false. **??** shows that $\omega \approx \omega \times \omega$, so $\omega \in \alpha$. Moreover, α is a cardinal: suppose otherwise, for reductio; then $|\alpha| \in \alpha$, so that $|\alpha| \approx |\alpha| \times |\alpha|$, by hypothesis; and $|\alpha| \approx \alpha$ by definition; so that $\alpha \approx \alpha \times \alpha$ by Proposition card-arithmetic.3.

Now, for each $\langle \gamma_1, \gamma_2 \rangle \in \alpha \times \alpha$, consider the segment:

$$\text{Seg}(\gamma_1, \gamma_2) = \{\langle \delta_1, \delta_2 \rangle \in \alpha \times \alpha : \langle \delta_1, \delta_2 \rangle \lhd \langle \gamma_1, \gamma_2 \rangle\}$$

[1] Cf. the naughtiness described in the footnote to **??**.

Let $\gamma = \max(\gamma_1, \gamma_2)$. When γ is infinite, observe:

$$\begin{aligned} \text{Seg}(\gamma_1, \gamma_2) &\precsim ((\gamma + 1) \cdot (\gamma + 1)), \text{ by the first clause defining } \lhd \\ &\approx (\gamma \cdot \gamma), \text{ by ?? and Proposition card-arithmetic.3} \\ &\approx \gamma, \text{ by the induction hypothesis} \\ &\prec \alpha, \text{ since } \alpha \text{ is a cardinal} \end{aligned}$$

So $\text{ord}(\alpha \times \alpha, \lhd) \leq \alpha$, and hence $\alpha \times \alpha \preceq \alpha$. Since of course $\alpha \preceq \alpha \times \alpha$, the result follows by Schröder-Bernstein. □

Finally, we get to our simplifying result:

Theorem card-arithmetic.5. *If $\mathfrak{a}, \mathfrak{b}$ are infinite cardinals, $\mathfrak{a} \otimes \mathfrak{b} = \mathfrak{a} \oplus \mathfrak{b} = max(\mathfrak{a}, \mathfrak{b})$.* *sth:card-arithmetic:simp:cardplustimesmax*

Proof. Without loss of generality, suppose $\mathfrak{a} = \max(\mathfrak{a}, \mathfrak{b})$. Then invoking Lemma card-arithmetic.4, $\mathfrak{a} \otimes \mathfrak{a} = \mathfrak{a} \leq \mathfrak{a} \oplus \mathfrak{b} \leq \mathfrak{a} \oplus \mathfrak{a} \leq \mathfrak{a} \otimes \mathfrak{a}$. □

Similarly, if $\mathfrak{a}$ is infinite, an $\mathfrak{a}$-sized union of $\leq \mathfrak{a}$-sized sets has size $\leq \mathfrak{a}$:

Proposition card-arithmetic.6. *Let $\mathfrak{a}$ be an infinite cardinal. For each ordinal $\beta \in \mathfrak{a}$, let X_β be a set with $|X_\beta| \leq \mathfrak{a}$. Then $\left|\bigcup_{\beta \in \mathfrak{a}} X_\beta\right| \leq \mathfrak{a}$.* *sth:card-arithmetic:simp:kappaunionkappasize*

Proof. For each $\beta \in \mathfrak{a}$, fix an injection $f_\beta \colon X_\beta \to \mathfrak{a}$. Define an injection $g \colon \bigcup_{\beta \in \mathfrak{a}} X_\beta \to \mathfrak{a} \times \mathfrak{a}$ by $g(v) = \langle \beta, f_\beta(v) \rangle$, where $v \in X_\beta$ and $v \notin X_\gamma$ for any $\gamma \in \beta$. Now $\bigcup_{\beta \in \mathfrak{a}} X_\beta \preceq \mathfrak{a} \times \mathfrak{a} \approx \mathfrak{a}$ by Theorem card-arithmetic.5. □

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Bibliography