card-arithmetic.1 Defining the Basic Operations

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Since we do not need to keep track of order, cardinal arithmetic is rather easier to define than ordinal arithmetic. We will define addition, multiplication, and exponentiation simultaneously.

Definition card-arithmetic.1. When $\mathfrak{a}$ and $\mathfrak{b}$ are cardinals:

$$\mathfrak{a} \oplus \mathfrak{b} = |\mathfrak{a} \sqcup \mathfrak{b}|$$
$$\mathfrak{a} \otimes \mathfrak{b} = |\mathfrak{a} \times \mathfrak{b}|$$
$$\mathfrak{a}^{\mathfrak{b}} = \left|{}^{\mathfrak{b}}\mathfrak{a}\right|$$

where ${}^{X}Y = \{f : f \text{ is a function } X \to Y\}$. (It is easy to show that ${}^{X}Y$ exists for any sets X and Y; we leave this as an exercise.)

Problem card-arithmetic.1. Prove in $\mathbf{Z}^{-}$ that ${}^{X}Y$ exists for any sets X and Y. Working in **ZF**, compute $\text{rank}({}^{X}Y)$ from $\text{rank}(X)$ and $\text{rank}(Y)$, in the manner of **??**.

It might help to explain this definition. Concerning addition: this uses the notion of disjoint sum, $\sqcup$, as defined in **??**; and it is easy to see that this definition gives the right verdict for finite cases. Concerning multiplication: **??** tells us that if A has n members and B has m members then $A \times B$ has $n \cdot m$ members, so our definition simply generalises the idea to transfinite multiplication. Exponentiation is similar: we are simply generalising the thought from the finite to the transfinite. Indeed, in certain ways, transfinite cardinal arithmetic looks much more like "ordinary" arithmetic than does transfinite ordinal arithmetic:

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Proposition card-arithmetic.2. *$\oplus$ and $\otimes$ are commutative and associative.*

Proof. For commutativity, by **??** it suffices to observe that $(\mathfrak{a} \sqcup \mathfrak{b}) \approx (\mathfrak{b} \sqcup \mathfrak{a})$ and $(\mathfrak{a} \times \mathfrak{b}) \approx (\mathfrak{b} \times \mathfrak{a})$. We leave associativity as an exercise. □

Problem card-arithmetic.2. Prove that $\oplus$ and $\otimes$ are associative.

Proposition card-arithmetic.3. *A is infinite iff $|A| \oplus 1 = 1 \oplus |A| = |A|$.*

Proof. As in **??**, from **??** and **??**. □

This explains why we need to use different symbols for ordinal versus cardinal addition/multiplication: these are genuinely *different* operations. This next pair of results shows that ordinal versus cardinal exponentiation are also different operations. (Recall that **??** entails that $2 = \{0, 1\}$):

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Lemma card-arithmetic.4. *$|\wp(A)| = 2^{|A|}$, for any A.*

Proof. For each subset $B \subseteq A$, let $\chi_B \in {}^A 2$ be given by:

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Now let $f(B) = \chi_B$; this defines a bijection $f \colon \wp(A) \to {}^A 2$. So $\wp(A) \approx {}^A 2$. Hence $\wp(A) \approx {}^{|A|}2$, so that $|\wp(A)| = \left|{}^{|A|}2\right| = 2^{|A|}$. □

This snappy proof essentially subsumes the discussion of **??**. There, we showed how to "reduce" the uncountability of $\wp(\omega)$ to the uncountability of the set of infinite binary strings, $\mathbb{B}^\omega$. In effect, $\mathbb{B}^\omega$ is just ${}^\omega 2$; and the preceding proof showed that the reasoning we went through in **??** will go through using any set A in place of ω. The result also yields a quick fact about cardinal exponentiation:

Corollary card-arithmetic.5. *$\mathfrak{a} < 2^{\mathfrak{a}}$ for any cardinal $\mathfrak{a}$.* sth:card-arithmetic:opps:cantorcor

Proof. From Cantor's Theorem (**??**) and Lemma card-arithmetic.4. □

So $\omega < 2^\omega$. But note: this is a result about *cardinal* exponentiation. It should be contrasted with *ordinal* exponentiation, since in the latter case $\omega = 2^{(\omega)}$ (see **??**).

Whilst we are on the topic of cardinal exponentiation, we can also be a bit more precise about the "way" in which $\mathbb{R}$ is non-enumerable.

Theorem card-arithmetic.6. $|\mathbb{R}| = 2^\omega$ sth:card-arithmetic:opps:continuumis2aleph0

Proof skeleton. There are plenty of ways to prove this. The most straightforward is to argue that $\wp(\omega) \preceq \mathbb{R}$ and $\mathbb{R} \preceq \wp(\omega)$, and then use Schröder-Bernstein to infer that $\mathbb{R} \approx \wp(\omega)$, and Lemma card-arithmetic.4 to infer that $|\mathbb{R}| = 2^\omega$. We leave it as an (illuminating) exercise to define injections $f \colon \wp(\omega) \to \mathbb{R}$ and $g \colon \mathbb{R} \to \wp(\omega)$. □

Problem card-arithmetic.3. Complete the proof of Theorem card-arithmetic.6, by showing that $\wp(\omega) \preceq \mathbb{R}$ and $\mathbb{R} \preceq \wp(\omega)$.

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Bibliography