card-arithmetic.1  Defining the Basic Operations

Since we do not need to keep track of order, cardinal arithmetic is rather easier to define than ordinal arithmetic. We will define addition, multiplication, and exponentiation simultaneously.

Definition card-arithmetic.1. When \(a\) and \(b\) are cardinals:

\[
\begin{align*}
  a \oplus b &= |a \sqcup b| \\
  a \otimes b &= |a \times b| \\
  a^b &= |^b a|
\end{align*}
\]

where \(^X Y = \{ f : f \text{ is a function } X \to Y \}\). (It is easy to show that \(^XY\) exists for any sets \(X\) and \(Y\); we leave this as an exercise.)

Problem card-arithmetic.1. Prove in \(\mathbb{Z}^-\) that \(^XY\) exists for any sets \(X\) and \(Y\). Working in \(\mathsf{ZF}\), compute \(\text{rank}(^XY)\) from \(\text{rank}(X)\) and \(\text{rank}(Y)\), in the manner of ??.

It might help to explain this definition. Concerning addition: this uses the notion of disjoint sum, \(\sqcup\), as defined in ??; and it is easy to see that this definition gives the right verdict for finite cases. Concerning multiplication: ?? tells us that if \(A\) has \(n\) members and \(B\) has \(m\) members then \(A \times B\) has \(n \cdot m\) members, so our definition simply generalises the idea to transfinite multiplication. Exponentiation is similar: we are simply generalising the thought from the finite to the transfinite. Indeed, in certain ways, transfinite cardinal arithmetic looks much more like “ordinary” arithmetic than does transfinite ordinal arithmetic:

Proposition card-arithmetic.2. \(\oplus\) and \(\otimes\) are commutative and associative.

Proof. For commutativity, by ?? it suffices to observe that \((a \sqcup b) \approx (b \sqcup a)\) and \((a \times b) \approx (b \times a)\). We leave associativity as an exercise.

Problem card-arithmetic.2. Prove that \(\oplus\) and \(\otimes\) are associative.

Proposition card-arithmetic.3. \(A\) is infinite iff \(|A| \oplus 1 = 1 \oplus |A| = |A|\).

Proof. As in ??, from ?? and ??.

This explains why we need to use different symbols for ordinal versus cardinal addition/multiplication: these are genuinely different operations. This next pair of results shows that ordinal versus cardinal exponentiation are also different operations. (Recall that ?? entails that \(2 = \{0,1\}\):

Lemma card-arithmetic.4. \(|p(A)| = 2^{|A|}\), for any \(A\).
Proof. For each subset $B \subseteq A$, let $\chi_B \in A^2$ be given by:

$$\chi_B(x) = \begin{cases} 
1 & \text{if } x \in B \\
0 & \text{otherwise.}
\end{cases}$$

Now let $f(B) = \chi_B$; this defines a bijection $f : \wp(A) \to A^2$. So $\wp(A) \approx A^2$. Hence $\wp(A) \approx |A|^2$, so that $|\wp(A)| = |A|^2 = 2^{|A|}$.

This snappy proof essentially subsumes the discussion of ?? There, we showed how to “reduce” the uncountability of $\wp(\omega)$ to the uncountability of the set of infinite binary strings, $B^\omega$. In effect, $B^\omega$ is just $\omega^2$; and the preceding proof showed that the reasoning we went through in ?? will go through using any set $A$ in place of $\omega$. The result also yields a quick fact about cardinal exponentiation:

**Corollary card-arithmetic.5.** $a < 2^a$ for any cardinal $a$.

*Proof.* From Cantor’s Theorem (?) and Lemma card-arithmetic.4.

So $\omega < 2^\omega$. But note: this is a result about *cardinal* exponentiation. It should be contrasted with *ordinal* exponentiation, since in the latter case $\omega = 2^{(\omega)}$ (see ??).

Whilst we are on the topic of cardinal exponentiation, we can also be a bit more precise about the “way” in which $\mathbb{R}$ is non-enumerable.

**Theorem card-arithmetic.6.** $|\mathbb{R}| = 2^\omega$

*Proof skeleton.* There are plenty of ways to prove this. The most straightforward is to argue that $\wp(\omega) \preceq \mathbb{R}$ and $\mathbb{R} \preceq \wp(\omega)$, and then use Schröder-Bernstein to infer that $\mathbb{R} \approx \wp(\omega)$, and Lemma card-arithmetic.4 to infer that $|\mathbb{R}| = 2^\omega$. We leave it as an (illuminating) exercise to define injections $f : \wp(\omega) \to \mathbb{R}$ and $g : \mathbb{R} \to \wp(\omega)$.

**Problem card-arithmetic.3.** Complete the proof of Theorem card-arithmetic.6, by showing that $\wp(\omega) \preceq \mathbb{R}$ and $\mathbb{R} \preceq \wp(\omega)$.