card-arithmetic.1 Some Simplification with Cardinal Exponentiation

Whilst defining $<$ was a little involved, the upshot is a useful result concerning cardinal addition and multiplication, $\leq$. Transfinite exponentiation, however, cannot be simplified so straightforwardly. To explain why, we start with a result which extends a familiar pattern from the finitary case (though its proof is at a high level of abstraction):

**Proposition card-arithmetic.1.** $a^{b+c} = a^b \otimes a^c$ and $(a^b)^c = a^{b\otimes c}$, for any cardinals $a, b, c$.

**Proof.**
For the first claim, consider a function $f: (b \mathbin{\uplus} c) \to a$. Now “split this”, by defining $f_b(\beta) = f(\beta, 0)$ for each $\beta \in b$, and $f_c(\gamma) = f(\gamma, 1)$ for each $\gamma \in c$. The map $f \mapsto (f_b \times f_c)$ is a bijection $b \mathbin{\uplus} c \to (b \mathbin{\uplus} c) \to b \otimes c$.

For the second claim, consider a function $f: c \to (b \mathbin{\uplus} c)$; so for each $\gamma \in c$ we have some function $f(\gamma): b \to a$. Now define $f^* (\beta, \gamma) = (f(\gamma))(\beta)$ for each $(\beta, \gamma) \in b \times c$. The map $f \mapsto f^*$ is a bijection $c \to b \otimes c$.

Now, what we would like is an easy way to compute $a^b$ when we are dealing with infinite cardinals. Here is a nice step in this direction:

**Proposition card-arithmetic.2.** If $2 \leq a \leq b$ and $b$ is infinite, then $a^b = 2^b$

**Proof.**

\[
2^b \leq a^b, \text{ as } 2 \leq a \\
\leq (2^a)^b, \text{ by } \leq \\
= 2^{a \otimes b}, \text{ by Proposition card-arithmetic.1} \\
= 2^b, \text{ by } \leq
\]

We should not really expect to be able to simplify this any further, since $b < 2^b$ by $\leq$. However, this does not tell us what to say about $a^b$ when $b < a$. Of course, if $b$ is finite, we know what to do.

**Proposition card-arithmetic.3.** If $a$ is infinite and $n \in \omega$ then $a^n = a$

**Proof.** $a^n = a \otimes a \otimes \ldots \otimes a = a$, by $\leq$.

Additionally, in some other cases, we can control the size of $a^b$:

**Proposition card-arithmetic.4.** If $2 \leq b \leq a \leq 2^b$ and $b$ is infinite, then $a^b = 2^b$

**Proof.** $2^b \leq a^b \leq (2^b)^b = 2^{b \otimes b} = 2^b$, reasoning as in Proposition card-arithmetic.2.
But, beyond this point, things become rather more subtle.

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Bibliography