card-arithmetic.1 Some Simplification with Cardinal Exponentiation

sth:card-arithmetic:expotough:sec

Whilst defining $\lhd$ was a little involved, the upshot is a useful result concerning cardinal addition and multiplication, **??**. Transfinite exponentiation, however, cannot be simplified so straightforwardly. To explain why, we start with a result which extends a familiar pattern from the finitary case (though its proof at quite a high level of abstraction):

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Proposition card-arithmetic.1. $\mathfrak{a}^{\mathfrak{b}\oplus\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}} \otimes \mathfrak{a}^{\mathfrak{c}}$ *and* $(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}\otimes\mathfrak{c}}$*, for any cardinals* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$*.*

Proof. For the first claim, consider a function $f\colon (\mathfrak{b} \sqcup \mathfrak{c}) \to \mathfrak{a}$. Now "split this", by defining $f_{\mathfrak{b}}(\beta) = f(\beta, 0)$ for each $\beta \in \mathfrak{b}$, and $f_{\mathfrak{c}}(\gamma) = f(\gamma, 1)$ for each $\gamma \in \mathfrak{c}$. The map $f \mapsto (f_{\mathfrak{b}} \times f_{\mathfrak{c}})$ is a bijection ${}^{\mathfrak{b}\sqcup\mathfrak{c}}\mathfrak{a} \to ({}^{\mathfrak{b}}\mathfrak{a} \times {}^{\mathfrak{c}}\mathfrak{a})$.

For the second claim, consider a function $f\colon \mathfrak{c} \to ({}^{\mathfrak{b}}\mathfrak{a})$; so for each $\gamma \in \mathfrak{c}$ we have some function $f(\gamma)\colon \mathfrak{b} \to \mathfrak{a}$. Now define $f^*(\beta, \gamma) = (f(\gamma))(\beta)$ for each $\langle \beta, \gamma \rangle \in \mathfrak{b} \times \mathfrak{c}$. The map $f \mapsto f^*$ is a bijection ${}^{\mathfrak{c}}({}^{\mathfrak{b}}\mathfrak{a}) \to {}^{\mathfrak{b}\otimes\mathfrak{c}}\mathfrak{a}$. □

Now, what we would *like* is an easy way to compute $\mathfrak{a}^{\mathfrak{b}}$ when we are dealing with infinite cardinals. Here is a nice step in this direction:

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Proposition card-arithmetic.2. *If* $2 \le \mathfrak{a} \le \mathfrak{b}$ *and* $\mathfrak{b}$ *is infinite, then* $\mathfrak{a}^{\mathfrak{b}} = 2^{\mathfrak{b}}$

Proof.

$$\begin{aligned} 2^{\mathfrak{b}} &\le \mathfrak{a}^{\mathfrak{b}}, \text{ as } 2 \le \mathfrak{a} \\ &\le (2^{\mathfrak{a}})^{\mathfrak{b}}, \text{ by ??} \\ &= 2^{\mathfrak{a}\otimes\mathfrak{b}}, \text{ by Proposition card-arithmetic.1} \\ &= 2^{\mathfrak{b}}, \text{ by ??} \end{aligned}$$

□

We should not really expect to be able to simplify this any further, since $\mathfrak{b} < 2^{\mathfrak{b}}$ by **??**. However, this does not tell us what to say about $\mathfrak{a}^{\mathfrak{b}}$ when $\mathfrak{b} < \mathfrak{a}$. Of course, if $\mathfrak{b}$ is *finite*, we know what to do.

Proposition card-arithmetic.3. *If* $\mathfrak{a}$ *is infinite and* $n \in \omega$ *then* $\mathfrak{a}^n = \mathfrak{a}$

Proof. $\mathfrak{a}^n = \mathfrak{a} \otimes \mathfrak{a} \otimes \ldots \otimes \mathfrak{a} = \mathfrak{a}$, by $n - 1$ applications of **??**. □

Additionally, in certain other cases, we can control the size of $\mathfrak{a}^{\mathfrak{b}}$:

Proposition card-arithmetic.4. *If* $2 \le \mathfrak{b} < \mathfrak{a} \le 2^{\mathfrak{b}}$ *and* $\mathfrak{b}$ *is infinite, then* $\mathfrak{a}^{\mathfrak{b}} = 2^{\mathfrak{b}}$

Proof. $2^{\mathfrak{b}} \le \mathfrak{a}^{\mathfrak{b}} \le (2^{\mathfrak{b}})^{\mathfrak{b}} = 2^{\mathfrak{b}\otimes\mathfrak{b}} = 2^{\mathfrak{b}}$, reasoning as in Proposition card-arithmetic.2. □

But, beyond this point, things become rather more subtle.

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Bibliography