Some Simplification with Cardinal Exponentiation

Whilst defining $<$ was a little involved, the upshot is a useful result concerning cardinal addition and multiplication, $\text{??}$. Transfinite exponentiation, however, cannot be simplified so straightforwardly. To explain why, we start with a result which extends a familiar pattern from the finitary case (though its proof at quite a high level of abstraction):

**Proposition card-arithmetic.1.** $a^{b+c} = a^b \otimes a^c$ and $(a^b)^c = a^{b\otimes c}$, for any cardinals $a, b, c$.

**Proof.** For the first claim, consider a function $f : (b \sqcup c) \to a$. Now “split this”, by defining $f_b(\beta) = f(\beta, 0)$ for each $\beta \in b$, and $f_c(\gamma) = f(\gamma, 1)$ for each $\gamma \in c$. The map $f \mapsto (f_b \times f_c)$ is a bijection $b \sqcup c \to (b^a \times c^a)$.

For the second claim, consider a function $f : c \to (b^a)$; so for each $\gamma \in c$ we have some function $f(\gamma) : b \to a$. Now define $f^\ast(\beta, \gamma) = (f(\gamma))(\beta)$ for each $(\beta, \gamma) \in b \times c$. The map $f \mapsto f^\ast$ is a bijection $c(b^a) \to b^\otimes c^a$. \hfill \Box

Now, what we would like is an easy way to compute $a^b$ when we are dealing with infinite cardinals. Here is a nice step in this direction:

**Proposition card-arithmetic.2.** If $2 \leq a \leq b$ and $b$ is infinite, then $a^b = 2^b$

**Proof.**

$2^b \leq a^b$, as $2 \leq a$

$\leq (2^a)^b$, by $\text{??}$

$= 2^{a\otimes b}$, by Proposition card-arithmetic.1

$= 2^b$, by $\text{??}$ \hfill \Box

We should not really expect to be able to simplify this any further, since $b < 2^b$ by $\text{??}$. However, this does not tell us what to say about $a^b$ when $b < a$. Of course, if $b$ is finite, we know what to do.

**Proposition card-arithmetic.3.** If $a$ is infinite and $n \in \omega$ then $a^n = a$

**Proof.** $a^n = a \otimes a \otimes \ldots \otimes a = a$, by $n - 1$ applications of $\text{??}$. \hfill \Box

Additionally, in certain other cases, we can control the size of $a^b$:

**Proposition card-arithmetic.4.** If $2 \leq b < a \leq 2^b$ and $b$ is infinite, then $a^b = 2^b$

**Proof.** $2^b \leq a^b \leq (2^b)^b = 2^{b\otimes b} = 2^b$, reasoning as in Proposition card-arithmetic.2. \hfill \Box
But, beyond this point, things become rather more subtle.

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Bibliography