Chapter udf

Cardinal Arithmetic

card-arithmetic.1 Defining the Basic Operations

sth:card-arithmetic:opps:sec Since we do not need to keep track of order, cardinal arithmetic is rather easier to define than ordinal arithmetic. We will define addition, multiplication, and exponentiation simultaneously.

Definition card-arithmetic.1. When $\mathfrak{a}$ and $\mathfrak{b}$ are cardinals:

$$\begin{aligned} \mathfrak{a} \oplus \mathfrak{b} &:= |\mathfrak{a} \sqcup \mathfrak{b}| \\ \mathfrak{a} \otimes \mathfrak{b} &:= |\mathfrak{a} \times \mathfrak{b}| \\ \mathfrak{a}^{\mathfrak{b}} &:= \left|{}^{\mathfrak{b}}\mathfrak{a}\right| \end{aligned}$$

where ${}^{X}Y = \{f : f \text{ is a function } X \to Y\}$. (It is easy to show that ${}^{X}Y$ exists for any sets X and Y; we leave this as an exercise.)

Problem card-arithmetic.1. Prove in $\mathbf{Z}^{-}$ that ${}^{X}Y$ exists for any sets X and Y. Working in **ZF**, compute $\text{rank}({}^{X}Y)$ from $\text{rank}(X)$ and $\text{rank}(Y)$, in the manner of **??**.

It might help to explain this definition. Concerning addition: this uses the notion of disjoint sum, $\sqcup$, as defined in **??**; and it is easy to see that this definition gives the right verdict for finite cases. Concerning multiplication: **??** tells us that if A has n members and B has m members then $A \times B$ has $n \cdot m$ members, so our definition simply generalises the idea to transfinite multiplication. Exponentiation is similar: we are simply generalising the thought from the finite to the transfinite. Indeed, in certain ways, transfinite cardinal arithmetic looks much more like "ordinary" arithmetic than does transfinite ordinal arithmetic:

sth:card-arithmetic:opps:cardplustimescommute **Proposition card-arithmetic.2.** *$\oplus$ and $\otimes$ are commutative and associative.*

Proof. For commutativity, by **??** it suffices to observe that $(\mathfrak{a} \sqcup \mathfrak{b}) \approx (\mathfrak{b} \sqcup \mathfrak{a})$ and $(\mathfrak{a} \times \mathfrak{b}) \approx (\mathfrak{b} \times \mathfrak{a})$. We leave associativity as an exercise. □

Problem card-arithmetic.2. Prove that $\oplus$ and $\otimes$ are associative.

Proposition card-arithmetic.3. *A is infinite iff $|A| \oplus 1 = 1 \oplus |A| = |A|$.*

Proof. As in **??**, from **??** and **??**. □

This explains why we need to use different symbols for ordinal versus cardinal addition/multiplication: these are genuinely *different* operations. This next pair of results shows that ordinal versus cardinal exponentiation are also different operations. (Recall that **??** entails that $2 = \{0, 1\}$):

Lemma card-arithmetic.4. *$|\wp(A)| = 2^{|A|}$, for any A.* sth:card-arithmetic:opps: lem:SizePowerset2Exp

Proof. For each subset $B \subseteq A$, let $\chi_B \in {}^{A}2$ be given by:

$$\chi_B(x) := \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Now let $f(B) = \chi_B$; this defines a bijection $f \colon \wp(A) \to {}^{A}2$. So $\wp(A) \approx {}^{A}2$. Hence $\wp(A) \approx {}^{|A|}2$, so that $|\wp(A)| = \left|{}^{|A|}2\right| = 2^{|A|}$. □

This snappy proof essentially subsumes the discussion of **??**. There, we showed how to "reduce" the uncountability of $\wp(\omega)$ to the uncountability of the set of infinite binary strings, $\mathbb{B}^\omega$. In effect, $\mathbb{B}^\omega$ is just ${}^{\omega}2$; and the preceding proof showed that the reasoning we went through in **??** will go through using any set A in place of ω. The result also yields a quick fact about cardinal exponentiation:

Corollary card-arithmetic.5. *$\mathfrak{a} < 2^{\mathfrak{a}}$ for any cardinal $\mathfrak{a}$.* sth:card-arithmetic:opps: cantorcor

Proof. From Cantor's Theorem (**??**) and Lemma card-arithmetic.4. □

So $\omega < 2^\omega$. But note: this is a result about *cardinal* exponentiation. It should be contrasted with *ordinal* exponentation, since in the latter case $\omega = 2^{(\omega)}$ (see **??**).

Whilst we are on the topic of cardinal exponentiation, we can also be a bit more precise about the "way" in which $\mathbb{R}$ is non-enumerable.

Theorem card-arithmetic.6. $|\mathbb{R}| = 2^\omega$ sth:card-arithmetic:opps: continuumis2aleph0

Proof skeleton. There are plenty of ways to prove this. The most straightforward is to argue that $\wp(\omega) \preceq \mathbb{R}$ and $\mathbb{R} \preceq \wp(\omega)$, and then use Schröder-Bernstein to infer that $\mathbb{R} \approx \wp(\omega)$, and Lemma card-arithmetic.4 to infer that $|\mathbb{R}| = 2^\omega$. We leave it as an (illuminating) exercise for the reader to define injections $f \colon \wp(\omega) \to \mathbb{R}$ and $g \colon \mathbb{R} \to \wp(\omega)$. □

Problem card-arithmetic.3. Complete the proof of Theorem card-arithmetic.6, by showing that $\wp(\omega) \preceq \mathbb{R}$ and $\mathbb{R} \preceq \wp(\omega)$.

card-arithmetic.2 Simplifying Addition and Multiplication

sth:card-arithmetic:simp:sec

It turns out that transfinite cardinal addition and multiplication is *extremely* easy. This follows from the fact that cardinals are (certain) ordinals, and so well-ordered, and so can be manipulated in a certain way. Showing this, though, is *not* so easy. To start, we need a tricksy definition:

Definition card-arithmetic.7. We define a *canonical ordering*, $\lhd$, on pairs of ordinals, by stipulating that $\langle \alpha_1, \alpha_2 \rangle \lhd \langle \beta_1, \beta_2 \rangle$ iff either:

1. $\max(\alpha_1, \alpha_2) < \max(\beta_1, \beta_2)$; or
2. $\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)$ and $\alpha_1 < \beta_1$; or
3. $\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)$ and $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$

Lemma card-arithmetic.8. *$\langle \alpha \times \alpha, \lhd \rangle$ is a well-order, for any ordinal α.*[1]

Proof. Evidently $\lhd$ is connected on $\alpha \times \alpha$. For suppose that neither $\langle \alpha_1, \alpha_2 \rangle$ nor $\langle \beta_1, \beta_2 \rangle$ is $\lhd$-less than the other. Then $\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2)$ and $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, so that $\langle \alpha_1, \alpha_2 \rangle = \langle \beta_1, \beta_2 \rangle$.

To show well-ordering, let $X \subseteq \alpha \times \alpha$ be non-empty. Since α is an ordinal, some δ is the least member of $\{\max(\gamma_1, \gamma_2) : \langle \gamma_1, \gamma_2 \rangle \in X\}$. Now discard all pairs from $\{\langle \gamma_1, \gamma_2 \rangle \in X : \max(\gamma_1, \gamma_2) = \delta\}$ except those with least first coordinate; from among these, the pair with least second coordinate is the $\lhd$-least element of X. □

Now for a teensy, simple observation:

sth:card-arithmetic:simp:simplecardproduct

Proposition card-arithmetic.9. *If $\alpha \approx \beta$, then $\alpha \times \alpha \approx \beta \times \beta$.*

Proof. Just let $f : \alpha \to \beta$ induce $\langle \gamma_1, \gamma_2 \rangle \mapsto \langle f(\gamma_1), f(\gamma_2) \rangle$. □

And now we will put all this to work, in proving a crucial lemma:

sth:card-arithmetic:simp:alphatimesalpha

Lemma card-arithmetic.10. *$\alpha \approx \alpha \times \alpha$, for any infinite ordinal α*

Proof. For reductio, let α be the least infinite ordinal for which this is false. **??** shows that $\omega \approx \omega \times \omega$, so $\omega \in \alpha$. Moreover, α is a cardinal: suppose otherwise, for reductio; then $|\alpha| \in \alpha$, so that $|\alpha| \approx |\alpha| \times |\alpha|$, by hypothesis; and $|\alpha| \approx \alpha$ by definition; so that $\alpha \approx \alpha \times \alpha$ by Proposition card-arithmetic.9.

Now, for each $\langle \gamma_1, \gamma_2 \rangle \in \alpha \times \alpha$, consider the segment:

$$\mathrm{Seg}(\gamma_1, \gamma_2) = \{\langle \delta_1, \delta_2 \rangle \in \alpha \times \alpha : \langle \delta_1, \delta_2 \rangle \lhd \langle \gamma_1, \gamma_2 \rangle\}$$

[1] Cf. the naughtiness described in the footnote to **??**.

Let $\gamma = \max(\gamma_1, \gamma_2)$. When γ is infinite, observe:

$$\begin{aligned} \text{Seg}(\gamma_1, \gamma_2) &\precsim ((\gamma + 1) \cdot (\gamma + 1)), \text{ by the first clause defining } \lhd \\ &\approx (\gamma \cdot \gamma), \text{ by ?? and Proposition card-arithmetic.9} \\ &\approx \gamma, \text{ by the induction hypothesis} \\ &\prec \alpha, \text{ since } \alpha \text{ is a cardinal} \end{aligned}$$

So $\text{ord}(\alpha \times \alpha, \lhd) \leq \alpha$, and hence $\alpha \times \alpha \preceq \alpha$. Since of course $\alpha \preceq \alpha \times \alpha$, the result follows by Schröder-Bernstein. □

Finally, we get to our simplifying result:

Theorem card-arithmetic.11. *If $\mathfrak{a}, \mathfrak{b}$ are infinite cardinals, $\mathfrak{a} \otimes \mathfrak{b} = \mathfrak{a} \oplus \mathfrak{b} = max(\mathfrak{a}, \mathfrak{b})$.* sth:card-arithmetic:simp:cardplustimesmax

Proof. Without loss of generality, suppose $\mathfrak{a} = \max(\mathfrak{a}, \mathfrak{b})$. Then invoking Lemma card-arithmetic.10, $\mathfrak{a} \otimes \mathfrak{a} = \mathfrak{a} \leq \mathfrak{a} \oplus \mathfrak{b} \leq \mathfrak{a} \oplus \mathfrak{a} \leq \mathfrak{a} \otimes \mathfrak{a}$. □

Similarly, if $\mathfrak{a}$ is infinite, an $\mathfrak{a}$-sized union of $\leq \mathfrak{a}$-sized sets has size $\leq \mathfrak{a}$:

Proposition card-arithmetic.12. *Let $\mathfrak{a}$ be an infinite cardinal. For each ordinal $\beta \in \mathfrak{a}$, let X_β be a set with $|X_\beta| \leq \mathfrak{a}$. Then $\left|\bigcup_{\beta \in \mathfrak{a}} X_\beta\right| \leq \mathfrak{a}$.* sth:card-arithmetic:simp:kappaunionkappasize

Proof. For each $\beta \in \mathfrak{a}$, fix an injection $f_\beta \colon X_\beta \to \mathfrak{a}$. Define an injection $g \colon \bigcup_{\beta \in \mathfrak{a}} X_\beta \to \mathfrak{a} \times \mathfrak{a}$ by $g(v) = \langle \beta, f_\beta(v) \rangle$, where $v \in X_\beta$ and $v \notin X_\gamma$ for any $\gamma \in \beta$. Now $\bigcup_{\beta \in \mathfrak{a}} X_\beta \preceq \mathfrak{a} \times \mathfrak{a} \approx \mathfrak{a}$ by Theorem card-arithmetic.11. □

card-arithmetic.3 Some Simplification with Cardinal Exponentiation

sth:card-arithmetic:expotough:sec

Whilst defining $\lhd$ was a little involved, the upshot is a useful result concerning cardinal addition and multiplication, Theorem card-arithmetic.11. Transfinite exponentiation, however, cannot be simplified so straightforwardly. To explain why, we start with a result which extends a familiar pattern from the finitary case (though its proof at quite a high level of abstraction):

Proposition card-arithmetic.13. $\mathfrak{a}^{\mathfrak{b} \oplus \mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}} \otimes \mathfrak{a}^{\mathfrak{c}}$ *and* $(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b} \otimes \mathfrak{c}}$*, for any cardinals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$.* sth:card-arithmetic:expotough:simplecardexpo

Proof. For the first claim, consider a function $f \colon (\mathfrak{b} \sqcup \mathfrak{c}) \to \mathfrak{a}$. Now "split this", by defining $f_{\mathfrak{b}}(\beta) = f(\beta, 0)$ for each $\beta \in \mathfrak{b}$, and $f_{\mathfrak{c}}(\gamma) = f(\gamma, 1)$ for each $\gamma \in \mathfrak{c}$. The map $f \mapsto (f_{\mathfrak{b}} \times f_{\mathfrak{c}})$ is a bijection ${}^{\mathfrak{b} \sqcup \mathfrak{c}}\mathfrak{a} \to ({}^{\mathfrak{b}}\mathfrak{a} \times {}^{\mathfrak{c}}\mathfrak{a})$.

For the second claim, consider a function $f \colon \mathfrak{c} \to ({}^{\mathfrak{b}}\mathfrak{a})$; so for each $\gamma \in \mathfrak{c}$ we have some function $f(\gamma) \colon \mathfrak{b} \to \mathfrak{a}$. Now define $f^*(\beta, \gamma) = (f(\gamma))(\beta)$ for each $\langle \beta, \gamma \rangle \in \mathfrak{b} \times \mathfrak{c}$. The map $f \mapsto f^*$ is a bijection ${}^{\mathfrak{c}}({}^{\mathfrak{b}}\mathfrak{a}) \to {}^{\mathfrak{b} \otimes \mathfrak{c}}\mathfrak{a}$. □

Now, what we would *like* is an easy way to compute $\mathfrak{a}^{\mathfrak{b}}$ when we are dealing with infinite cardinals. Here is a nice step in this direction:

sth:card-arithmetic:expotough: cardexpo2reduct

Proposition card-arithmetic.14. *If $2 \leq \mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b}$ is infinite, then $\mathfrak{a}^{\mathfrak{b}} = 2^{\mathfrak{b}}$*

Proof.

$$\begin{aligned} 2^{\mathfrak{b}} &\leq \mathfrak{a}^{\mathfrak{b}}, \text{ as } 2 \leq \mathfrak{a} \\ &\leq (2^{\mathfrak{a}})^{\mathfrak{b}}, \text{ by Lemma card-arithmetic.4} \\ &= 2^{\mathfrak{a} \otimes \mathfrak{b}}, \text{ by Proposition card-arithmetic.13} \\ &= 2^{\mathfrak{b}}, \text{ by Theorem card-arithmetic.11} \end{aligned}$$

□

We should not really expect to be able to simplify this any further, since $\mathfrak{b} < 2^{\mathfrak{b}}$ by Lemma card-arithmetic.4. However, this does not tell us what to say about $\mathfrak{a}^{\mathfrak{b}}$ when $\mathfrak{b} < \mathfrak{a}$. Of course, if $\mathfrak{b}$ is *finite*, we know what to do.

Proposition card-arithmetic.15. *If $\mathfrak{a}$ is infinite and $n \in \omega$ then $\mathfrak{a}^n = \mathfrak{a}$*

Proof. $\mathfrak{a}^n = \mathfrak{a} \otimes \mathfrak{a} \otimes \ldots \otimes \mathfrak{a} = \mathfrak{a}$, by $n - 1$ applications of Theorem card-arithmetic.11. □

Additionally, in certain other cases, we can control the size of $\mathfrak{a}^{\mathfrak{b}}$:

Proposition card-arithmetic.16. *If $2 \leq \mathfrak{b} < \mathfrak{a} \leq 2^{\mathfrak{b}}$ and $\mathfrak{b}$ is infinite, then $\mathfrak{a}^{\mathfrak{b}} = 2^{\mathfrak{b}}$*

Proof. $2^{\mathfrak{b}} \leq \mathfrak{a}^{\mathfrak{b}} \leq (2^{\mathfrak{b}})^{\mathfrak{b}} = 2^{\mathfrak{b} \otimes \mathfrak{b}} = 2^{\mathfrak{b}}$, reasoning as in Proposition card-arithmetic.14. □

But, beyond this point, things become rather more subtle.

card-arithmetic.4 The Continuum Hypothesis

sth:card-arithmetic:ch: sec

The previous result hints (correctly) that cardinal exponentiation would be quite *easy*, if infinite cardinals are guaranteed to "play straightforwardly" with powers of 2, i.e., (by Lemma card-arithmetic.4) with taking powersets. But we cannot assume that infinite cardinals *do* play nicely with powersets. This section is dedicated to explaining all of this. (Although, to be honest, it's more of a *gesture* in the direction of something fascinating.)

We will start by introducing some nice notation.

Definition card-arithmetic.17. Where $\mathfrak{a}^{\oplus}$ is the least cardinal strictly greater than $\mathfrak{a}$, we define two infinite sequences:

$$\begin{aligned} \aleph_0 &:= \omega & \beth_0 &:= \omega \\ \aleph_{\alpha+1} &:= (\aleph_\alpha)^{\oplus} & \beth_{\alpha+1} &:= 2^{\beth_\alpha} \\ \aleph_\alpha &:= \bigcup_{\beta<\alpha} \aleph_\beta & \beth_\alpha &:= \bigcup_{\beta<\alpha} \beth_\beta & &\text{when } \alpha \text{ is a limit ordinal.} \end{aligned}$$

The definition of $\mathfrak{a}^{\oplus}$ is in order, since **??** tells us that, for each cardinal $\mathfrak{a}$, there is some cardinal greater than $\mathfrak{a}$, and Transfinite Induction guarantees that there is a *least* cardinal greater than $\mathfrak{a}$. The rest of the definition of $\mathfrak{a}$ is provided by transfinite recursion.

Cantor introduced this "$\aleph$" notation; this is *aleph*, the first letter in the Hebrew alphabet and the first letter in the Hebrew word for "infinite". Peirce introduced the "$\beth$" notation; this is *beth*, which is the second letter in the Hebrew alphabet.[2] Now, these notations provide us with infinite cardinals.

Proposition card-arithmetic.18. *Both $\aleph_\alpha$ and $\beth_\alpha$ are cardinals, for every ordinal α.*

Proof. Both results hold by a simple transfinite induction. $\aleph_0 = \beth_0 = \omega$ is a cardinal by **??**. Assuming $\aleph_\alpha$ and $\beth_\alpha$ are both cardinals, $\aleph_{\alpha+1}$ and $\beth_{\alpha+1}$ are explicitly defined as cardinals. And the union of a set of cardinals is a cardinal, by **??**. □

Moreover, every infinite cardinal is an $\aleph$:

Proposition card-arithmetic.19. *If $\mathfrak{a}$ is an infinite cardinal, then $\mathfrak{a} = \aleph_\gamma$ for some γ.*

Proof. By transfinite induction on cardinals. For induction, suppose that if $\mathfrak{b} < \mathfrak{a}$ then $\mathfrak{b} = \aleph_{\gamma_\mathfrak{b}}$. If $\mathfrak{a} = \mathfrak{b}^{\oplus}$ for some $\mathfrak{b}$, then $\mathfrak{a} = \aleph_{\gamma_\mathfrak{b}}^{\oplus} = \aleph_{\gamma_\mathfrak{b}+1}$. If $\mathfrak{a}$ is not the successor of any cardinal, then since cardinals are ordinals $\mathfrak{a} = \bigcup_{\mathfrak{b}<\mathfrak{a}} \mathfrak{b} = \bigcup_{\mathfrak{b}<\mathfrak{a}} \aleph_{\gamma_\mathfrak{b}}$, so $\mathfrak{a} = \aleph_\gamma$ where $\gamma = \bigcup_{\mathfrak{b}<\mathfrak{a}} \gamma_\mathfrak{b}$. □

Since every infinite cardinal is an $\aleph$, this prompts us to ask: is every infinite cardinal a $\beth$? Certainly if that *were* the case, then the infinite cardinals would "play straightforwardly" with the operation of taking powersets. Indeed, we would have the following:

General Continuum Hypothesis (GCH). $\aleph_\alpha = \beth_\alpha$, for all α.

Moreover, if GCH held, then we could make some considerable simplifications with cardinal exponentiation. In particular, we could show that when $\mathfrak{b} < \mathfrak{a}$, the value of $\mathfrak{a}^\mathfrak{b}$ is trapped by $\mathfrak{a} \leq \mathfrak{a}^\mathfrak{b} \leq \mathfrak{a}^{\oplus}$. We could then go on to give precise conditions which determine which of the two possibilities obtains (i.e., whether $\mathfrak{a} = \mathfrak{a}^\mathfrak{b}$ or $\mathfrak{a}^\mathfrak{b} = \mathfrak{a}^{\oplus}$).[3]

But GCH is a *hypothesis*, not a *theorem*. In fact, Gödel (1938) proved that if **ZFC** is consistent, then so is **ZFC** + GCH. But it later turned out that we can equally add ¬GCH to **ZFC**. Indeed, consider the simplest non-trivial *instance* of GCH, namely:

[2] Peirce used this notation in a letter to Cantor of December 1900. Unfortunately, Peirce also gave a bad argument there that $\beth_\alpha$ does not exist for $\alpha \geq \omega$.

[3] The condition is dictated by *cofinality*.

Continuum Hypothesis (CH). $\aleph_1 = \beth_1$.

Cohen (1963) proved that if **ZFC** is consistent then so is **ZFC** + ¬CH.

The Continuum Hypothesis is so-called, since "the continuum" is another name for the real line, $\mathbb{R}$. Theorem card-arithmetic.6 tells us that $|\mathbb{R}| = \beth_1$. So the Continuum Hypothesis states that there is no cardinal between the cardinality of the natural numbers, $\aleph_0 = \beth_0$, and the cardinality of the continuum, $\beth_1$.

Given the *independence* of (G)CH from **ZFC**, what should say about their *truth*? Well, there is *much* to say. Indeed, and much fertile recent work in set theory has been directed at investigating these issues. But two quick points are certainly worth emphasising.

First: it does not *immediately* follow from these formal independence results that either GCH or CH is *indeterminate* in truth value. After all, maybe we just need to add more axioms, which strike us as natural, and which will settle the question one way or another. Gödel himself suggested that this was the right response.

Second: the independence of CH from **ZFC** is certainly *striking*, but it is certainly not *incredible* (in the literal sense). The point is simply that, for all **ZFC** tells us, moving from cardinals to their successors may involve a less blunt tool than simply taking powersets.

With those two observations made, if you want to know more, you will now have to turn to the various philosophers and mathematicians with horses in the race. (Though you may want to start with the very nice discussion in Potter 2004, §15.6.)

card-arithmetic.5 $\aleph$-Fixed Points

sth:card-arithmetic:fix:sec

In **??**, we suggested that Replacement stands in need of justification, because it forces the hierarchy to be rather tall. Having done some cardinal arithmetic, we can give a little illustration of the height of the hierarchy.

Evidently $0 < \aleph_0$, and $1 < \aleph_1$, and $2 < \aleph_2 \ldots$ and, indeed, the difference in size only gets *bigger* with every step. So it is tempting to conjecture that $\kappa < \aleph_\kappa$ for every ordinal κ.

But this conjecture is *false*, given **ZFC**. In fact, we can easily prove that there are $\aleph$-*fixed-points*, i.e., cardinals κ such that $\kappa = \aleph_\kappa$.

sth:card-arithmetic:fix:alephfixed

Proposition card-arithmetic.20. *There is an $\aleph$-fixed-point.*

Proof. Using recursion, define:

$$\kappa_0 = 0$$
$$\kappa_{n+1} = \aleph_{\kappa_n}$$
$$\kappa = \bigcup_{n<\omega} \kappa_n$$

Now κ is a cardinal by **??**. But now:

$$\kappa = \bigcup_{n<\omega} \kappa_{n+1} = \bigcup_{n<\omega} \aleph_{\kappa_n} = \bigcup_{\alpha<\kappa} \aleph_\alpha = \aleph_\kappa$$

□

Boolos once wrote an article about exactly the $\aleph$-fixed-point we just constructed. After noting the existence of κ, at the start of his article, he said:

> [κ is] a *pretty big* number, by the lights of those with no previous exposure to set theory, so big, it seems to me, that it calls into question the truth of any theory, one of whose assertions is the claim that there are at least κ objects. (Boolos, 2000, p. 257)

And he ultimately concluded his paper by asking:

> [do] we suspect that, however it may have been at the beginning of the story, by the time we have come thus far the wheels are spinning and we are no longer listening to a description of anything that is the case? (Boolos, 2000, p. 268)

If we have, indeed, outrun "anything that is the case", then we must point the finger of blame directly at Replacement. For it is this axiom which allows our proof to work. In which case, one assumes, Boolos would need to revisit the claim he made, a few decades earlier, that Replacement has "no undesirable" consequences (see **??**).

But is the existence of κ so bad? It might help, here, to consider Russell's *Tristram Shandy paradox*. Tristram Shandy documents his life in his diary, but it takes him a year to record a single day. With every passing year, Tristram falls further and further behind: after one year, he has recorded only one day, and has lived 364 days unrecorded days; after two years, he has only recorded two days, and has lived 728 unrecorded days; after three years, he has only recorded three days, and lived 1092 unrecorded days ...[4] Still, if Tristram is *immortal*, Tristram will manage to record every day, for he will record the nth day on the nth year of his life. And so, "at the end of time", Tristram will have a complete diary.

Now: why is this so different from the thought that α is smaller than $\aleph_\alpha$—and indeed, increasingly, desperately smaller—up until κ, at which point, we catch up, and $\kappa = \aleph_\kappa$?

Setting that aside, and assuming we accept **ZFC**, let's close with a little more fun concerning fixed-point constructions. The next three results establish, intuitively, that there is a (non-trivial) point at which the hierarchy is as wide as it is tall:

Proposition card-arithmetic.21. *There is a $\beth$-fixed-point, i.e., a κ such that $\kappa = \beth_\kappa$* *sth:card-arithmetic:fix:bethfixed*

[4] Forgetting about leap years.

Proof. As in Proposition card-arithmetic.20, using "ℶ" in place of "ℵ". ☐

sth:card-arithmetic:fix:stagesize **Proposition card-arithmetic.22.** $|V_{\omega+\alpha}| = \beth_\alpha$. *If* $\omega \cdot \omega \leq \alpha$, *then* $|V_\alpha| = \beth_\alpha$.

Proof. The first claim holds by a simple transfinite induction. The second claim follows, since if $\omega \cdot \omega \leq \alpha$ then $\omega + \alpha = \alpha$. To establish this, we use facts about ordinal arithmetic from **??**. First note that $\omega \cdot \omega = \omega \cdot (1 + \omega) = (\omega \cdot 1) + (\omega \cdot \omega) = \omega + (\omega \cdot \omega)$. Now if $\omega \cdot \omega \leq \alpha$, i.e., $\alpha = (\omega \cdot \omega) + \beta$ for some β, then $\omega + \alpha = \omega + ((\omega \cdot \omega) + \beta) = (\omega + (\omega \cdot \omega)) + \beta = (\omega \cdot \omega) + \beta = \alpha$. ☐

Corollary card-arithmetic.23. *There is a* κ *such that* $|V_\kappa| = \kappa$.

Proof. Let κ be a ℶ-fixed point, as given by Proposition card-arithmetic.21. Clearly $\omega \cdot \omega < \kappa$. So $|V_\kappa| = \beth_\kappa = \kappa$ by Proposition card-arithmetic.22. ☐

There are as many stages beneath V_κ as there are elements of V_κ. Intuitively, then, V_κ is as wide as it is tall. This is very Tristram-Shandy-esque: we move from one stage to the next by taking *powersets*, thereby making our hierarchy *much* bigger with each step. But, "in the end", i.e., at stage κ, the hierarchy's width catches up with its height.

One might ask: *How often does the hierarchy's width match its height?* The answer is: *As often as there are ordinals.* But this needs a little explanation.

We define a term τ as follows. For any A, let $\tau_0(A) = |A|$, let $\tau_{n+1}(A) = \beth_{\kappa_n}$, and let $\tau(A) = \bigcup_{n<\omega} \kappa_n$. As in Proposition card-arithmetic.21, $\tau(A)$ is a ℶ-fixed point for any A, and trivially $|A| < \tau(A)$. So now consider this recursive definition of ℸ-fixed-points:[5]

$$\begin{aligned} \daleth_0 &= 0 \\ \daleth_{\alpha+1} &= \tau(\daleth_\alpha) \\ \daleth_\beta &= \bigcup_{\alpha<\beta} \daleth_\alpha && \text{if } \beta \text{ is a limit} \end{aligned}$$

The construction is defined for all ordinals. Intuitively, then, ℸ is an injection from the ordinals to ℶ-fixed points. And, exactly as before, for any ordinal α, the stage $V_{\daleth_\alpha}$ is as wide as it is tall.

Photo Credits

[5] We're using the Hebrew letter "ℸ"; it has no standard definition in set theory.

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