

## syn.1 Terms and Formulas

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sec

Like in first-order logic, expressions of second-order logic are built up from a basic vocabulary containing *variables*, *constant symbols*, *predicate symbols* and sometimes *function symbols*. From them, together with logical connectives, quantifiers, and punctuation symbols such as parentheses and commas, *terms* and *formulas* are formed. The difference is that in addition to variables for objects, second-order logic also contains variables for relations and functions, and allows quantification over them. So the logical symbols of second-order logic are those of first-order logic, plus:

1. A **denumerable** set of second-order relation **variables** of every arity  $n$ :  $V_0^n, V_1^n, V_2^n, \dots$
2. A **denumerable** set of second-order function **variables**:  $u_0^n, u_1^n, u_2^n, \dots$

Just as we use  $x, y, z$  as meta-variables for first-order variables  $v_i$ , we'll use  $X, Y, Z$ , etc., as metavariables for  $V_i^n$  and  $u, v$ , etc., as meta-variables for  $u_i^n$ .

The non-logical symbols of a second-order language are specified the same way a first-order language is: by listing its **constant symbols**, **function symbols**, and **predicate symbols**

explanation

In first-order logic, the **identity predicate**  $=$  is usually included. In first-order logic, the non-logical symbols of a language  $\mathcal{L}$  are crucial to allow us to express anything interesting. There are of course **sentences** that use no non-logical symbols, but with only  $=$  it is hard to say anything interesting. In second-order logic, since we have an unlimited supply of relation and function variables, we can say anything we can say in a first-order language even without a special supply of non-logical symbols.

**Definition syn.1** (Second-order Terms). The set of *second-order terms* of  $\mathcal{L}$ ,  $\text{Trm}^2(\mathcal{L})$ , is defined by adding to ?? the clause

1. If  $u$  is an  $n$ -place function variable and  $t_1, \dots, t_n$  are terms, then  $u(t_1, \dots, t_n)$  is a term.

So, a second-order term looks just like a first-order term, except that where a first-order term contains a **function symbol**  $f_i^n$ , a second-order term may contain a function variable  $u_i^n$  in its place.

explanation

**Definition syn.2** (Second-order formula). The set of *second-order formulas*  $\text{Frm}^2(\mathcal{L})$  of the language  $\mathcal{L}$  is defined by adding to ?? the clauses

1. If  $X$  is an  $n$ -place predicate variable and  $t_1, \dots, t_n$  are second-order terms of  $\mathcal{L}$ , then  $X(t_1, \dots, t_n)$  is an atomic **formula**.
2. If  $\varphi$  is a **formula** and  $u$  is a function variable, then  $\forall u \varphi$  is a **formula**.
3. If  $\varphi$  is a **formula** and  $X$  is a predicate variable, then  $\forall X \varphi$  is a **formula**.
4. If  $\varphi$  is a **formula** and  $u$  is a function variable, then  $\exists u \varphi$  is a **formula**.

5. If  $\varphi$  is a **formula** and  $X$  is a predicate variable, then  $\exists X \varphi$  is a **formula**.

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## **Bibliography**