syn.1 Terms and Formulas

Like in first-order logic, expressions of second-order logic are built up from a basic vocabulary containing variables, constant symbols, predicate symbols and sometimes function symbols. From them, together with logical connectives, quantifiers, and punctuation symbols such as parentheses and commas, terms and formulas are formed. The difference is that in addition to variables for objects, second-order logic also contains variables for relations and functions, and allows quantification over them. So the logical symbols of second-order logic are those of first-order logic, plus:

1. A denumerable set of second-order relation variables of every arity \( n \): \( V^n_0, V^n_1, V^n_2, \ldots \)
2. A denumerable set of second-order function variables: \( u^n_0, u^n_1, u^n_2, \ldots \)

Just as we use \( x, y, z \) as meta-variables for first-order variables \( v_i \), we’ll use \( X, Y, Z, \) etc., as metavariables for \( V^n_i \) and \( u, v, \) etc., as meta-variables for \( u^n_i \).

The non-logical symbols of a second-order language are specified the same way a first-order language is: by listing its constant symbols, function symbols, and predicate symbols.

In first-order logic, the identity predicate \( = \) is usually included. In first-order logic, the non-logical symbols of a language \( L \) are crucial to allow us to express anything interesting. There are of course sentences that use no non-logical symbols, but with only \( = \) it is hard to say anything interesting. In second-order logic, since we have an unlimited supply of relation and function variables, we can say anything we can say in a first-order language even without a special supply of non-logical symbols.

Definition syn.1 (Second-order Terms). The set of second-order terms of \( L \), \( \text{Trm}^2(L) \), is defined by adding to the clause

1. If \( u \) is an \( n \)-place function variable and \( t_1, \ldots, t_n \) are terms, then \( u(t_1, \ldots, t_n) \) is a term.

So, a second-order term looks just like a first-order term, except that where a first-order term contains a function symbol \( f^n_i \), a second-order term may contain a function variable \( u^n_i \) in its place.

Definition syn.2 (Second-order formulas). The set of second-order formulas \( \text{Frm}^2(L) \) of the language \( L \) is defined by adding to the clauses

1. If \( X \) is an \( n \)-place predicate variable and \( t_1, \ldots, t_n \) are second-order terms of \( L \), then \( X(t_1, \ldots, t_n) \) is an atomic formula.
2. If \( \varphi \) is a formula and \( u \) is a function variable, then \( \forall u \varphi \) is a formula.
3. If \( \varphi \) is a formula and \( X \) is a predicate variable, then \( \forall X \varphi \) is a formula.
4. If \( \varphi \) is a formula and \( u \) is a function variable, then \( \exists u \varphi \) is a formula.
5. If ϕ is a formula and X is a predicate variable, then ∃X ϕ is a formula.