

# Chapter udf

## Syntax and Semantics

### syn.1 Introduction

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In first-order logic, we combine the non-logical symbols of a given language, i.e., its **constant symbols**, **function symbols**, and **predicate symbols**, with the logical symbols to express things about first-order **structures**. This is done using the notion of satisfaction, which relates a **structure**  $\mathfrak{M}$ , together with a variable assignment  $s$ , and a **formula**  $\varphi$ :  $\mathfrak{M}, s \models \varphi$  holds iff what  $\varphi$  expresses when its **constant symbols**, **function symbols**, and **predicate symbols** are interpreted as  $\mathfrak{M}$  says, and its free variables are interpreted as  $s$  says, is true. The interpretation of the **identity predicate**  $=$  is built into the definition of  $\mathfrak{M}, s \models \varphi$ , as is the interpretation of  $\forall$  and  $\exists$ . The former is always interpreted as the identity relation on the **domain**  $|\mathfrak{M}|$  of the structure, and the quantifiers are always interpreted as ranging over the entire **domain**. But, crucially, quantification is only allowed over elements of the **domain**, and so only **object variables** are allowed to follow a quantifier.

In second-order logic, both the language and the definition of satisfaction are extended to include free and bound function and predicate variables, and quantification over them. These variables are related to **function symbols** and **predicate symbols** the same way that object variables are related to **constant symbols**. They play the same role in the formation of terms and **formulas** of second-order logic, and quantification over them is handled in a similar way. In the *standard* semantics, the second-order quantifiers range over all possible objects of the right type ( $n$ -place functions from  $|\mathfrak{M}|$  to  $|\mathfrak{M}|$  for function variables,  $n$ -place relations for predicate variables). For instance, while  $\forall v_0 (P_0^1(v_0) \vee \neg P_0^1(v_0))$  is a formula in both first- and second-order logic, in the latter we can also consider  $\forall V_0^1 \forall v_0 (V_0^1(v_0) \vee \neg V_0^1(v_0))$  and  $\exists V_0^1 \forall v_0 (V_0^1(v_0) \vee \neg V_0^1(v_0))$ . Since these contain no free variables, they are **sentences** of second-order logic. Here,  $V_0^1$  is a second-order 1-place predicate variable. The allowable interpretations of  $V_0^1$  are the same that we can assign to a 1-place **predicate symbol** like  $P_0^1$ , i.e., subsets of  $|\mathfrak{M}|$ . Quantification over them then amounts to saying that  $\forall v_0 (V_0^1(v_0) \vee \neg V_0^1(v_0))$  holds for all ways

of assigning a subset of  $|\mathfrak{M}|$  as the value of  $V_0^1$ , or for at least one. Since every set either contains or fails to contain a given object, both are true in any **structure**.

## syn.2 Terms and Formulas

Like in first-order logic, expressions of second-order logic are built up from a basic vocabulary containing *variables*, *constant symbols*, *predicate symbols* and sometimes *function symbols*. From them, together with logical connectives, quantifiers, and punctuation symbols such as parentheses and commas, *terms* and *formulas* are formed. The difference is that in addition to variables for objects, second-order logic also contains variables for relations and functions, and allows quantification over them. So the logical symbols of second-order logic are those of first-order logic, plus:

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1. A **denumerable** set of second-order relation **variables** of every arity  $n$ :  $V_0^n, V_1^n, V_2^n, \dots$
2. A **denumerable** set of second-order function **variables**:  $u_0^n, u_1^n, u_2^n, \dots$

Just as we use  $x, y, z$  as meta-variables for first-order variables  $v_i$ , we'll use  $X, Y, Z$ , etc., as metavariables for  $V_i^n$  and  $u, v$ , etc., as meta-variables for  $u_i^n$ .

explanation

The non-logical symbols of a second-order language are specified the same way a first-order language is: by listing its **constant symbols**, **function symbols**, and **predicate symbols**

In first-order logic, the **identity predicate**  $=$  is usually included. In first-order logic, the non-logical symbols of a language  $\mathcal{L}$  are crucial to allow us to express anything interesting. There are of course **sentences** that use no non-logical symbols, but with only  $=$  it is hard to say anything interesting. In second-order logic, since we have an unlimited supply of relation and function variables, we can say anything we can say in a first-order language even without a special supply of non-logical symbols.

**Definition syn.1** (Second-order Terms). The set of *second-order terms* of  $\mathcal{L}$ ,  $\text{Trm}^2(\mathcal{L})$ , is defined by adding to ?? the clause

1. If  $u$  is an  $n$ -place function variable and  $t_1, \dots, t_n$  are terms, then  $u(t_1, \dots, t_n)$  is a term.

explanation

So, a second-order term looks just like a first-order term, except that where a first-order term contains a **function symbol**  $f_i^n$ , a second-order term may contain a function variable  $u_i^n$  in its place.

**Definition syn.2** (Second-order formula). The set of *second-order formulas*  $\text{Frm}^2(\mathcal{L})$  of the language  $\mathcal{L}$  is defined by adding to ?? the clauses

1. If  $X$  is an  $n$ -place predicate variable and  $t_1, \dots, t_n$  are second-order terms of  $\mathcal{L}$ , then  $X(t_1, \dots, t_n)$  is an atomic **formula**.

2. If  $\varphi$  is a **formula** and  $u$  is a function variable, then  $\forall u \varphi$  is a **formula**.
3. If  $\varphi$  is a **formula** and  $X$  is a predicate variable, then  $\forall X \varphi$  is a **formula**.
4. If  $\varphi$  is a **formula** and  $u$  is a function variable, then  $\exists u \varphi$  is a **formula**.
5. If  $\varphi$  is a **formula** and  $X$  is a predicate variable, then  $\exists X \varphi$  is a **formula**.

### syn.3 Satisfaction

sol:syn:sat:  
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**Definition syn.3** (Variable Assignment). A *variable assignment*  $s$  for a **structure**  $\mathfrak{M}$  is a function which maps each

1. object **variable**  $v_i$  to an element of  $|\mathfrak{M}|$ , i.e.,  $s(v_i) \in |\mathfrak{M}|$
2.  $n$ -place relation variable  $V_i^n$  to an  $n$ -place relation on  $|\mathfrak{M}|$ , i.e.,  $s(V_i^n) \subseteq |\mathfrak{M}|^n$ ;
3.  $n$ -place function variable  $u_i^n$  to an  $n$ -place function from  $|\mathfrak{M}|$  to  $|\mathfrak{M}|$ , i.e.,  $s(u_i^n): |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$ ;

A **structure** assigns a **value** to each **constant symbol** and **function symbol**, explanation and a second-order variable assigns objects and functions to each object and function variable. Together, they let us assign a value to every term.

**Definition syn.4** (Value of a Term). If  $t$  is a term of the language  $\mathcal{L}$ ,  $\mathfrak{M}$  is a **structure** for  $\mathcal{L}$ , and  $s$  is a **variable** assignment for  $\mathfrak{M}$ , the *value*  $\text{Val}_s^{\mathfrak{M}}(t)$  is defined as for first-order terms, plus the following clause:

$$t \equiv u(t_1, \dots, t_n):$$

$$\text{Val}_s^{\mathfrak{M}}(t) = s(u)(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n)).$$

**Definition syn.5** (Satisfaction). For second-order **formulas**  $\varphi$ , the definition of satisfaction is like ?? with the addition of:

1.  $\varphi \equiv X^n t_1, \dots, t_n$ :  $\mathfrak{M}, s \models \varphi$  iff  $\langle \text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n) \rangle \in s(X^n)$ .
2.  $\varphi \equiv \forall X \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for every  $X$ -variant  $s'$  of  $s$ ,  $\mathfrak{M}, s' \models \psi$ .
3.  $\varphi \equiv \exists X \psi$ :  $\mathfrak{M}, s \models \varphi$  iff there is an  $X$ -variant  $s'$  of  $s$  so that  $\mathfrak{M}, s' \models \psi$ .
4.  $\varphi \equiv \forall u \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for every  $u$ -variant  $s'$  of  $s$ ,  $\mathfrak{M}, s' \models \psi$ .
5.  $\varphi \equiv \exists u \psi$ :  $\mathfrak{M}, s \models \varphi$  iff there is an  $u$ -variant  $s'$  of  $s$  so that  $\mathfrak{M}, s' \models \psi$ .

**Example syn.6.**  $\mathfrak{M}, s \models \forall z (Xz \leftrightarrow \neg Yz)$  whenever  $s(Y) = |\mathfrak{M}| \setminus s(X)$ . So for instance, let  $|\mathfrak{M}| = \{1, 2, 3\}$ ,  $s(X) = \{1, 2\}$  and  $s(Y) = \{3\}$ .

$\mathfrak{M}, s \models \exists Y (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$  if there is an  $s' \sim_Y s$  such that  $\mathfrak{M}, s' \models (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$ . And that is the case iff  $s'(Y) \neq \emptyset$  (so that  $\mathfrak{M}, s' \models \exists y Yy$ ) and, as before,  $s'(Y) = |\mathfrak{M}| \setminus s'(X)$ . In other words,  $\mathfrak{M}, s \models \exists Y (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$  iff  $|\mathfrak{M}| \setminus s(X)$  is non-empty, or,  $s(X) \neq |\mathfrak{M}|$ . So, the formula is satisfied, e.g., if  $s(X) = \{1, 2\}$  but not if  $s(X) = \{1, 2, 3\}$ .

## syn.4 Semantic Notions

explanation

The central logical notions of *validity*, *entailment*, and *satisfiability* are defined the same way for second-order logic as they are for first-order logic, except that the underlying satisfaction relation is now that for second-order formulas. A second-order sentence, of course, is a formula in which all variables, including predicate and function variables, are bound.

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**Definition syn.7** (Validity). A sentence  $\varphi$  is *valid*,  $\models \varphi$ , iff  $\mathfrak{M} \models \varphi$  for every structure  $\mathfrak{M}$ .

**Definition syn.8** (Entailment). A set of sentences  $\Gamma$  *entails* a sentence  $\varphi$ ,  $\Gamma \models \varphi$ , iff for every structure  $\mathfrak{M}$  with  $\mathfrak{M} \models \Gamma$ ,  $\mathfrak{M} \models \varphi$ .

**Definition syn.9** (Satisfiability). A set of sentences  $\Gamma$  is *satisfiable* if  $\mathfrak{M} \models \Gamma$  for some structure  $\mathfrak{M}$ . If  $\Gamma$  is not satisfiable it is called *unsatisfiable*.

## syn.5 Expressive Power

explanation

Quantification over second-order variables is responsible for an immense increase in the expressive power of the language over that of first-order logic. Second-order existential quantification lets us say that functions or relations with certain properties exist. In first-order logic, the only way to do that is to specify non-logical symbol (i.e., a function symbol or predicate symbol) for this purpose. Second-order universal quantification lets us say that all subsets of, relations on, or functions from the domain to the domain have a property. In first-order logic, we can only say that the subsets, relations, or functions assigned to one of the non-logical symbols of the language have a property. And when we say that subsets, relations, functions exist that have a property, or that all of them have it, we can use second-order quantification in specifying this property as well. This lets us define relations not definable in first-order logic, and express properties of the domain not expressible in first-order logic.

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**Example syn.10.** If  $\mathfrak{M}$  is a structure for a language  $\mathcal{L}$ , a relation  $R \subseteq |\mathfrak{M}|^2$  is definable in  $\mathcal{L}$  if there is some formula  $\varphi_R(v_0, v_1)$  with only the variables  $v_0$  and  $v_1$  free, such that  $R(x, y)$  holds (i.e.,  $\langle x, y \rangle \in R$ ) iff  $\mathfrak{M}, s \models \varphi_R(v_0, v_1)$  for  $s(v_0) = x$  and  $s(v_1) = y$ . For instance, in first-order logic we can define the identity relation  $\text{Id}_{|\mathfrak{M}|}$  (i.e.,  $\{\langle x, x \rangle : x \in |\mathfrak{M}|\}$ ) by the formula  $v_0 = v_1$ . In second-order logic, we can define this relation *without*  $=$ . For if  $x$  and  $y$  are the

same **element** of  $|\mathfrak{M}|$ , then they are **elements** of the same subsets of  $|\mathfrak{M}|$  (since sets are determined by their **elements**). Conversely, if  $x$  and  $y$  are different, then they are not **elements** of the same subsets: e.g.,  $x \in \{x\}$  but  $y \notin \{x\}$  if  $x \neq y$ . So “being **elements** of the same subsets of  $|\mathfrak{M}|$ ” is a relation that holds of  $x$  and  $y$  iff  $x = y$ . It is a relation that can be expressed in second-order logic, since we can quantify over all subsets of  $|\mathfrak{M}|$ . Hence, the following **formula** defines  $\text{Id}_{|\mathfrak{M}|}$ :

$$\forall X (X(v_0) \leftrightarrow X(v_1))$$

**Problem syn.1.** Show that  $\forall X (X(v_0) \rightarrow X(v_1))$  (note:  $\rightarrow$  not  $\leftrightarrow$ !) defines  $\text{Id}_{|\mathfrak{M}|}$ .

**Example syn.11.** If  $R$  is a two-place **predicate symbol**,  $R^{\mathfrak{M}}$  is a two-place relation on  $|\mathfrak{M}|$ . Its *transitive closure*  $R^*$  is the relation that holds between  $x$  and  $y$  if for some  $z_1, \dots, z_k$ ,  $R(x, z_1), R(z_1, z_2), \dots, R(z_k, y)$  holds. This includes the case if  $k = 0$ , i.e., if  $R(x, y)$  holds. This means that  $R \subseteq R^*$ . In fact,  $R^*$  is the smallest relation that includes  $R$  and that is transitive. We can say in second-order logic that  $X$  is a transitive relation that includes  $R$ :

$$\begin{aligned} \psi_R(X) \equiv & \forall x \forall y (R(x, y) \rightarrow X(x, y)) \wedge \\ & \forall x \forall y \forall z ((X(x, y) \wedge X(y, z)) \rightarrow X(x, z)) \end{aligned}$$

Here, somewhat confusingly, we use  $R$  as the **predicate symbol** for  $R$ . The first conjunct says that  $R \subseteq X$  and the second that  $X$  is transitive.

To say that  $X$  is the smallest such relation is to say that it is itself included in every relation that includes  $R$  and is transitive. So we can define the transitive closure of  $R$  by the **formula**

$$R^*(X) \equiv \psi_R(X) \wedge \forall Y (\psi_R(Y) \rightarrow \forall x \forall y (X(x, y) \rightarrow Y(x, y)))$$

$\mathfrak{M}, s \models R^*(X)$  iff  $s(X) = R^*$ . The transitive closure of  $R$  cannot be expressed in first-order logic.

## syn.6 Describing Infinite and Enumerable Domains

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A set  $M$  is (Dedekind) infinite iff there is an **injective** function  $f: M \rightarrow M$  which is not **surjective**, i.e., with  $\text{dom}(f) \neq M$ . In first-order logic, we can consider a one-place **function symbol**  $f$  and say that the function  $f^{\mathfrak{M}}$  assigned to it in a **structure**  $\mathfrak{M}$  is **injective** and  $\text{ran}(f) \neq |\mathfrak{M}|$ :

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y) \wedge \exists y \forall x y \neq f(x)$$

If  $\mathfrak{M}$  satisfies this **sentence**,  $f^{\mathfrak{M}}: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$  is **injective**, and so  $|\mathfrak{M}|$  must be infinite. If  $|\mathfrak{M}|$  is infinite, and hence such a function exists, we can let  $f^{\mathfrak{M}}$  be that function and  $\mathfrak{M}$  will satisfy the sentence. However, this requires that our

language contains the non-logical symbol  $f$  we use for this purpose. In second-order logic, we can simply say that such a function *exists*. This no-longer requires  $f$ , and we have the **sentence** in pure second-order logic

$$\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x))$$

$\mathfrak{M} \models \text{Inf}$  iff  $|\mathfrak{M}|$  is infinite. We can then define  $\text{Fin} \equiv \neg \text{Inf}$ ;  $\mathfrak{M} \models \text{Fin}$  iff  $|\mathfrak{M}|$  is finite. No single **sentence** of pure first-order logic can express that the **domain** is infinite although an infinite set of them can. There is no set of **sentences** of pure first-order logic that is satisfied in a **structure** iff its domain is finite.

**Proposition syn.12.**  $\mathfrak{M} \models \text{Inf}$  iff  $|\mathfrak{M}|$  is infinite.

*Proof.*  $\mathfrak{M} \models \text{Inf}$  iff  $\mathfrak{M}, s \models \forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x)$  for some  $s$ . If it does,  $s(u)$  is an **injective** function, and some  $y \in |\mathfrak{M}|$  is not in the domain of  $s(u)$ . Conversely, if there is an **injective**  $f: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$  with  $\text{dom}(f) \neq |\mathfrak{M}|$ , then  $s(u) = f$  is such a variable assignment.  $\square$

A set  $M$  is **enumerable** if there is an enumeration

$$m_0, m_1, m_2, \dots$$

of its **elements** (without repetitions). Such an enumeration exists iff there is an **element**  $z \in M$  and a function  $f: M \rightarrow M$  such that  $z, f(z), f(f(z))$  are all the **elements** of  $M$ . For if the enumeration exists,  $z = m_0$  and  $f(m_k) = m_{k+1}$  (or  $f(m_k) = m_k$  if  $m_k$  is the last **element** of the enumeration) are the requisite **element** and function. On the other hand, if such a  $z$  and  $f$  exist, then  $z, f(z), f(f(z)), \dots$ , is an enumeration of  $M$ , and  $M$  is **enumerable**. We can express the existence of  $z$  and  $f$  in second-order logic to produce a **sentence** true in a **structure** iff the structure is **enumerable**:

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

**Proposition syn.13.**  $\mathfrak{M} \models \text{Count}$  iff  $|\mathfrak{M}|$  is **enumerable**.

*Proof.* Suppose  $|\mathfrak{M}|$  is enumerable, and let  $m_0, m_1, \dots$ , be an enumeration. By removing repetitions we can guarantee that no  $m_k$  appears twice. Define  $f(m_k) = m_{k+1}$  and let  $s(z) = m_0$  and  $s(u) = f$ . We show that

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

Suppose  $s' \sim_X s$ , and  $M = s'(X)$ . Suppose further that  $\mathfrak{M}, s' \models (X(z) \wedge \forall x (X(x) \rightarrow X(u(x))))$ . Then  $s'(z) \in M$  and whenever  $x \in M$ , also  $s'(u)(x) \in M$ . In other words, since  $s' \sim_X s$ ,  $m_0 \in M$  and if  $x \in M$  then  $f(x) \in M$ , so  $m_0 \in M$ ,  $m_1 = f(m_0) \in M$ ,  $m_2 = f(f(m_0)) \in M$ , etc. Thus,  $M = |\mathfrak{M}|$ , and  $\mathfrak{M} \models \forall x X(x)s'$ . Since  $s'$  was an arbitrary  $X$ -variant of  $s$ , we are done.

Now assume that

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

for some  $s$ . Let  $m = s(z)$  and  $f = s(u)$  and consider  $M = \{m, f(m), f(f(m)), \dots\}$ . Let  $s'$  be the  $X$ -variant of  $s$  with  $s'(X) = M$ . Then

$$\mathfrak{M}, s' \models ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

by assumption. Also,  $\mathfrak{M}, s' \models X(z)$  since  $s'(X) = M \ni m = s'(z)$ , and also  $\mathfrak{M}, s' \models \forall x (X(x) \rightarrow X(u(x)))$  since whenever  $x \in M$  also  $f(x) \in M$ . So, since both antecedent and conditional are satisfied, the consequent must also be:  $\mathfrak{M}, s' \models \forall x X(x)$ . But that means that  $M = |\mathfrak{M}|$ , and so  $|\mathfrak{M}|$  is **enumerable**.  $\square$

**Problem syn.2.** The sentence  $\text{Inf} \wedge \text{Count}$  is true in all and only **enumerable** domains. Adjust the definition of  $\text{Count}$  so that it becomes a different **sentence** that directly expresses that the domain is **enumerable**, and prove that it does.

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# Bibliography