

Chapter udf

Syntax and Semantics

syn.1 Introduction

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sec

In first-order logic, we combine the non-logical symbols of a given language, i.e., its **constant symbols**, **function symbols**, and **predicate symbols**, with the logical symbols to express things about first-order **structures**. This is done using the notion of satisfaction, which relates a **structure** \mathfrak{M} , together with a variable assignment s , and a **formula** φ : $\mathfrak{M}, s \models \varphi$ holds iff what φ expresses when its **constant symbols**, **function symbols**, and **predicate symbols** are interpreted as \mathfrak{M} says, and its free variables are interpreted as s says, is true. The interpretation of the **identity predicate** $=$ is built into the definition of $\mathfrak{M}, s \models \varphi$, as is the interpretation of \forall and \exists . The former is always interpreted as the identity relation on the **domain** $|\mathfrak{M}|$ of the structure, and the quantifiers are always interpreted as ranging over the entire **domain**. But, crucially, quantification is only allowed over elements of the **domain**, and so only **object variables** are allowed to follow a quantifier.

In second-order logic, both the language and the definition of satisfaction are extended to include free and bound function and predicate variables, and quantification over them. These variables are related to **function symbols** and **predicate symbols** the same way that object variables are related to **constant symbols**. They play the same role in the formation of terms and **formulas** of second-order logic, and quantification over them is handled in a similar way. In the *standard* semantics, the second-order quantifiers range over all possible objects of the right type (n -place functions from $|\mathfrak{M}|$ to $|\mathfrak{M}|$ for function variables, n -place relations for predicate variables). For instance, while $\forall v_0 (P_0^1(v_0) \vee \neg P_0^1(v_0))$ is a formula in both first- and second-order logic, in the latter we can also consider $\forall V_0^1 \forall v_0 (V_0^1(v_0) \vee \neg V_0^1(v_0))$ and $\exists V_0^1 \forall v_0 (V_0^1(v_0) \vee \neg V_0^1(v_0))$. Since these contain no free variables, they are **sentences** of second-order logic. Here, V_0^1 is a second-order 1-place predicate variable. The allowable interpretations of V_0^1 are the same that we can assign to a 1-place **predicate symbol** like P_0^1 , i.e., subsets of $|\mathfrak{M}|$. Quantification over them then amounts to saying that $\forall v_0 (V_0^1(v_0) \vee \neg V_0^1(v_0))$ holds for all ways

of assigning a subset of $|\mathfrak{M}|$ as the value of V_0^1 , or for at least one. Since every set either contains or fails to contain a given object, both are true in any **structure**.

syn.2 Terms and Formulas

Like in first-order logic, expressions of second-order logic are built up from a basic vocabulary containing *variables*, *constant symbols*, *predicate symbols* and sometimes *function symbols*. From them, together with logical connectives, quantifiers, and punctuation symbols such as parentheses and commas, *terms* and *formulas* are formed. The difference is that in addition to variables for objects, second-order logic also contains variables for relations and functions, and allows quantification over them. So the logical symbols of second-order logic are those of first-order logic, plus:

1. A **denumerable** set of second-order relation **variables** of every arity n : $V_0^n, V_1^n, V_2^n, \dots$
2. A **denumerable** set of second-order function **variables**: $u_0^n, u_1^n, u_2^n, \dots$

Just as we use x, y, z as meta-variables for first-order variables v_i , we'll use X, Y, Z , etc., as metavariables for V_i^n and u, v , etc., as meta-variables for u_i^n .

explanation

The non-logical symbols of a second-order language are specified the same way a first-order language is: by listing its **constant symbols**, **function symbols**, and **predicate symbols**

In first-order logic, the **identity predicate** $=$ is usually included. In first-order logic, the non-logical symbols of a language \mathcal{L} are crucial to allow us to express anything interesting. There are of course **sentences** that use no non-logical symbols, but with only $=$ it is hard to say anything interesting. In second-order logic, since we have an unlimited supply of relation and function variables, we can say anything we can say in a first-order language even without a special supply of non-logical symbols.

Definition syn.1 (Second-order Terms). The set of *second-order terms* of \mathcal{L} , $\text{Trm}^2(\mathcal{L})$, is defined by adding to ?? the clause

1. If u is an n -place function variable and t_1, \dots, t_n are terms, then $u(t_1, \dots, t_n)$ is a term.

explanation

So, a second-order term looks just like a first-order term, except that where a first-order term contains a **function symbol** f_i^n , a second-order term may contain a function variable u_i^n in its place.

Definition syn.2 (Second-order formula). The set of *second-order formulas* $\text{Frm}^2(\mathcal{L})$ of the language \mathcal{L} is defined by adding to ?? the clauses

1. If X is an n -place predicate variable and t_1, \dots, t_n are second-order terms of \mathcal{L} , then $X(t_1, \dots, t_n)$ is an atomic **formula**.

2. If φ is a **formula** and u is a function variable, then $\forall u \varphi$ is a **formula**.
3. If φ is a **formula** and X is a predicate variable, then $\forall X \varphi$ is a **formula**.
4. If φ is a **formula** and u is a function variable, then $\exists u \varphi$ is a **formula**.
5. If φ is a **formula** and X is a predicate variable, then $\exists X \varphi$ is a **formula**.

syn.3 Satisfaction

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sec To define the satisfaction relation $\mathfrak{M}, s \models \varphi$ for second-order **formulas**, we explanation have to extend the definitions to cover second-order variables.

Definition syn.3 (Variable Assignment). A *variable assignment* s for a **structure** \mathfrak{M} is a function which maps each

1. object **variable** v_i to an element of $|\mathfrak{M}|$, i.e., $s(v_i) \in |\mathfrak{M}|$
2. n -place relation variable V_i^n to an n -place relation on $|\mathfrak{M}|$, i.e., $s(V_i^n) \subseteq |\mathfrak{M}|^n$;
3. n -place function variable u_i^n to an n -place function from $|\mathfrak{M}|$ to $|\mathfrak{M}|$, i.e., $s(u_i^n): |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$;

A **structure** assigns a **value** to each **constant symbol** and **function symbol**, explanation and a second-order variable assigns objects and functions to each object and function variable. Together, they let us assign a value to every term.

Definition syn.4 (Value of a Term). If t is a term of the language \mathcal{L} , \mathfrak{M} is a **structure** for \mathcal{L} , and s is a **variable** assignment for \mathfrak{M} , the *value* $\text{Val}_s^{\mathfrak{M}}(t)$ is defined as for first-order terms, plus the following clause:

$$t \equiv u(t_1, \dots, t_n):$$

$$\text{Val}_s^{\mathfrak{M}}(t) = s(u)(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n)).$$

Definition syn.5 (Satisfaction). For second-order **formulas** φ , the definition of satisfaction is like ?? with the addition of:

1. $\varphi \equiv X^n t_1, \dots, t_n$: $\mathfrak{M}, s \models \varphi$ iff $\langle \text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n) \rangle \in s(X^n)$.
2. $\varphi \equiv \forall X \psi$: $\mathfrak{M}, s \models \varphi$ iff for every X -variant s' of s , $\mathfrak{M}, s' \models \psi$.
3. $\varphi \equiv \exists X \psi$: $\mathfrak{M}, s \models \varphi$ iff there is an X -variant s' of s so that $\mathfrak{M}, s' \models \psi$.
4. $\varphi \equiv \forall u \psi$: $\mathfrak{M}, s \models \varphi$ iff for every u -variant s' of s , $\mathfrak{M}, s' \models \psi$.
5. $\varphi \equiv \exists u \psi$: $\mathfrak{M}, s \models \varphi$ iff there is an u -variant s' of s so that $\mathfrak{M}, s' \models \psi$.

Example syn.6. $\mathfrak{M}, s \models \forall z (Xz \leftrightarrow \neg Yz)$ whenever $s(Y) = |\mathfrak{M}| \setminus s(X)$. So for instance, let $|\mathfrak{M}| = \{1, 2, 3\}$, $s(X) = \{1, 2\}$ and $s(Y) = \{3\}$.

$\mathfrak{M}, s \models \exists Y (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$ if there is an $s' \sim_Y s$ such that $\mathfrak{M}, s' \models (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$. And that is the case iff $s'(Y) \neq \emptyset$ (so that $\mathfrak{M}, s' \models \exists y Yy$) and, as before, $s'(Y) = |\mathfrak{M}| \setminus s'(X)$. In other words, $\mathfrak{M}, s \models \exists Y (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$ iff $|\mathfrak{M}| \setminus s(X)$ is non-empty, or, $s(X) \neq |\mathfrak{M}|$. So, the formula is satisfied, e.g., if $s(X) = \{1, 2\}$ but not if $s(X) = \{1, 2, 3\}$.

syn.4 Semantic Notions

explanation

The central logical notions of *validity*, *entailment*, and *satisfiability* are defined the same way for second-order logic as they are for first-order logic, except that the underlying satisfaction relation is now that for second-order formulas. A second-order sentence, of course, is a formula in which all variables, including predicate and function variables, are bound.

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Definition syn.7 (Validity). A sentence φ is *valid*, $\models \varphi$, iff $\mathfrak{M} \models \varphi$ for every structure \mathfrak{M} .

Definition syn.8 (Entailment). A set of sentences Γ *entails* a sentence φ , $\Gamma \models \varphi$, iff for every structure \mathfrak{M} with $\mathfrak{M} \models \Gamma$, $\mathfrak{M} \models \varphi$.

Definition syn.9 (Satisfiability). A set of sentences Γ is *satisfiable* if $\mathfrak{M} \models \Gamma$ for some structure \mathfrak{M} . If Γ is not satisfiable it is called *unsatisfiable*.

syn.5 Expressive Power

explanation

Quantification over second-order variables is responsible for an immense increase in the expressive power of the language over that of first-order logic. Second-order existential quantification lets us say that functions or relations with certain properties exist. In first-order logic, the only way to do that is to specify non-logical symbol (i.e., a function symbol or predicate symbol) for this purpose. Second-order universal quantification lets us say that all subsets of, relations on, or functions from the domain to the domain have a property. In first-order logic, we can only say that the subsets, relations, or functions assigned to one of the non-logical symbols of the language have a property. And when we say that subsets, relations, functions exist that have a property, or that all of them have it, we can use second-order quantification in specifying this property as well. This lets us define relations not definable in first-order logic, and express properties of the domain not expressible in first-order logic.

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Example syn.10. If \mathfrak{M} is a structure for a language \mathcal{L} , a relation $R \subseteq |\mathfrak{M}|^2$ is definable in \mathcal{L} if there is some formula $\varphi_R(v_0, v_1)$ with only the variables v_0 and v_1 free, such that $R(x, y)$ holds (i.e., $\langle x, y \rangle \in R$) iff $\mathfrak{M}, s \models \varphi_R(v_0, v_1)$ for $s(v_0) = x$ and $s(v_1) = y$. For instance, in first-order logic we can define the identity relation $\text{Id}_{|\mathfrak{M}|}$ (i.e., $\{\langle x, x \rangle : x \in |\mathfrak{M}|\}$) by the formula $v_0 = v_1$. In second-order logic, we can define this relation *without* $=$. For if x and y are the

same **element** of $|\mathfrak{M}|$, then they are **elements** of the same subsets of $|\mathfrak{M}|$ (since sets are determined by their **elements**). Conversely, if x and y are different, then they are not **elements** of the same subsets: e.g., $x \in \{x\}$ but $y \notin \{x\}$ if $x \neq y$. So “being **elements** of the same subsets of $|\mathfrak{M}|$ ” is a relation that holds of x and y iff $x = y$. It is a relation that can be expressed in second-order logic, since we can quantify over all subsets of $|\mathfrak{M}|$. Hence, the following **formula** defines $\text{Id}_{|\mathfrak{M}|}$:

$$\forall X (X(v_0) \leftrightarrow X(v_1))$$

Problem syn.1. Show that $\forall X (X(v_0) \rightarrow X(v_1))$ (note: \rightarrow not \leftrightarrow !) defines $\text{Id}_{|\mathfrak{M}|}$.

Example syn.11. If R is a two-place **predicate symbol**, $R^{\mathfrak{M}}$ is a two-place relation on $|\mathfrak{M}|$. Its *transitive closure* R^* is the relation that holds between x and y if for some z_1, \dots, z_k , $R(x, z_1), R(z_1, z_2), \dots, R(z_k, y)$ holds. This includes the case if $k = 0$, i.e., if $R(x, y)$ holds. This means that $R \subseteq R^*$. In fact, R^* is the smallest relation that includes R and that is transitive. We can say in second-order logic that X is a transitive relation that includes R :

$$\begin{aligned} \psi_R(X) \equiv & \forall x \forall y (R(x, y) \rightarrow X(x, y)) \wedge \\ & \forall x \forall y \forall z ((X(x, y) \wedge X(y, z)) \rightarrow X(x, z)) \end{aligned}$$

Here, somewhat confusingly, we use R as the **predicate symbol** for R . The first conjunct says that $R \subseteq X$ and the second that X is transitive.

To say that X is the smallest such relation is to say that it is itself included in every relation that includes R and is transitive. So we can define the transitive closure of R by the **formula**

$$R^*(X) \equiv \psi_R(X) \wedge \forall Y (\psi_R(Y) \rightarrow \forall x \forall y (X(x, y) \rightarrow Y(x, y)))$$

$\mathfrak{M}, s \models R^*(X)$ iff $s(X) = R^*$. The transitive closure of R cannot be expressed in first-order logic.

syn.6 Describing Infinite and Enumerable Domains

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A set M is (Dedekind) infinite iff there is an **injective** function $f: M \rightarrow M$ which is not **surjective**, i.e., with $\text{dom}(f) \neq M$. In first-order logic, we can consider a one-place **function symbol** f and say that the function $f^{\mathfrak{M}}$ assigned to it in a **structure** \mathfrak{M} is **injective** and $\text{ran}(f) \neq |\mathfrak{M}|$:

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y) \wedge \exists y \forall x y \neq f(x)$$

If \mathfrak{M} satisfies this **sentence**, $f^{\mathfrak{M}}: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$ is **injective**, and so $|\mathfrak{M}|$ must be infinite. If $|\mathfrak{M}|$ is infinite, and hence such a function exists, we can let $f^{\mathfrak{M}}$ be that function and \mathfrak{M} will satisfy the sentence. However, this requires that our

language contains the non-logical symbol f we use for this purpose. In second-order logic, we can simply say that such a function *exists*. This no-longer requires f , and we have the **sentence** in pure second-order logic

$$\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x))$$

$\mathfrak{M} \models \text{Inf}$ iff $|\mathfrak{M}|$ is infinite. We can then define $\text{Fin} \equiv \neg \text{Inf}$; $\mathfrak{M} \models \text{Fin}$ iff $|\mathfrak{M}|$ is finite. No single **sentence** of pure first-order logic can express that the **domain** is infinite although an infinite set of them can. There is no set of **sentences** of pure first-order logic that is satisfied in a **structure** iff its domain is finite.

Proposition syn.12. $\mathfrak{M} \models \text{Inf}$ iff $|\mathfrak{M}|$ is infinite.

Proof. $\mathfrak{M} \models \text{Inf}$ iff $\mathfrak{M}, s \models \forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x)$ for some s . If it does, $s(u)$ is an **injective** function, and some $y \in |\mathfrak{M}|$ is not in the domain of $s(u)$. Conversely, if there is an **injective** $f: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$ with $\text{dom}(f) \neq |\mathfrak{M}|$, then $s(u) = f$ is such a variable assignment. \square

A set M is **enumerable** if there is an enumeration

$$m_0, m_1, m_2, \dots$$

of its **elements** (without repetitions). Such an enumeration exists iff there is an **element** $z \in M$ and a function $f: M \rightarrow M$ such that $z, f(z), f(f(z))$ are all the **elements** of M . For if the enumeration exists, $z = m_0$ and $f(m_k) = m_{k+1}$ (or $f(m_k) = m_k$ if m_k is the last **element** of the enumeration) are the requisite **element** and function. On the other hand, if such a z and f exist, then $z, f(z), f(f(z)), \dots$, is an enumeration of M , and M is **enumerable**. We can express the existence of z and f in second-order logic to produce a **sentence** true in a **structure** iff the structure is **enumerable**:

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

Proposition syn.13. $\mathfrak{M} \models \text{Count}$ iff $|\mathfrak{M}|$ is **enumerable**.

Proof. Suppose $|\mathfrak{M}|$ is enumerable, and let m_0, m_1, \dots , be an enumeration. By removing repetitions we can guarantee that no m_k appears twice. Define $f(m_k) = m_{k+1}$ and let $s(z) = m_0$ and $s(u) = f$. We show that

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

Suppose $s' \sim_X s$, and $M = s'(X)$. Suppose further that $\mathfrak{M}, s' \models (X(z) \wedge \forall x (X(x) \rightarrow X(u(x))))$. Then $s'(z) \in M$ and whenever $x \in M$, also $s'(u)(x) \in M$. In other words, since $s' \sim_X s$, $m_0 \in M$ and if $x \in M$ then $f(x) \in M$, so $m_0 \in M$, $m_1 = f(m_0) \in M$, $m_2 = f(f(m_0)) \in M$, etc. Thus, $M = |\mathfrak{M}|$, and $\mathfrak{M} \models \forall x X(x)s'$. Since s' was an arbitrary X -variant of s , we are done.

Now assume that

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

for some s . Let $m = s(z)$ and $f = s(u)$ and consider $M = \{m, f(m), f(f(m)), \dots\}$. Let s' be the X -variant of s with $s'(X) = M$. Then

$$\mathfrak{M}, s' \models ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

by assumption. Also, $\mathfrak{M}, s' \models X(z)$ since $s'(X) = M \ni m = s'(z)$, and also $\mathfrak{M}, s' \models \forall x (X(x) \rightarrow X(u(x)))$ since whenever $x \in M$ also $f(x) \in M$. So, since both antecedent and conditional are satisfied, the consequent must also be: $\mathfrak{M}, s' \models \forall x X(x)$. But that means that $M = |\mathfrak{M}|$, and so $|\mathfrak{M}|$ is **enumerable**. \square

Problem syn.2. The sentence $\text{Inf} \wedge \text{Count}$ is true in all and only **enumerable** domains. Adjust the definition of Count so that it becomes a different **sentence** that directly expresses that the domain is **enumerable**, and prove that it does.

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Bibliography