

syn.1 Satisfaction

sol:syn:sat:
sec To define the satisfaction relation $\mathfrak{M}, s \models \varphi$ for second-order **formulas**, we explanation have to extend the definitions to cover second-order variables.

Definition syn.1 (Variable Assignment). A *variable assignment* s for a **structure** \mathfrak{M} is a function which maps each

1. object **variable** v_i to an element of $|\mathfrak{M}|$, i.e., $s(v_i) \in |\mathfrak{M}|$
2. n -place relation variable V_i^n to an n -place relation on $|\mathfrak{M}|$, i.e., $s(V_i^n) \subseteq |\mathfrak{M}|^n$;
3. n -place function variable u_i^n to an n -place function from $|\mathfrak{M}|$ to $|\mathfrak{M}|$, i.e., $s(u_i^n): |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$;

A **structure** assigns a **value** to each **constant symbol** and **function symbol**, explanation and a second-order variable assigns objects and functions to each object and function variable. Together, they let us assign a value to every term.

Definition syn.2 (Value of a Term). If t is a term of the language \mathcal{L} , \mathfrak{M} is a **structure** for \mathcal{L} , and s is a **variable assignment** for \mathfrak{M} , the *value* $\text{Val}_s^{\mathfrak{M}}(t)$ is defined as for first-order terms, plus the following clause:

$$t \equiv u(t_1, \dots, t_n):$$

$$\text{Val}_s^{\mathfrak{M}}(t) = s(u)(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n)).$$

Definition syn.3 (Satisfaction). For second-order **formulas** φ , the definition of satisfaction is like ?? with the addition of:

1. $\varphi \equiv X^n t_1, \dots, t_n$: $\mathfrak{M}, s \models \varphi$ iff $\langle \text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n) \rangle \in s(X^n)$.
2. $\varphi \equiv \forall X \psi$: $\mathfrak{M}, s \models \varphi$ iff for every X -variant s' of s , $\mathfrak{M}, s' \models \psi$.
3. $\varphi \equiv \exists X \psi$: $\mathfrak{M}, s \models \varphi$ iff there is an X -variant s' of s so that $\mathfrak{M}, s' \models \psi$.
4. $\varphi \equiv \forall u \psi$: $\mathfrak{M}, s \models \varphi$ iff for every u -variant s' of s , $\mathfrak{M}, s' \models \psi$.
5. $\varphi \equiv \exists u \psi$: $\mathfrak{M}, s \models \varphi$ iff there is an u -variant s' of s so that $\mathfrak{M}, s' \models \psi$.

Example syn.4. $\mathfrak{M}, s \models \forall z (Xz \leftrightarrow \neg Yz)$ whenever $s(Y) = |\mathfrak{M}| \setminus s(X)$. So for instance, let $|\mathfrak{M}| = \{1, 2, 3\}$, $s(X) = \{1, 2\}$ and $s(Y) = \{3\}$.

$\mathfrak{M}, s \models \exists Y (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$ if there is an $s' \sim_Y s$ such that $\mathfrak{M}, s' \models (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$. And that is the case iff $s'(Y) \neq \emptyset$ (so that $\mathfrak{M}, s' \models \exists y Yy$) and, as before, $s'(Y) = |\mathfrak{M}| \setminus s'(X)$. In other words, $\mathfrak{M}, s \models \exists Y (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$ iff $|\mathfrak{M}| \setminus s(X)$ is non-empty, or, $s(X) \neq |\mathfrak{M}|$. So, the **formula** is satisfied, e.g., if $s(X) = \{1, 2\}$ but not if $s(X) = \{1, 2, 3\}$.

Photo Credits

Bibliography