syn.1 Satisfaction

sol:syn:sat:sec

To define the satisfaction relation $\mathfrak{M}, s \vDash \varphi$ for second-order formulas, we have to extend the definitions to cover second-order variables. explanation

Definition syn.1 (Variable Assignment)**.** A *variable assignment* s for a structure $\mathfrak{M}$ is a function which maps each

1. object variable v_i to an element of $|\mathfrak{M}|$, i.e., $s(v_i) \in |\mathfrak{M}|$
2. n-place relation variable V_i^n to an n-place relation on $|\mathfrak{M}|$, i.e., $s(V_i^n) \subseteq |\mathfrak{M}|^n$;
3. n-place function variable u_i^n to an n-place function from $|\mathfrak{M}|$ to $|\mathfrak{M}|$, i.e., $s(u_i^n)\colon |\mathfrak{M}|^n \to |\mathfrak{M}|$;

A structure assigns a value to each constant symbol and function symbol, and a second-order variable assigns objects and functions to each object and function variable. Together, they let us assign a value to very term. explanation

Definition syn.2 (Value of a Term)**.** If t is a term of the language $\mathcal{L}$, $\mathfrak{M}$ is a structure for $\mathcal{L}$, and s is a variable assignment for $\mathfrak{M}$, the *value* $\mathrm{Val}_s^{\mathfrak{M}}(t)$ is defined as for first-order terms, plus the following clause:

$t \equiv u(t_1, \dots, t_n)$:

$$\mathrm{Val}_s^{\mathfrak{M}}(t) = s(u)(\mathrm{Val}_s^{\mathfrak{M}}(t_1), \dots, \mathrm{Val}_s^{\mathfrak{M}}(t_n)).$$

Definition syn.3 (Satisfaction)**.** For second-order formulas φ, the definition of satisfaction is like **??** with the addition of:

1. $\varphi \equiv X^n t_1, \dots, t_n$: $\mathfrak{M}, s \vDash \varphi$ iff $\langle \mathrm{Val}_s^{\mathfrak{M}}(t_1), \dots, \mathrm{Val}_s^{\mathfrak{M}}(t_n)\rangle \in s(X^n)$.
2. $\varphi \equiv \forall X\, \psi$: $\mathfrak{M}, s \vDash \varphi$ iff for every X-variant s' of s, $\mathfrak{M}, s' \vDash \psi$.
3. $\varphi \equiv \exists X\, \psi$: $\mathfrak{M}, s \vDash \varphi$ iff there is an X-variant s' of s so that $\mathfrak{M}, s' \vDash \psi$.
4. $\varphi \equiv \forall u\, \psi$: $\mathfrak{M}, s \vDash \varphi$ iff for every u-variant s' of s, $\mathfrak{M}, s' \vDash \psi$.
5. $\varphi \equiv \exists u\, \psi$: $\mathfrak{M}, s \vDash \varphi$ iff there is an u-variant s' of s so that $\mathfrak{M}, s' \vDash \psi$.

Example syn.4. $\mathfrak{M}, s \vDash \forall z\,(Xz \leftrightarrow \lnot Yz)$ whenever $s(Y) = |\mathfrak{M}| \setminus s(X)$. So for instance, let $|\mathfrak{M}| = \{1, 2, 3\}$, $s(X) = \{1, 2\}$ and $s(Y) = \{3\}$.

$\mathfrak{M}, s \vDash \exists Y\,(\exists y\, Yy \land \forall z\,(Xz \leftrightarrow \lnot Yz))$ if there is an $s' \sim_Y s$ such that $\mathfrak{M}, s \vDash (\exists y\, Yy \land \forall z\,(Xz \leftrightarrow \lnot Yz))$. And that is the case iff $s'(Y) \neq \emptyset$ (so that $\mathfrak{M}, s' \vDash \exists y\, Yy$) and, as before, $s'(Y) = |\mathfrak{M}| \setminus s'(X)$. In other words, $\mathfrak{M}, s \vDash \exists Y\,(\exists y\, Yy \land \forall z\,(Xz \leftrightarrow \lnot Yz))$ iff $|\mathfrak{M}| \setminus s(X)$ is non-empty, or, $s(X) \neq |\mathfrak{M}|$. So, the formula is satisfied, e.g., if $s(X) = \{1, 2\}$ but not if $s(X) = \{1, 2, 3\}$.

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Bibliography