

## syn.1 Satisfaction

sol:syn:sat:  
sec To define the satisfaction relation  $\mathfrak{M}, s \models \varphi$  for second-order **formulas**, we explanation have to extend the definitions to cover second-order variables.

**Definition syn.1** (Variable Assignment). A *variable assignment*  $s$  for a **structure**  $\mathfrak{M}$  is a function which maps each

1. object **variable**  $v_i$  to an element of  $|\mathfrak{M}|$ , i.e.,  $s(v_i) \in |\mathfrak{M}|$
2.  $n$ -place relation variable  $V_i^n$  to an  $n$ -place relation on  $|\mathfrak{M}|$ , i.e.,  $s(V_i^n) \subseteq |\mathfrak{M}|^n$ ;
3.  $n$ -place function variable  $u_i^n$  to an  $n$ -place function from  $|\mathfrak{M}|$  to  $|\mathfrak{M}|$ , i.e.,  $s(u_i^n): |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|$ ;

A **structure** assigns a **value** to each **constant symbol** and **function symbol**, explanation and a second-order variable assigns objects and functions to each object and function variable. Together, they let us assign a value to every term.

**Definition syn.2** (Value of a Term). If  $t$  is a term of the language  $\mathcal{L}$ ,  $\mathfrak{M}$  is a **structure** for  $\mathcal{L}$ , and  $s$  is a **variable assignment** for  $\mathfrak{M}$ , the *value*  $\text{Val}_s^{\mathfrak{M}}(t)$  is defined as for first-order terms, plus the following clause:

$$t \equiv u(t_1, \dots, t_n):$$

$$\text{Val}_s^{\mathfrak{M}}(t) = s(u)(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n)).$$

**Definition syn.3** (Satisfaction). For second-order **formulas**  $\varphi$ , the definition of satisfaction is like ?? with the addition of:

1.  $\varphi \equiv X^n t_1, \dots, t_n$ :  $\mathfrak{M}, s \models \varphi$  iff  $\langle \text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n) \rangle \in s(X^n)$ .
2.  $\varphi \equiv \forall X \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for every  $X$ -variant  $s'$  of  $s$ ,  $\mathfrak{M}, s' \models \psi$ .
3.  $\varphi \equiv \exists X \psi$ :  $\mathfrak{M}, s \models \varphi$  iff there is an  $X$ -variant  $s'$  of  $s$  so that  $\mathfrak{M}, s' \models \psi$ .
4.  $\varphi \equiv \forall u \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for every  $u$ -variant  $s'$  of  $s$ ,  $\mathfrak{M}, s' \models \psi$ .
5.  $\varphi \equiv \exists u \psi$ :  $\mathfrak{M}, s \models \varphi$  iff there is an  $u$ -variant  $s'$  of  $s$  so that  $\mathfrak{M}, s' \models \psi$ .

**Example syn.4.**  $\mathfrak{M}, s \models \forall z (Xz \leftrightarrow \neg Yz)$  whenever  $s(Y) = |\mathfrak{M}| \setminus s(X)$ . So for instance, let  $|\mathfrak{M}| = \{1, 2, 3\}$ ,  $s(X) = \{1, 2\}$  and  $s(Y) = \{3\}$ .

$\mathfrak{M}, s \models \exists Y (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$  if there is an  $s' \sim_Y s$  such that  $\mathfrak{M}, s' \models (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$ . And that is the case iff  $s'(Y) \neq \emptyset$  (so that  $\mathfrak{M}, s' \models \exists y Yy$ ) and, as before,  $s'(Y) = |\mathfrak{M}| \setminus s'(X)$ . In other words,  $\mathfrak{M}, s \models \exists Y (\exists y Yy \wedge \forall z (Xz \leftrightarrow \neg Yz))$  iff  $|\mathfrak{M}| \setminus s(X)$  is non-empty, or,  $s(X) \neq |\mathfrak{M}|$ . So, the **formula** is satisfied, e.g., if  $s(X) = \{1, 2\}$  but not if  $s(X) = \{1, 2, 3\}$ .

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**Bibliography**