syn.1  Satisfaction

To define the satisfaction relation $\mathfrak{M}, s \vDash \varphi$ for second-order formulas, we have to extend the definitions to cover second-order variables.

**Definition syn.1 (Variable Assignment).** A variable assignment $s$ for a structure $\mathfrak{M}$ is a function which maps each

1. object variable $v_i$ to an element of $|\mathfrak{M}|$, i.e., $s(v_i) \in |\mathfrak{M}|$
2. $n$-place relation variable $V^n_i$ to an $n$-place relation on $|\mathfrak{M}|$, i.e., $s(V^n_i) \subseteq |\mathfrak{M}|^n$;
3. $n$-place function variable $u^n_i$ to an $n$-place function from $|\mathfrak{M}|$ to $|\mathfrak{M}|$, i.e.,

$$s(u^n_i): |\mathfrak{M}|^n \rightarrow |\mathfrak{M}|;$$

A structure assigns a value to each constant symbol and function symbol, and a second-order variable assigns objects and functions to each object and function variable. Together, they let us assign a value to very term.

**Definition syn.2 (Value of a Term).** If $t$ is a term of the language $L$, $\mathfrak{M}$ is a structure for $L$, and $s$ is a variable assignment for $\mathfrak{M}$, the value $\text{Val}_s^\mathfrak{M}(t)$ is defined as for first-order terms, plus the following clause:

$$t \equiv u(t_1, \ldots, t_n);$$

$$\text{Val}_s^\mathfrak{M}(t) = s(u)(\text{Val}_s^\mathfrak{M}(t_1), \ldots, \text{Val}_s^\mathfrak{M}(t_n)).$$

**Definition syn.3 (Satisfaction).** For second-order formulas $\varphi$, the definition of satisfaction is like ?? with the addition of:

1. $\varphi \equiv X^n t_1, \ldots, t_n$: $\mathfrak{M}, s \vDash \varphi$ iff $(\text{Val}_s^\mathfrak{M}(t_1), \ldots, \text{Val}_s^\mathfrak{M}(t_n)) \in s(X^n)$.
2. $\varphi \equiv \forall X \psi$: $\mathfrak{M}, s \vDash \varphi$ iff for every $X$-variant $s'$ of $s$, $\mathfrak{M}, s' \vDash \psi$.
3. $\varphi \equiv \exists X \psi$: $\mathfrak{M}, s \vDash \varphi$ iff there is an $X$-variant $s'$ of $s$ so that $\mathfrak{M}, s' \vDash \psi$.
4. $\varphi \equiv \forall u \psi$: $\mathfrak{M}, s \vDash \varphi$ iff for every $u$-variant $s'$ of $s$, $\mathfrak{M}, s' \vDash \psi$.
5. $\varphi \equiv \exists u \psi$: $\mathfrak{M}, s \vDash \varphi$ iff there is an $u$-variant $s'$ of $s$ so that $\mathfrak{M}, s' \vDash \psi$.

**Example syn.4.** $\mathfrak{M}, s \vDash \forall z (X z \leftrightarrow \neg Y z)$ whenever $s(Y) = |\mathfrak{M}| \setminus s(X)$. So for instance, let $|\mathfrak{M}| = \{1, 2, 3\}$, $s(X) = \{1, 2\}$ and $s(Y) = \{3\}$.

$\mathfrak{M}, s \vDash \exists Y (\exists y Y y \land \forall z (X z \leftrightarrow \neg Y z))$ if there is an $s'$ of $s$ such that $\mathfrak{M}, s' \vDash (\exists y Y y \land \forall z (X z \leftrightarrow \neg Y z))$. And that is the case if $s'(Y) \neq 0$ (so that $\mathfrak{M}, s' \vDash \exists y Y y$) and, as before, $s'(Y) = |\mathfrak{M}| \setminus s'(X)$. In other words, $\mathfrak{M}, s \vDash \exists Y (\exists y Y y \land \forall z (X z \leftrightarrow \neg Y z))$ iff $|\mathfrak{M}| \setminus s(X)$ is non-empty, or, $s(X) \neq |\mathfrak{M}|$. So, the formula is satisfied, e.g., if $s(X) = \{1, 2\}$ but not if $s(X) = \{1, 2, 3\}$.