## syn.1 Introduction

sol:syn:int:sec In first-order logic, we combine the non-logical symbols of a given language, i.e., its constant symbols, function symbols, and predicate symbols, with the logical symbols to express things about first-order structures. This is done using the notion of satisfaction, which relates a structure $\mathfrak{M}$, together with a variable assignment $s$, and a formula $\varphi$: $\mathfrak{M}, s \vDash \varphi$ holds iff what $\varphi$ expresses when its constant symbols, function symbols, and predicate symbols are interpreted as $\mathfrak{M}$ says, and its free variables are interpreted as $s$ says, is true. The interpretation of the identity predicate $=$ is built into the definition of $\mathfrak{M}, s \vDash \varphi$, as is the interpretation of $\forall$ and $\exists$. The former is always interpreted as the identity relation on the domain $|\mathfrak{M}|$ of the structure, and the quantifiers are always interpreted as ranging over the entire domain. But, crucially, quantification is only allowed over elements of the domain, and so only object variables are allowed to follow a quantifier.

In second-order logic, both the language and the definition of satisfaction are extended to include free and bound function and predicate variables, and quantification over them. These variables are related to function symbols and predicate symbols the same way that object variables are related to constant symbols. They play the same role in the formation of terms and formulas of second-order logic, and quantification over them is handled in a similar way. In the *standard* semantics, the second-order quantifiers range over all possible objects of the right type ($n$-place functions from $|\mathfrak{M}|$ to $|\mathfrak{M}|$ for function variables, $n$-place relations for predicate variables). For instance, while $\forall v_0 \, (P^1_0(v_0) \lor \lnot P^1_0(v_0))$ is a formula in both first- and second-order logic, in the latter we can also consider $\forall V^1_0 \, \forall v_0 \, (V^1_0(v_0) \lor \lnot V^1_0(v_0))$ and $\exists V^1_0 \, \forall v_0 \, (V^1_0(v_0) \lor \lnot V^1_0(v_0))$. Since these contain no free variables, they are sentences of second-order logic. Here, $V^1_0$ is a second-order 1-place predicate variable. The allowable interpretations of $V^1_0$ are the same that we can assign to a 1-place predicate symbol like $P^1_0$, i.e., subsets of $|\mathfrak{M}|$. Quantification over them then amounts to saying that $\forall v_0 \, (V^1_0(v_0) \lor \lnot V^1_0(v_0))$ holds for all ways of assigning a subset of $|\mathfrak{M}|$ as the value of $V^1_0$, or for at least one. Since every set either contains or fails to contain a given object, both are true in any structure.

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## Bibliography