

syn.1 Describing Infinite and Enumerable Domains

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A set M is (Dedekind) infinite iff there is an **injective** function $f: M \rightarrow M$ which is not **surjective**, i.e., with $\text{dom}(f) \neq M$. In first-order logic, we can consider a one-place **function symbol** f and say that the function $f^{\mathfrak{M}}$ assigned to it in a **structure** \mathfrak{M} is **injective** and $\text{ran}(f) \neq |\mathfrak{M}|$:

$$\forall x \forall y (f(x) = f(y) \rightarrow x = y) \wedge \exists y \forall x y \neq f(x)$$

If \mathfrak{M} satisfies this **sentence**, $f^{\mathfrak{M}}: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$ is **injective**, and so $|\mathfrak{M}|$ must be infinite. If $|\mathfrak{M}|$ is infinite, and hence such a function exists, we can let $f^{\mathfrak{M}}$ be that function and \mathfrak{M} will satisfy the sentence. However, this requires that our language contains the non-logical symbol f we use for this purpose. In second-order logic, we can simply say that such a function *exists*. This no-longer requires f , and we have the **sentence** in pure second-order logic

$$\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x))$$

$\mathfrak{M} \models \text{Inf}$ iff $|\mathfrak{M}|$ is infinite. We can then define $\text{Fin} \equiv \neg \text{Inf}$; $\mathfrak{M} \models \text{Fin}$ iff $|\mathfrak{M}|$ is finite. No single **sentence** of pure first-order logic can express that the **domain** is infinite although an infinite set of them can. There is no set of **sentences** of pure first-order logic that is satisfied in a **structure** iff its domain is finite.

Proposition syn.1. $\mathfrak{M} \models \text{Inf}$ iff $|\mathfrak{M}|$ is infinite.

Proof. $\mathfrak{M} \models \text{Inf}$ iff $\mathfrak{M}, s \models \forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \exists y \forall x y \neq u(x)$ for some s . If it does, $s(u)$ is an **injective** function, and some $y \in |\mathfrak{M}|$ is not in the domain of $s(u)$. Conversely, if there is an **injective** $f: |\mathfrak{M}| \rightarrow |\mathfrak{M}|$ with $\text{dom}(f) \neq |\mathfrak{M}|$, then $s(u) = f$ is such a variable assignment. \square

A set M is **enumerable** if there is an enumeration

$$m_0, m_1, m_2, \dots$$

of its **elements** (without repetitions). Such an enumeration exists iff there is an **element** $z \in M$ and a function $f: M \rightarrow M$ such that $z, f(z), f(f(z))$ are all the **elements** of M . For if the enumeration exists, $z = m_0$ and $f(m_k) = m_{k+1}$ (or $f(m_k) = m_k$ if m_k is the last **element** of the enumeration) are the requisite **element** and function. On the other hand, if such a z and f exist, then $z, f(z), f(f(z)), \dots$, is an enumeration of M , and M is **enumerable**. We can express the existence of z and f in second-order logic to produce a **sentence** true in a **structure** iff the structure is **enumerable**:

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

Proposition syn.2. $\mathfrak{M} \models \text{Count}$ iff $|\mathfrak{M}|$ is **enumerable**.

Proof. Suppose $|\mathfrak{M}|$ is enumerable, and let m_0, m_1, \dots , be an enumeration. By removing repetitions we can guarantee that no m_k appears twice. Define $f(m_k) = m_{k+1}$ and let $s(z) = m_0$ and $s(u) = f$. We show that

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

Suppose $s' \sim_X s$, and $M = s'(X)$. Suppose further that $\mathfrak{M}, s' \models (X(z) \wedge \forall x (X(x) \rightarrow X(u(x))))$. Then $s'(z) \in M$ and whenever $x \in M$, also $s'(u)(x) \in M$. In other words, since $s' \sim_X s$, $m_0 \in M$ and if $x \in M$ then $f(x) \in M$, so $m_0 \in M$, $m_1 = f(m_0) \in M$, $m_2 = f(f(m_0)) \in M$, etc. Thus, $M = |\mathfrak{M}|$, and $\mathfrak{M} \models \forall x X(x)s'$. Since s' was an arbitrary X -variant of s , we are done.

Now assume that

$$\mathfrak{M}, s \models \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

for some s . Let $m = s(z)$ and $f = s(u)$ and consider $M = \{m, f(m), f(f(m)), \dots\}$. Let s' be the X -variant of s with $s'(X) = M$. Then

$$\mathfrak{M}, s' \models ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

by assumption. Also, $\mathfrak{M}, s' \models X(z)$ since $s'(X) = M \ni m = s'(z)$, and also $\mathfrak{M}, s' \models \forall x (X(x) \rightarrow X(u(x)))$ since whenever $x \in M$ also $f(x) \in M$. So, since both antecedent and conditional are satisfied, the consequent must also be: $\mathfrak{M}, s' \models \forall x X(x)$. But that means that $M = |\mathfrak{M}|$, and so $|\mathfrak{M}|$ is **enumerable**. \square

Problem syn.1. The sentence $\text{Inf} \wedge \text{Count}$ is true in all and only **denumerable** domains. Adjust the definition of Count so that it becomes a different **sentence** that directly expresses that the domain is **denumerable**, and prove that it does.

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Bibliography