A set \( M \) is (Dedekind) infinite iff there is an injective function \( f: M \to M \) which is not surjective, i.e., with \( \text{dom}(f) \neq M \). In first-order logic, we can consider a one-place function symbol \( f \) and say that the function \( f^\mathfrak{M} \) assigned to it in a structure \( \mathfrak{M} \) is injective and \( \text{ran}(f) \neq |\mathfrak{M}| \):

\[
\forall x \forall y (f(x) = f(y) \to x = y) \land \exists y \forall x y \neq f(x).
\]

If \( \mathfrak{M} \) satisfies this sentence, \( f^\mathfrak{M}: |\mathfrak{M}| \to |\mathfrak{M}| \) is injective, and so \( |\mathfrak{M}| \) must be infinite. If \( |\mathfrak{M}| \) is infinite, and hence such a function exists, we can let \( f^\mathfrak{M} \) be that function and \( \mathfrak{M} \) will satisfy the sentence. However, this requires that our language contains the non-logical symbol \( f \) we use for this purpose. In second-order logic, we can simply say that such a function exists. This no-longer requires \( f \), and we obtain the sentence in pure second-order logic:

\[
\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \to x = y) \land \exists y \forall x y \neq u(x)).
\]

\( \mathfrak{M} \models \text{Inf} \) if \( |\mathfrak{M}| \) is infinite. We can then define \( \text{Fin} \equiv \neg \text{Inf} \); \( \mathfrak{M} \models \text{Fin} \) iff \( |\mathfrak{M}| \) is finite. No single sentence of pure first-order logic can express that the domain is infinite although an infinite set of them can. There is no set of sentences of pure first-order logic that is satisfied in a structure iff its domain is finite.

Proposition syn.1. \( \mathfrak{M} \models \text{Inf} \) iff \( |\mathfrak{M}| \) is infinite.

Proof. \( \mathfrak{M} \models \text{Inf} \iff \mathfrak{M} \models \forall x \forall y (u(x) = u(y) \to x = y) \land \exists y \forall x y \neq u(x) \) for some \( s \). If it does, \( s(u) \) is an injective function, and some \( y \in |\mathfrak{M}| \) is not in the domain of \( s(u) \). Conversely, if there is an injective \( f: |\mathfrak{M}| \to |\mathfrak{M}| \) with \( \text{dom}(f) \neq |\mathfrak{M}| \), then \( s(u) = f \) is such a variable assignment.

A set \( M \) is enumerable if there is an enumeration

\[
m_0, m_1, m_2, \ldots
\]

of its elements (without repetitions but possibly finite). Such an enumeration exists iff there is an element \( z \in M \) and a function \( f: M \to M \) such that \( z, f(z), f(f(z)), \ldots, \) are all the elements of \( M \). For if the enumeration exists, \( z = m_0 \) and \( f(m_k) = m_{k+1} \) (or \( f(m_k) = m_k \) if \( m_k \) is the last element of the enumeration) are the requisite element and function. On the other hand, if such a \( z \) and \( f \) exist, then \( z, f(z), f(f(z)), \ldots \) is an enumeration of \( M \), and \( M \) is enumerable. We can express the existence of \( z \) and \( f \) in second-order logic to produce a sentence true in a structure iff the structure is enumerable:

\[
\text{Count} \equiv \exists z \exists u \forall X ((X(z) \land \forall x (X(x) \to X(u(x)))) \to \forall x X(x))
\]

Proposition syn.2. \( \mathfrak{M} \models \text{Count} \) iff \( |\mathfrak{M}| \) is enumerable.
Proof. Suppose $|\mathcal{M}|$ is enumerable, and let $m_0, m_1, \ldots$, be an enumeration. By removing repetitions we can guarantee that no $m_k$ appears twice. Define $f(m_k) = m_{k+1}$ and let $s(z) = m_0$ and $s(u) = f$. We show that $\mathcal{M}, s \models \forall X ((X(z) \land \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$

Suppose $s' \sim_X s$ is arbitrary, and let $M = s'(X)$. Suppose further that $\mathcal{M}, s' \models (X(z) \land \forall x (X(x) \rightarrow X(u(x))))$. Then $s'(z) \in M$ and whenever $x \in M$, also $s'(u)(x) \in M$. In other words, since $s' \sim_X s$, $m_0 \in M$ and if $x \in M$ then $f(x) \in M$, so $m_0 \in M$, $m_1 = f(m_0) \in M$, $m_2 = f(f(m_0)) \in M$, etc. Thus, $M = |\mathcal{M}|$, and so $\mathcal{M}, s' \models \forall x X(x)$. Since $s'$ was an arbitrary $X$-variant of $s$, we are done: $\mathcal{M} \models \text{Count}$.

Now assume that $\mathcal{M} \models \text{Count}$, i.e.,

$\mathcal{M}, s \models \forall X ((X(z) \land \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$

for some $s$. Let $m = s(z)$ and $f = s(u)$ and consider $M = \{m, f(m), f(f(m)), \ldots\}$. Let $s'$ be the $X$-variant of $s$ with $s(X) = M$. Then

$\mathcal{M}, s' \models (X(z) \land \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x)$

by assumption. Also, $\mathcal{M}, s' \models X(z)$ since $s'(X) = M \ni m = s'(z)$, and also $\mathcal{M}, s' \models \forall x (X(x) \rightarrow X(u(x)))$ since whenever $x \in M$ also $f(x) \in M$. So, since both antecedent and conditional are satisfied, the consequent must also be: $\mathcal{M}, s' \models \forall x X(x)$. But that means that $M = |\mathcal{M}|$, and so $|\mathcal{M}|$ is enumerable since $M$ is, by definition. \qed

**Problem syn.1.** The sentence $\text{Inf} \land \text{Count}$ is true in all and only denumerable domains. Adjust the definition of $\text{Count}$ so that it becomes a different sentence that directly expresses that the domain is denumerable, and prove that it does.

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**Bibliography**