syn.1 Expressive Power

sol:syn:exp:sec Quantification over second-order variables is responsible for an immense increase in the expressive power of the language over that of first-order logic. Second-order existential quantification lets us say that functions or relations with certain properties exists. In first-order logic, the only way to do that is to specify a non-logical symbol (i.e., a function symbol or predicate symbol) for this purpose. Second-order universal quantification lets us say that all subsets of, relations on, or functions from the domain to the domain have a property. In first-order logic, we can only say that the subsets, relations, or functions assigned to one of the non-logical symbols of the language have a property. And when we say that subsets, relations, functions exist that have a property, or that all of them have it, we can use second-order quantification in specifying this property as well. This lets us define relations not definable in first-order logic, and express properties of the domain not expressible in first-order logic. explanation

Definition syn.1. If $\mathfrak{M}$ is a structure for a language $\mathcal{L}$, a relation $R \subseteq |\mathfrak{M}|^2$ is *definable* in $\mathcal{L}$ if there is some formula $\varphi_R(x, y)$ with only the variables x and y free, such that $R(a, b)$ holds (i.e., $\langle a, b \rangle \in R$) iff $\mathfrak{M}, s \vDash \varphi_R(x, y)$ for $s(x) = a$ and $s(y) = b$.

Example syn.2. In first-order logic we can define the identity relation $\mathrm{Id}_{|\mathfrak{M}|}$ (i.e., $\{\langle a, a \rangle : a \in |\mathfrak{M}|\}$) by the formula $x = y$. In second-order logic, we can define this relation *without* $=$. For if a and b are the same element of $|\mathfrak{M}|$, then they are elements of the same subsets of $|\mathfrak{M}|$ (since sets are determined by their elements). Conversely, if a and b are different, then they are not elements of the same subsets: e.g., $a \in \{a\}$ but $b \notin \{a\}$ if $a \neq b$. So "being elements of the same subsets of $|\mathfrak{M}|$" is a relation that holds of a and b iff $a = b$. It is a relation that can be expressed in second-order logic, since we can quantify over all subsets of $|\mathfrak{M}|$. Hence, the following formula defines $\mathrm{Id}_{|\mathfrak{M}|}$:

$$\forall X\,(X(x) \leftrightarrow X(y))$$

Problem syn.1. Show that $\forall X\,(X(x) \rightarrow X(y))$ (note: $\rightarrow$ not $\leftrightarrow$!) defines $\mathrm{Id}_{|\mathfrak{M}|}$.

Example syn.3. If R is a two-place predicate symbol, $R^{\mathfrak{M}}$ is a two-place relation on $|\mathfrak{M}|$. Perhaps somewhat confusingly, we'll use R as the predicate symbol for R and for the relation $R^{\mathfrak{M}}$ itself. The *transitive closure* R^* of R is the relation that holds between a and b iff for some $c_1, \ldots, c_k$, $R(a, c_1)$, $R(c_1, c_2)$, $\ldots$, $R(c_k, b)$ holds. This includes the case if $k = 0$, i.e., if $R(a, b)$ holds, so does $R^*(a, b)$. This means that $R \subseteq R^*$. In fact, R^* is the smallest relation that includes R and that is transitive. We can say in second-order logic that X is a transitive relation that includes R:

$$\psi_R(X) \equiv \forall x\,\forall y\,(R(x, y) \rightarrow X(x, y)) \land \\ \forall x\,\forall y\,\forall z\,((X(x, y) \land X(y, z)) \rightarrow X(x, z)).$$

The first conjunct says that $R \subseteq X$ and the second that X is transitive.

To say that X is the smallest such relation is to say that it is itself included in every relation that includes R and is transitive. So we can define the transitive closure of R by the formula

$$R^*(X) \equiv \psi_R(X) \land \forall Y\, (\psi_R(Y) \to \forall x\, \forall y\, (X(x,y) \to Y(x,y))).$$

We have $\mathfrak{M}, s \vDash R^*(X)$ iff $s(X) = R^*$. The transitive closure of R cannot be expressed in first-order logic.

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Bibliography