Quantification over second-order variables is responsible for an immense increase in the expressive power of the language over that of first-order logic. Second-order existential quantification lets us say that functions or relations with certain properties exist. In first-order logic, the only way to do that is to specify a non-logical symbol (i.e., a function symbol or predicate symbol) for this purpose. Second-order universal quantification lets us say that all subsets of, relations on, or functions from the domain to the domain have a property. And when we say that subsets, relations, functions exist that have a property, or that all of them have it, we can use second-order quantification in specifying this property as well. This lets us define relations not definable in first-order logic, and express properties of the domain not expressible in first-order logic.

Definition syn.1. If $\mathcal{M}$ is a structure for a language $\mathcal{L}$, a relation $R \subseteq |\mathcal{M}|^2$ is \textit{definable} in $\mathcal{L}$ if there is some formula $\varphi_R(x, y)$ with only the variables $x$ and $y$ free, such that $R(a, b)$ holds (i.e., $\langle a, b \rangle \in R$) if $\mathcal{M}, s \models \varphi_R(x, y)$ for $s(x) = a$ and $s(y) = b$.

Example syn.2. In first-order logic we can define the identity relation $\text{Id}_{|\mathcal{M}|}$ (i.e., $\{\langle a, a \rangle : a \in |\mathcal{M}|\}$) by the formula $x = y$. In second-order logic, we can define this relation \textit{without} $=$. For if $a$ and $b$ are the same element of $|\mathcal{M}|$, then they are elements of the same subsets of $|\mathcal{M}|$ (since sets are determined by their elements). Conversely, if $a$ and $b$ are different, then they are not elements of the same subsets: e.g., $a \in \{a\}$ but $b \notin \{a\}$ if $a \neq b$. So “being elements of the same subsets of $|\mathcal{M}|$" is a relation that holds of $a$ and $b$ iff $a = b$. It is a relation that can be expressed in second-order logic, since we can quantify over all subsets of $|\mathcal{M}|$. Hence, the following formula defines $\text{Id}_{|\mathcal{M}|}$:

$$\forall X \left( X(x) \leftrightarrow X(y) \right)$$

Problem syn.1. Show that $\forall X \left( X(x) \rightarrow X(y) \right)$ (note: $\rightarrow \not= \leftrightarrow$!) defines $\text{Id}_{|\mathcal{M}|}$.

Example syn.3. If $R$ is a two-place predicate symbol, $R^{\mathcal{M}}$ is a two-place relation on $|\mathcal{M}|$. Perhaps somewhat confusingly, we’ll use $R$ as the predicate symbol for $R$ and for the relation $R^{\mathcal{M}}$ itself. The \textit{transitive closure} $R^*$ of $R$ is the relation that holds between $a$ and $b$ iff for some $c_1, \ldots, c_k$, $R(a, c_1), R(c_1, c_2), \ldots, R(c_k, b)$ holds. This includes the case if $k = 0$, i.e., if $R(a, b)$ holds, so does $R^*(a, b)$. This means that $R \subseteq R^*$. In fact, $R^*$ is the smallest relation that includes $R$ and that is transitive. We can say in second-order logic that $X$ is a transitive relation that includes $R$:

$$\psi_R(X) \equiv \forall x \forall y \left( R(x, y) \rightarrow X(x, y) \right) \land \forall x \forall y \forall z \left( \left( X(x, y) \land X(y, z) \right) \rightarrow X(x, z) \right).$$
The first conjunct says that $R \subseteq X$ and the second that $X$ is transitive.

To say that $X$ is the smallest such relation is to say that it is itself included in every relation that includes $R$ and is transitive. So we can define the transitive closure of $R$ by the formula

$$R^*(X) \equiv \psi_R(X) \land \forall Y (\psi_R(Y) \rightarrow \forall x \forall y (X(x, y) \rightarrow Y(x, y))).$$

We have $\mathfrak{M}, s \models R^*(X)$ iff $s(X) = R^*$. The transitive closure of $R$ cannot be expressed in first-order logic.

Photo Credits

Bibliography