

syn.1 Expressive Power

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sec

Quantification over second-order variables is responsible for an immense increase in the expressive power of the language over that of first-order logic. Second-order existential quantification lets us say that functions or relations with certain properties exist. In first-order logic, the only way to do that is to specify non-logical symbol (i.e., a **function symbol** or **predicate symbol**) for this purpose. Second-order universal quantification lets us say that all subsets of, relations on, or functions from the **domain** to the **domain** have a property. In first-order logic, we can only say that the subsets, relations, or functions assigned to one of the non-logical symbols of the language have a property. And when we say that subsets, relations, functions exist that have a property, or that all of them have it, we can use second-order quantification in specifying this property as well. This lets us define relations not definable in first-order logic, and express properties of the domain not expressible in first-order logic.

explanation

Example syn.1. If \mathfrak{M} is a **structure** for a language \mathcal{L} , a relation $R \subseteq |\mathfrak{M}|^2$ is definable in \mathcal{L} if there is some **formula** $\varphi_R(v_0, v_1)$ with only the variables v_0 and v_1 free, such that $R(x, y)$ holds (i.e., $\langle x, y \rangle \in R$) iff $\mathfrak{M}, s \models \varphi_R(v_0, v_1)$ for $s(v_0) = x$ and $s(v_1) = y$. For instance, in first-order logic we can define the identity relation $\text{Id}_{|\mathfrak{M}|}$ (i.e., $\{\langle x, x \rangle : x \in |\mathfrak{M}|\}$) by the formula $v_0 = v_1$. In second-order logic, we can define this relation *without* $=$. For if x and y are the same **element** of $|\mathfrak{M}|$, then they are **elements** of the same subsets of $|\mathfrak{M}|$ (since sets are determined by their **elements**). Conversely, if x and y are different, then they are not **elements** of the same subsets: e.g., $x \in \{x\}$ but $y \notin \{x\}$ if $x \neq y$. So “being **elements** of the same subsets of $|\mathfrak{M}|$ ” is a relation that holds of x and y iff $x = y$. It is a relation that can be expressed in second-order logic, since we can quantify over all subsets of $|\mathfrak{M}|$. Hence, the following **formula** defines $\text{Id}_{|\mathfrak{M}|}$:

$$\forall X (X(v_0) \leftrightarrow X(v_1))$$

Problem syn.1. Show that $\forall X (X(v_0) \rightarrow X(v_1))$ (note: \rightarrow not \leftrightarrow !) defines $\text{Id}_{|\mathfrak{M}|}$.

Example syn.2. If R is a two-place **predicate symbol**, $R^{\mathfrak{M}}$ is a two-place relation on $|\mathfrak{M}|$. Its *transitive closure* R^* is the relation that holds between x and y if for some z_1, \dots, z_k , $R(x, z_1), R(z_1, z_2), \dots, R(z_k, y)$ holds. This includes the case if $k = 0$, i.e., if $R(x, y)$ holds. This means that $R \subseteq R^*$. In fact, R^* is the smallest relation that includes R and that is transitive. We can say in second-order logic that X is a transitive relation that includes R :

$$\begin{aligned} \psi_R(X) \equiv & \forall x \forall y (R(x, y) \rightarrow X(x, y)) \wedge \\ & \forall x \forall y \forall z ((X(x, y) \wedge X(y, z)) \rightarrow X(x, z)) \end{aligned}$$

Here, somewhat confusingly, we use R as the **predicate symbol** for R . The first conjunct says that $R \subseteq X$ and the second that X is transitive.

To say that X is the smallest such relation is to say that it is itself included in every relation that includes R and is transitive. So we can define the transitive closure of R by the **formula**

$$R^*(X) \equiv \psi_R(X) \wedge \forall Y (\psi_R(Y) \rightarrow \forall x \forall y (X(x, y) \rightarrow Y(x, y)))$$

$\mathfrak{M}, s \models R^*(X)$ iff $s(X) = R^*$. The transitive closure of R cannot be expressed in first-order logic.

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Bibliography