

set.1 The Power of the Continuum

sol:set:pow:
sec

In second-order logic we can quantify over subsets of the **domain**, but not over sets of subsets of the **domain**. To do this directly, we would need *third-order* logic. For instance, if we wanted to state Cantor's Theorem that there is no **injective** function from the power set of a set to the set itself, we might try to formulate it as “for every set X , and every set P , if P is the power set of X , then not $P \preceq X$ ”. And to say that P is the power set of X would require formalizing that the **elements** of P are all and only the subsets of X , so something like $\forall Y (P(Y) \leftrightarrow Y \subseteq X)$. The problem lies in $P(Y)$: that is not a **formula** of second-order logic, since only terms can be arguments to one-place relation variables like P .

explanation

We can, however, *simulate* quantification over sets of sets, if the domain is large enough. The idea is to make use of the fact that two-place relations R relates **elements** of the **domain** to **elements** of the **domain**. Given such an R , we can collect all the **elements** to which some x is R -related: $\{y \in |\mathfrak{M}| : R(x, y)\}$ is the set “coded by” x . Conversely, if $Z \subseteq \wp(|\mathfrak{M}|)$ is some collection of subsets of $|\mathfrak{M}|$, and there are at least as many **elements** of $|\mathfrak{M}|$ as there are sets in Z , then there is also a relation $R \subseteq |\mathfrak{M}|^2$ such that every $Y \in Z$ is coded by some x using R .

Definition set.1. If $R \subseteq |\mathfrak{M}|^2$, then x *R-codes* $\{y \in |\mathfrak{M}| : R(x, y)\}$.

If an **element** $x \in |\mathfrak{M}|$ *R-codes* a set $Z \subseteq |\mathfrak{M}|$, then a set $Y \subseteq |\mathfrak{M}|$ codes a set of sets, namely the sets coded by the **elements** of Y . So a set Y can *R-code* $\wp(X)$. It does so iff for every $Z \subseteq X$, some $x \in Y$ *R-codes* Z , and every $x \in Y$ *R-codes* a $Z \subseteq X$.

Proposition set.2. *The formula*

$$\text{Codes}(x, R, Z) \equiv \forall y (Z(y) \leftrightarrow R(x, y))$$

expresses that $s(x)$ *s(R)-codes* $s(Z)$. *The formula*

$$\begin{aligned} \text{Pow}(Y, R, X) \equiv \\ \forall Z (Z \subseteq X \rightarrow \exists x (Y(x) \wedge \text{Codes}(x, R, Z))) \wedge \\ \forall x (Y(x) \rightarrow \forall Z (\text{Codes}(x, R, Z) \rightarrow Z \subseteq X)) \end{aligned}$$

expresses that $s(Y)$ *s(R)-codes the power set of* $s(X)$, *i.e., the elements of* $s(Y)$ *s(R)-code exactly the subsets of* $s(X)$.

With this trick, we can express statements about the power set by quantifying over the codes of subsets rather than the subsets themselves. For instance, Cantor's Theorem can now be expressed by saying that there is no **injective** function from the domain of any relation that codes the power set of X to X itself.

explanation

Proposition set.3. *The sentence*

$$\forall X \forall Y \forall R (\text{Pow}(Y, R, X) \rightarrow \neg \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \forall x (Y(x) \rightarrow X(u(x))))))$$

is valid.

explanation

The power set of a **denumerable** set is **non-enumerable**, and so its cardinality is larger than that of any **denumerable** set (which is \aleph_0). The size of $\wp(\mathbb{N})$ is called the “power of the continuum,” since it is the same size as the points on the real number line, \mathbb{R} . If the **domain** is large enough to code the power set of a **denumerable** set, we can express that a set is the size of the continuum by saying that it is equinumerous with any set Y that codes the power set of set X of size \aleph_0 . (If the domain is not large enough, i.e., it contains no subset equinumerous with \mathbb{R} , then there can also be no relation that codes $\wp(X)$.)

Proposition set.4. *If $\mathbb{R} \preceq |\mathfrak{M}|$, then the formula*

$$\text{Cont}(Y) \equiv \exists X \exists R ((\text{Aleph}_0(X) \wedge \text{Pow}(Y, R, X)) \wedge \forall x \forall y ((Y(x) \wedge Y(y) \wedge \forall z R(x, z) \leftrightarrow R(y, z)) \rightarrow x = y))$$

expresses that $s(Y) \approx \mathbb{R}$.

Proof. $\text{Pow}(Y, R, X)$ expresses that $s(Y)$ $s(R)$ -codes the power set of $s(X)$, which $\text{Aleph}_0(X)$ says is countable. So $s(Y)$ is at least as large as the power of the continuum, although it may be larger (if multiple elements of $s(Y)$ code the same subset of X). This is ruled out by the last conjunct, which requires the association between elements of $s(Y)$ and subsets of $s(X)$ via $s(R)$ to be **injective**. \square

Proposition set.5. $|\mathfrak{M}| \approx \mathbb{R}$ *iff*

$$\mathfrak{M} \models \exists X \exists Y \exists R (\text{Aleph}_0(X) \wedge \text{Pow}(Y, R, X) \wedge \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \forall y (Y(y) \rightarrow \exists x y = u(x))))).$$

explanation

The Continuum Hypothesis is the statement that the size of the continuum is the first **non-enumerable** cardinality, i.e., that $\wp(\mathbb{N})$ has size \aleph_1 .

Proposition set.6. *The Continuum Hypothesis is true iff*

$$\text{CH} \equiv \forall X (\text{Aleph}_1(X) \leftrightarrow \text{Cont}(X))$$

is valid.

Note that it isn't true that $\neg\text{CH}$ is valid iff the Continuum Hypothesis is false. In an **enumerable** domain, there are no subsets of size \aleph_1 and also no subsets of the size of the continuum, so CH is always true in an **enumerable** domain. However, we can give a different sentence that is valid iff the Continuum Hypothesis is false:

Proposition set.7. *The Continuum Hypothesis is false iff*

$$\text{NCH} \equiv \forall X (\text{Cont}(X) \rightarrow \exists Y (Y \subseteq X \wedge \neg\text{Count}(Y) \wedge \neg X \approx Y))$$

is valid.

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Bibliography