

set.1 Comparing Sets

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Proposition set.1. The formula $\forall x (X(x) \rightarrow Y(x))$ defines the subset relation, i.e., $\mathfrak{M}, s \models \forall x (X(x) \rightarrow Y(x))$ iff $s(X) \subseteq S(y)$.

Proposition set.2. The formula $\forall x (X(x) \leftrightarrow Y(x))$ defines the identity relation on sets, i.e., $\mathfrak{M}, s \models \forall x (X(x) \leftrightarrow Y(x))$ iff $s(X) = S(y)$.

Proposition set.3. The formula $\exists x X(x)$ defines the property of being non-empty, i.e., $\mathfrak{M}, s \models \exists x X(x)$ iff $s(X) \neq \emptyset$.

A set X is no larger than a set Y , $X \preceq Y$, iff there is an injective function $f: X \rightarrow Y$. Since we can express that a function is injective, and also that its values for arguments in X are in Y , we can also define the relation of being no larger than on subsets of the domain.

Proposition set.4. The formula

$$\exists u (\forall x (X(x) \rightarrow Y(u(x))) \wedge \forall x \forall y (u(x) = u(y) \rightarrow x = y))$$

defines the relation of being no larger than.

Two sets are the same size, or “equinumerous,” $X \approx Y$, iff there is a bijective function $f: X \rightarrow Y$.

Proposition set.5. The formula

$$\begin{aligned} \exists u (\forall x (X(x) \rightarrow Y(u(x))) \wedge \\ \forall x \forall y (u(x) = u(y) \rightarrow x = y) \wedge \\ \forall y (Y(y) \rightarrow \exists x (X(x) \wedge y = u(x)))) \end{aligned}$$

defines the relation of being equinumerous with.

We will abbreviate these formulas, respectively, as $X \subseteq Y$, $X = Y$, $X \neq \emptyset$, $X \preceq Y$, and $X \approx Y$. (This may be slightly confusing, since we use the same notation when we speak informally about sets X and Y —but here the notation is an abbreviation for formulas in second-order logic involving one-place relation variables X and Y .)

Proposition set.6. The sentence $\forall X \forall Y ((X \preceq Y \wedge Y \preceq X) \rightarrow X \approx Y)$ is valid.

Proof. The sentence is satisfied in a structure \mathfrak{M} if, for any subsets $X \subseteq |\mathfrak{M}|$ and $Y \subseteq |\mathfrak{M}|$, if $X \preceq Y$ and $Y \preceq X$ then $X \approx Y$. But this holds for any sets X and Y —it is the Schröder-Bernstein Theorem. \square

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Bibliography