set.1 Comparing Sets

**Proposition set.1.** The formula $\forall x (X(x) \rightarrow Y(x))$ defines the subset relation, i.e., $\mathcal{M}, s \models \forall x (X(x) \rightarrow Y(x))$ iff $s(X) \subseteq s(Y)$.

**Proposition set.2.** The formula $\forall x (X(x) \leftrightarrow Y(x))$ defines the identity relation on sets, i.e., $\mathcal{M}, s \models \forall x (X(x) \leftrightarrow Y(x))$ iff $s(X) = s(Y)$.

**Proposition set.3.** The formula $\exists x X(x)$ defines the property of being non-empty, i.e., $\mathcal{M}, s \models \exists x X(x)$ iff $s(X) \neq \emptyset$.

A set $X$ is no larger than a set $Y$, $X \preceq Y$, iff there is an injective function $f : X \rightarrow Y$. Since we can express that a function is injective, and also that its values for arguments in $X$ are in $Y$, we can also define the relation of being no larger than on subsets of the domain.

**Proposition set.4.** The formula

$$\exists u (\forall x (X(x) \rightarrow Y(u(x))) \land \forall x \forall y (u(x) = u(y) \rightarrow x = y))$$

defines the relation of being no larger than.

Two sets are the same size, or “equinumerous,” $X \approx Y$, iff there is a bijective function $f : X \rightarrow Y$. We will abbreviate these formulas, respectively, as $X \subseteq Y$, $X = Y$, $X \neq \emptyset$, $X \preceq Y$, and $X \approx Y$. (This may be slightly confusing, since we use the same notation when we speak informally about sets $X$ and $Y$—but here the notation is an abbreviation for formulas in second-order logic involving one-place relation variables $X$ and $Y$.)

**Proposition set.5.** The formula

$$\exists u (\forall x (X(x) \rightarrow Y(u(x))) \land \forall x \forall y (u(x) = u(y) \rightarrow x = y) \land \forall y (Y(y) \rightarrow \exists x (X(x) \land y = u(x))))$$

defines the relation of being equinumerous with.

We will abbreviate these formulas, respectively, as $X \subseteq Y$, $X = Y$, $X \neq \emptyset$, $X \preceq Y$, and $X \approx Y$. (This may be slightly confusing, since we use the same notation when we speak informally about sets $X$ and $Y$—but here the notation is an abbreviation for formulas in second-order logic involving one-place relation variables $X$ and $Y$.)

**Proposition set.6.** The sentence $\forall X \forall Y ((X \preceq Y \land Y \preceq X) \rightarrow X \approx Y)$ is valid.

**Proof.** The sentence is satisfied in a structure $\mathcal{M}$ if, for any subsets $X \subseteq |\mathcal{M}|$ and $Y \subseteq |\mathcal{M}|$, if $X \preceq Y$ and $Y \preceq X$ then $X \approx Y$. But this holds for any sets $X$ and $Y$—it is the Schröder-Bernstein Theorem. \qed
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Bibliography