set.1 Comparing Sets

Proposition set.1. The formula \( \forall x (X(x) \rightarrow Y(x)) \) defines the subset relation, i.e., \( \mathfrak{M}, s \models \forall x (X(x) \rightarrow Y(x)) \) iff \( s(X) \subseteq S(y) \).

Proposition set.2. The formula \( \forall x (X(x) \leftrightarrow Y(x)) \) defines the identity relation on sets, i.e., \( \mathfrak{M}, s \models \forall x (X(x) \leftrightarrow Y(x)) \) iff \( s(X) = S(y) \).

Proposition set.3. The formula \( \exists x X(x) \) defines the property of being non-empty, i.e., \( \mathfrak{M}, s \models \exists x X(x) \) iff \( s(X) \neq \emptyset \).

A set \( X \) is no larger than a set \( Y \), \( X \preceq Y \), iff there is an injective function \( f: X \rightarrow Y \). Since we can express that a function is injective, and also that its values for arguments in \( X \) are in \( Y \), we can also define the relation of being no larger than on subsets of the domain.

Proposition set.4. The formula
\[
\exists u (\forall x (X(x) \rightarrow Y(u(x)))) \land \forall x \forall y (u(x) = u(y) \rightarrow x = y))
\]
defines the relation of being no larger than.

Two sets are the same size, or “equinumerous,” \( X \approx Y \), iff there is a bijective function \( f: X \rightarrow Y \). Since we can express that a function is bijective, and also that its values for arguments in \( X \) are in \( Y \), we can also define the relation of being equinumerous.

Proposition set.5. The formula
\[
\exists u (\forall x (X(x) \rightarrow Y(u(x)))) \land \\
\forall x \forall y (u(x) = u(y) \rightarrow x = y)) \land \\
\forall y (Y(y) \rightarrow \exists x (X(x) \land y = u(x))))
\]
defines the relation of being equinumerous with.

We will abbreviate these formulas, respectively, as \( X \subseteq Y \), \( X = Y \), \( X \neq \emptyset \), \( X \preceq Y \), and \( X \approx Y \). (This may be slightly confusing, since we use the same notation when we speak informally about sets \( X \) and \( Y \)—but here the notation is an abbreviation for formulas in second-order logic involving one-place relation variables \( X \) and \( Y \).)

Proposition set.6. The sentence \( \forall X \forall Y ((X \preceq Y \land Y \preceq X) \rightarrow X \approx Y) \) is valid.

Proof. The is satisfied in a structure \( \mathfrak{M} \) if, for any subsets \( X \subseteq |X| \) and \( Y \subseteq |\mathfrak{M}| \), if \( X \preceq Y \) and \( Y \preceq X \) then \( X \approx Y \). But this holds for any sets \( X \) and \( Y \)—it is the Schröder-Bernstein Theorem. \( \square \)