set.1 Cardinalities of Sets

sol:set:crd:sec Just as we can express that the domain is finite or infinite, enumerable or non-enumerable, we can define the property of a subset of $|\mathfrak{M}|$ being finite or infinite, enumerable or non-enumerable. explanation

Proposition set.1. *The formula* $\mathrm{Inf}(X) \equiv$

$$\exists u\,(\forall x\,\forall y\,(u(x) = u(y) \rightarrow x = y) \land \exists y\,(X(y) \land \forall x\,(X(x) \rightarrow y \neq u(x)))$$

is satisfied with respect to a variable assignment s iff $s(X)$ is infinite.

Proposition set.2. *The formula* $\mathrm{Count}(X) \equiv$

$$\exists z\,\exists u\,(X(z) \land \forall x\,(X(x) \rightarrow X(u(x))) \land \forall Y\,((Y(z) \land \forall x\,(Y(x) \rightarrow Y(u(x)))) \rightarrow X = Y))$$

is satisfied with respect to a variable assignment s iff $s(X)$ is enumerable

We know from Cantor's Theorem that there are non-enumerable sets, and in fact, that there are infinitely many different levels of infinite sizes. Set theory develops an entire arithmetic of sizes of sets, and assigns infinite cardinal numbers to sets. The natural numbers serve as the cardinal numbers measuring the sizes of finite sets. The cardinality of denumerable sets is the first infinite cardinality, called $\aleph_0$ ("aleph-nought" or "aleph-zero"). The next infinite size is $\aleph_1$. It is the smallest size a set can be without being countable (i.e., of size $\aleph_0$). We can define "X has size $\aleph_0$" as $\mathrm{Aleph}_0(X) \leftrightarrow \mathrm{Inf}(X) \land \mathrm{Count}(X)$. X has size $\aleph_1$ iff all its subsets are finite or have size $\aleph_0$, but is not itself of size $\aleph_0$. Hence we can express this by the formula $\mathrm{Aleph}_1(X) \equiv \forall Y\,(Y \subseteq X \rightarrow (\lnot \mathrm{Inf}(Y) \lor \mathrm{Aleph}_0(Y))) \land \lnot \mathrm{Aleph}_0(X)$. Being of size $\aleph_2$ is defined similarly, etc.

There is one size of special interest, the so-called cardinality of the continuum. It is the size of $\wp(\mathbb{N})$, or, equivalently, the size of $\mathbb{R}$. That a set is the size of the continuum can also be expressed in second-order logic, but requires a bit more work.

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Bibliography