Second-order Logic is not Axiomatizable

Theorem met.1. Second-order logic is undecidable.

Proof. A first-order sentence is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable. □

Theorem met.2. There is no sound and complete derivation system for second-order logic.

Proof. Let \( \varphi \) be a sentence in the language of arithmetic. \( \mathfrak{N} \models \varphi \) iff \( \mathbf{PA}^2 \models \varphi \). Let \( P \) be the conjunction of the nine axioms of \( \mathbf{PA}^2 \). \( \mathbf{PA}^2 \models \varphi \) iff \( \models P \rightarrow \varphi \), i.e., \( \mathfrak{M} \models P \rightarrow \varphi \). Now consider the sentence \( \forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi') \) resulting by replacing \( 0 \) by \( z \), \( \cdot \) by the one-place function variable \( u \), \( + \) and \( \times \) by the two-place function-variables \( u' \) and \( u'' \), respectively, and \( < \) by the two-place relation variable \( L \) and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff \( \mathbf{PA}^2 \models \varphi \) iff \( \mathfrak{N} \models \varphi \).

Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration \( f : \mathbb{N} \rightarrow \text{Sent}(\mathcal{L}_A) \) of the sentences true in \( \mathfrak{N} \). This function would be representable in \( \mathbb{Q} \) by some first-order formula \( \psi_f(x,y) \). Then the formula \( \exists x \psi_f(x,y) \) would define the set of true first-order sentences of \( \mathfrak{N} \), contradicting Tarski’s Theorem. □

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Bibliography