Second-order Logic is not Axiomatizable

Theorem met.1. Second-order logic is undecidable.

Proof. A first-order sentence is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable.

Theorem met.2. There is no sound and complete proof system for second-order logic.

Proof. Let $\varphi$ be a sentence in the language of arithmetic. $\mathfrak{N} \models \varphi$ iff $\text{PA}^2 \models \varphi$. Let $P$ be the conjunction of the nine axioms of $\text{PA}^2$. $\text{PA}^2 \models \varphi$ iff $P \rightarrow \varphi$, i.e., $\mathfrak{M} \models P \rightarrow \varphi$. Now consider the sentence $\forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi')$ resulting by replacing $\circ$ by $z$, $i$ by the one-place function variable $u$, $+$ and $\times$ by the two-place function-variables $u'$ and $u''$, respectively, and $<$ by the two-place relation variable $L$ and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff $\text{PA}^2 \models \varphi$ iff $\mathfrak{N} \models \varphi$. Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L}_A)$ of the sentences true in $\mathfrak{N}$. This function would be representable in $\textbf{Q}$ by some first-order formula $\psi_f(x,y)$. Then the formula $\exists x \psi_f(x,y)$ would define the set of true first-order sentences of $\mathfrak{N}$, contradicting Tarski’s Theorem.

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Bibliography