

met.1 Second-order Logic is not Axiomatizable

sol:met:nax:
sec

sol:met:nax: **Theorem met.1.** *Second-order logic is undecidable.*

thm:sol-undecidable

Proof. A first-order **sentence** is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable. \square

sol:met:nax:

cor:sol-not-axiomatizable

Theorem met.2. *There is no sound and complete proof system for second-order logic.*

Proof. Let φ be a **sentence** in the language of arithmetic. $\mathfrak{N} \models \varphi$ iff $\mathbf{PA}^2 \models \varphi$. Let P be the conjunction of the nine axioms of \mathbf{PA}^2 . $\mathbf{PA}^2 \models \varphi$ iff $\models P \rightarrow \varphi$, i.e., $\mathfrak{M} \models P \rightarrow \varphi$. Now consider the **sentence** $\forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi')$ resulting by replacing 0 by z , $!$ by the one-place function variable u , $+$ and \times by the two-place function-variables u' and u'' , respectively, and $<$ by the two-place relation variable L and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff $\mathbf{PA}^2 \models \varphi$ iff $\mathfrak{N} \models \varphi$. Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L}_A)$ of the **sentences** true in \mathfrak{N} . This function would be representable in \mathbf{Q} by some first-order formula $\psi_f(x, y)$. Then the **formula** $\exists x \psi_f(x, y)$ would define the set of true first-order **sentences** of \mathfrak{N} , contradicting Tarski's Theorem. \square

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Bibliography