Recall that the theory \( \mathbf{PA} \) of Peano arithmetic includes the eight axioms of \( \mathbf{Q} \),

\[
\begin{align*}
\forall x x' &\neq 0 \\
\forall x \forall y (x' = y' \rightarrow x = y) \\
\forall x \forall y (x < y &\iff \exists z (x + z') = y) \\
\forall x (x + 0) &= x \\
\forall x \forall y (x + y') &= (x + y)' \\
\forall x (x \times 0) &= 0 \\
\forall x \forall y (x \times y') &= ((x \times y) + x)
\end{align*}
\]

plus all sentences of the form

\[
(\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \varphi(x)
\]

The latter is a “schema,” i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula \( \varphi(x) \). We call this schema the (first-order) axiom schema of induction. In second-order Peano arithmetic \( \mathbf{PA}^2 \), induction can be stated as a single sentence. \( \mathbf{PA}^2 \) consists of the first eight axioms above plus the (second-order) induction axiom:

\[
\forall X (X(0) \land \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x)
\]

It says that if a subset \( X \) of the domain contains \( 0 \) and with any \( x \in |\mathfrak{M}| \) also contains \( X(x) \) (i.e., it is “closed under successor”) it contains everything in the domain (i.e., \( X = |\mathfrak{M}| \)).

The induction axiom guarantees that any structure satisfying it contains only those elements of \( |\mathfrak{M}| \) the axioms require to be there, i.e., the values of \( \pi \) for \( n \in \mathbb{N} \). A model of \( \mathbf{PA}^2 \) contains no non-standard numbers.

**Theorem met.1.** If \( \mathfrak{M} \models \mathbf{PA}^2 \) then \( |\mathfrak{M}| = \{ \text{Val}^\mathfrak{M}(M) : n \in \mathbb{N} \} \).

*Proof.* Let \( N = \{ \text{Val}^\mathfrak{M} (\pi) : n \in \mathbb{N} \} \), and suppose \( \mathfrak{M} \models \mathbf{PA}^2 \). Of course, for any \( n \in \mathbb{N} \), \( \text{Val}^\mathfrak{M} (\pi) \in |\mathfrak{M}| \), so \( N \subseteq |\mathfrak{M}| \).

Now for inclusion in the other direction. Consider a variable assignment \( s \) with \( s(X) = N \). By assumption,

\[
\begin{align*}
\mathfrak{M} &\models \forall X (X(o) \land \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x), \\
\mathfrak{M}, s &\models (X(o) \land \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x).
\end{align*}
\]

Consider the antecedent of this conditional. \( \text{Val}^\mathfrak{M} (o) \in N \), and so \( \mathfrak{M}, s \models X(o) \). The second conjunct, \( \forall x (X(x) \rightarrow X(x')) \) is also satisfied. For suppose \( x \in N \). By definition of \( N \), \( x = \text{Val}^\mathfrak{M} (\pi) \) for some \( n \). That gives \( \pi^\mathfrak{M} (x) = \text{Val}^\mathfrak{M} (n + 1) \in N \). So, \( \pi^\mathfrak{M} (x) \in N \).
We have that $\mathfrak{M}, s \models X(x) \land \forall x (X(x) \rightarrow X(x'))$. Consequently, $\mathfrak{M}, s \not\models \forall x X(x)$. But that means that for every $x \in |\mathfrak{M}|$ we have $x \in s(X) = N$. So, $|\mathfrak{M}| \subseteq N$.

**Corollary met.2.** Any two models of $\text{PA}^2$ are isomorphic.

*Proof.* By Theorem met.1, the domain of any model of $\text{PA}^2$ is exhausted by $\text{Val}^\mathfrak{M}(\pi)$. Any such model is also a model of $\text{Q}$. By ??, any such model is standard, i.e., isomorphic to $\mathfrak{N}$.

Above we defined $\text{PA}^2$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the first two of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

**Proposition met.3.** Let $\text{PA}^{2!}$ be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. $\text{\textgreater, }\text{+}, \text{and }\times$ are definable in $\text{PA}^{2!}$.

*Proof.* Exercise.

**Problem met.1.** Prove Proposition met.3.

**Corollary met.4.** $\mathfrak{M} \models \text{PA}^2$ iff $\mathfrak{M} \models \text{PA}^{2!}$.

*Proof.* Immediate from Proposition met.3.

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**Bibliography**