met.1 Second-order Arithmetic

Recall that the theory \( \text{PA} \) of Peano arithmetic includes the eight axioms of \( \mathbb{Q} \),
\[
\begin{align*}
\forall x \; x' &\neq 0 \\
\forall x \forall y \; (x' = y' \rightarrow x = y) \\
\forall x \; (x = 0 \lor \exists y \; x = y') \\
\forall x \; (x + 0) & = x \\
\forall x \forall y \; (x + y') = (x + y)' \\
\forall x \; (x \times 0) & = 0 \\
\forall x \forall y \; (x \times y') = ((x \times y) + x) \\
\forall x \forall y \; (x < y \leftrightarrow \exists z \; (z' + x) = y)
\end{align*}
\]
plus all sentences of the form
\[
(\varphi(0) \land \forall x \; (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \; \varphi(x).
\]
The latter is a “schema,” i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula \( \varphi(x) \). We call this schema the (first-order) \textit{axiom schema of induction}. In second-order Peano arithmetic \( \text{PA}^2 \), induction can be stated as a single sentence. \( \text{PA}^2 \) consists of the first eight axioms above plus the (second-order) \textit{induction axiom}:
\[
\forall X \; (X(0) \land \forall x \; (X(x) \rightarrow X(x'))) \rightarrow \forall x \; X(x).
\]
It says that if a subset \( X \) of the domain contains \( 0 \) and with any \( x \in |M| \) also contains \( x' \) (i.e., it is “closed under successor”) it contains everything in the domain (i.e., \( X = |M| \)).

The induction axiom guarantees that any \textit{structure} satisfying it contains only those \textit{elements} of \( |M| \) the axioms require to be there, i.e., the values of \( \pi \) for \( n \in \mathbb{N} \). A model of \( \text{PA}^2 \) contains no non-standard numbers.

**Theorem met.1.** If \( M \models \text{PA}^2 \) then \( |M| = \{ \text{Val}^{\text{M}}(\pi) : n \in \mathbb{N} \} \).

**Proof.** Let \( N = \{ \text{Val}^{\text{M}}(\pi) : n \in \mathbb{N} \} \), and suppose \( M \models \text{PA}^2 \). Of course, for any \( n \in \mathbb{N} \), \( \text{Val}^{\text{M}}(\pi) \in |M| \), so \( N \subseteq |M| \).

Now for inclusion in the other direction. Consider a variable assignment \( s \) with \( s(X) = N \). By assumption,
\[
M \models \forall X \; (X(0) \land \forall x \; (X(x) \rightarrow X(x'))) \rightarrow \forall x \; X(x),
\]
\[
M, s \models (X(0) \land \forall x \; (X(x) \rightarrow X(x'))) \rightarrow \forall x \; X(x).
\]
Consider the antecedent of this conditional. \( \text{Val}^{\text{M}}(0) \in N \), and so \( M, s \models X(0) \). The second conjunct, \( \forall x \; (X(x) \rightarrow X(x')) \) is also satisfied. For suppose \( x \in N \). By definition of \( N \), \( x = \text{Val}^{\text{M}}(\pi) \) for some \( n \). That gives \( \text{Val}^{\text{M}}(x) = \text{Val}^{\text{M}}(n+1) \in N \). So, \( \text{Val}^{\text{M}}(x) \in N \).
We have that $\mathcal{M}, s \models X(0) \land \forall x (X(x) \rightarrow X(x'))$. Consequently, $\mathcal{M}, s \models \forall x X(x)$. But that means that for every $x \in |\mathcal{M}|$ we have $x \in s(X) = N$. So, $|\mathcal{M}| \subseteq N$.

**Corollary met.2.** Any two models of $\text{PA}^2$ are isomorphic.

*Proof.* By Theorem met.1, the domain of any model of $\text{PA}^2$ is exhausted by $\text{Val}^\mathcal{M}(\pi)$. Any such model is also a model of $\text{Q}$. By ??, any such model is standard, i.e., isomorphic to $\mathfrak{N}$.

Above we defined $\text{PA}^2$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the first two of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

**Proposition met.3.** Let $\text{PA}^{2\dagger}$ be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then $\leq$, $+$, and $\times$ are definable in $\text{PA}^{2\dagger}$.

*Proof.* To show that $\leq$ is definable, we have to find a formula $\varphi_{\leq}(x, y)$ such that $\mathfrak{N} \models \varphi(\pi, m)$ iff $n < m$. Consider the formula

$$\psi(x, Y) \equiv Y(x) \land \forall y (Y(y) \rightarrow Y(y'))$$

Clearly, $\psi(\pi, Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m\} \subseteq Y$, so we can take $\varphi_{\leq}(x, y) \equiv \forall Y (\psi(x, Y) \rightarrow Y(y))$.

**Problem met.1.** Complete the proof of Proposition met.3.

**Corollary met.4.** $\mathcal{M} \models \text{PA}^2$ iff $\mathcal{M} \models \text{PA}^{2\dagger}$.

*Proof.* Immediate from Proposition met.3.

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**Bibliography**