Recall that the theory $\text{PA}$ of Peano arithmetic includes the eight axioms of $\mathbb{Q}$,

\[
\begin{align*}
\forall x \ x' \neq 0 \\
\forall x \forall y \ (x' = y' \to x = y) \\
\forall x \ (x = 0 \lor \exists y \ x = y') \\
\forall x \ (x + 0) = x \\
\forall x \forall y \ (x + y') = (x + y)' \\
\forall x \forall y \ (x \times y') = ((x \times y) + x) \\
\forall x \forall y \ (x < y \leftrightarrow \exists z \ (z' + x) = y)
\end{align*}
\]

plus all sentences of the form

\[\left(\varphi(0) \land \forall x \ (\varphi(x) \to \varphi(x'))\right) \to \forall x \varphi(x).\]

The latter is a "schema," i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula $\varphi(x)$. We call this schema the (first-order) axiom schema of induction. In second-order Peano arithmetic $\text{PA}^2$, induction can be stated as a single sentence. $\text{PA}^2$ consists of the first eight axioms above plus the (second-order) induction axiom:

\[\forall X \ (X(0) \land \forall x \ (X(x) \to X(x'))) \to \forall x \ X(x).\]

It says that if a subset $X$ of the domain contains $0$ and with any $x \in |\mathcal{M}|$ also contains $x'$ (i.e., it is "closed under successor") it contains everything in the domain (i.e., $X = |\mathcal{M}|$).

The induction axiom guarantees that any structure satisfying it contains only those elements of $|\mathcal{M}|$ the axioms require to be there, i.e., the values of $\pi$ for $n \in \mathbb{N}$. A model of $\text{PA}^2$ contains no non-standard numbers.

**Theorem met.1.** If $\mathcal{M} \models \text{PA}^2$ then $|\mathcal{M}| = \{\text{Val}^\mathcal{M}(\pi) : n \in \mathbb{N}\}$.

**Proof.** Let $N = \{\text{Val}^\mathcal{M}(\pi) : n \in \mathbb{N}\}$, and suppose $\mathcal{M} \models \text{PA}^2$. Of course, for any $n \in \mathbb{N}$, $\text{Val}^\mathcal{M}(\pi) \in |\mathcal{M}|$, so $N \subseteq |\mathcal{M}|$.

Now for inclusion in the other direction. Consider a variable assignment $s$ with $s(X) = N$. By assumption,

\[\mathcal{M} \models \forall X \ (X(0) \land \forall x \ (X(x) \to X(x'))) \to \forall x \ X(x),\]

thus

\[\mathcal{M}, s \models (X(0) \land \forall x \ (X(x) \to X(x'))) \to \forall x \ X(x).\]

Consider the antecedent of this conditional. $\text{Val}^\mathcal{M}(0) \in N$, and so $\mathcal{M}, s \models X(0)$. The second conjunct, $\forall x \ (X(x) \to X(x'))$ is also satisfied. For suppose $x \in N$. By definition of $N$, $x = \text{Val}^\mathcal{M}(\pi)$ for some $n$. That gives $\text{Val}^\mathcal{M}(x) = \text{Val}^\mathcal{M}(n + 1) \in N$. So, $\pi(n) \in N$.\[\square\]
We have that $\mathcal{M}, s \models X(o) \land \forall x (X(x) \rightarrow X(x'))$. Consequently, $\mathcal{M}, s \models \forall x X(x)$. But that means that for every $x \in |\mathcal{M}|$ we have $x \in s(X) = N$. So, $|\mathcal{M}| \subseteq N$.

Corollary met.2. Any two models of $\text{PA}^2$ are isomorphic.

Proof. By Theorem met.1, the domain of any model of $\text{PA}^2$ is exhausted by $\text{Val}^\mathcal{M}(\pi)$. Any such model is also a model of $\text{Q}$. By ??, any such model is standard, i.e., isomorphic to $\mathcal{N}$.

Above we defined $\text{PA}^2$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the first two of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

Proposition met.3. Let $\text{PA}^{2\dagger}$ be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then $\leq$, $\oplus$, and $\times$ are definable in $\text{PA}^{2\dagger}$.

Proof. To show that $\leq$ is definable, we have to find a formula $\varphi_{\leq}(x, y)$ such that $\mathcal{N} \models \varphi_{\leq}(\pi, m)$ iff $n \leq m$. Consider the formula

$$\psi(x, Y) \equiv Y(x) \land \forall y (Y(y) \rightarrow Y(y'))$$

Clearly, $\psi(\pi, Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m\} \subseteq Y$, so we can take $\varphi_{\leq}(x, y) \equiv \forall Y (\psi(x, Y) \rightarrow Y(y))$

To see that addition is definable observe that $k + l = m$ iff there is a function $u$ such that $u(0) = k$, $u(n') = u(n)'$ for all $n$, and $m = u(l)$. We can use this equivalence to define addition in $\text{PA}^{2\dagger}$ by the following formula:

$$\varphi_{+}(x, y, z) \equiv \exists u (u(0) = x \land \forall w (u(x') = u(x)' \land u(y) = z)$$

It should be clear that $\mathcal{N} \models \varphi_{+}(\overline{k}, \overline{l}, \overline{m})$ iff $k + l = m$.

Problem met.1. Complete the proof of Proposition met.3.

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Bibliography