Recall that the theory \( \mathbf{PA} \) of Peano arithmetic includes the eight axioms of \( \mathbf{Q} \),

\[
\begin{align*}
\forall x \ x' \neq 0 \\
\forall x \forall y (x' = y' \to x = y) \\
\forall x (x = 0 \lor \exists y \ x = y') \\
\forall x (x + 0) = x \\
\forall x \forall y (x + y') = (x + y)' \\
\forall x (x \times 0) = 0 \\
\forall x \forall y (x \times y') = ((x \times y) + x) \\
\forall x \forall y (x < y \leftrightarrow \exists z \ (z' + x) = y)
\end{align*}
\]

plus all sentences of the form

\[
(\varphi(0) \land \forall x (\varphi(x) \to \varphi(x'))) \to \forall x \varphi(x).
\]

The latter is a “schema,” i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula \( \varphi(x) \). We call this schema the (first-order) axiom schema of induction. In second-order Peano arithmetic \( \mathbf{PA}^2 \), induction can be stated as a single sentence. \( \mathbf{PA}^2 \) consists of the first eight axioms above plus the (second-order) induction axiom:

\[
\forall X \ (X(0) \land \forall x (X(x) \to X(x'))) \to \forall x \ X(x).
\]

It says that if a subset \( X \) of the domain contains \( 0^\mathfrak{M} \) and with any \( x \in |\mathfrak{M}| \) also contains \( \mathfrak{M}(x) \) (i.e., it is “closed under successor”) it contains everything in the domain (i.e., \( X = |\mathfrak{M}| \)).

The induction axiom guarantees that any structure satisfying it contains only those elements of \( |\mathfrak{M}| \) the axioms require to be there, i.e., the values of \( \pi \) for \( n \in \mathbb{N} \). A model of \( \mathbf{PA}^2 \) contains no non-standard numbers.

**Theorem met.1.** If \( \mathfrak{M} \models \mathbf{PA}^2 \) then \( |\mathfrak{M}| = \{ \text{Val}^\mathfrak{M}(\pi) : n \in \mathbb{N} \} \).

**Proof.** Let \( N = \{ \text{Val}^\mathfrak{M}(\pi) : n \in \mathbb{N} \} \), and suppose \( \mathfrak{M} \models \mathbf{PA}^2 \). Of course, for any \( n \in \mathbb{N} \), \( \text{Val}^\mathfrak{M}(\pi) \in |\mathfrak{M}| \), so \( N \subseteq |\mathfrak{M}| \).

Now for inclusion in the other direction. Consider a variable assignment \( s \) with \( s(X) = N \). By assumption,

\[
\mathfrak{M}, s \models (X(0) \land \forall x (X(x) \to X(x'))) \to \forall x \ X(x).
\]

Consider the antecedent of this conditional. \( \text{Val}^\mathfrak{M}(\pi) \in N \), and so \( \mathfrak{M}, s \models X(0) \). The second conjunct, \( \forall x (X(x) \to X(x')) \) is also satisfied. For suppose \( x \in N \). By definition of \( N \), \( x = \text{Val}^\mathfrak{M}(\pi) \) for some \( n \). That gives \( \mathfrak{M}(x) = \text{Val}^\mathfrak{M}(n + 1) \in N \). So, \( \mathfrak{M}(x) \in N \).
We have that $\mathcal{M}, s \models X(0) \land \forall x (X(x) \rightarrow X(x'))$. Consequently, $\mathcal{M}, s \models \forall x X(x)$. But that means that for every $x \in |\mathcal{M}|$ we have $x \in s(X) = N$. So, $|\mathcal{M}| \subseteq N$.

**Corollary met.2.** Any two models of PA$^2$ are isomorphic.

**Proof.** By Theorem met.1, the domain of any model of PA$^2$ is exhausted by $Val^\mathcal{M}(\pi)$. Any such model is also a model of Q. By ??, any such model is standard, i.e., isomorphic to $\mathcal{N}$.

Above we defined PA$^2$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the first two of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

**Proposition met.3.** Let PA$^{2\dagger}$ be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then $\leq$, $+$, and $\times$ are definable in PA$^{2\dagger}$.

**Proof.** To show that $\leq$ is definable, we have to find a formula $\varphi_{\leq}(x, y)$ such that $\mathcal{M} \models \varphi_{\leq}(\pi, m)$ iff $n \leq m$. Consider the formula

$$
\psi(x, Y) \equiv Y(x) \land \forall y (Y(y) \rightarrow Y(y'))
$$

Clearly, $\psi(\pi, Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m\} \subseteq Y$, so we can take $\varphi_{\leq}(x, y) \equiv \forall Y (\psi(x, Y) \rightarrow Y(y))$.

To see that addition is definable observe that $k+l = m$ iff there is a function $u$ such that $u(0) = k$, $u(n') = u(n)'$ for all $n$, and $m = u(l)$. We can use this equivalence to define addition in PA$^{2\dagger}$ by the following formula:

$$
\varphi_+(x, y, z) \equiv \exists u (u(0) = x \land \forall w u(x') = u(x)' \land u(y) = z)
$$

It should be clear that $\mathcal{M} \models \varphi_+(\overline{k}, \overline{l}, \overline{m})$ iff $k + l = m$.

**Problem met.1.** Complete the proof of Proposition met.3.

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**Bibliography**