met.1 Second-order Arithmetic

sol:met:spa:sec

Recall that the theory **PA** of Peano arithmetic includes the eight axioms of **Q**,

$$\begin{aligned}
&\forall x\, x' \neq \mathsf{o} \\
&\forall x\, \forall y\, (x' = y' \rightarrow x = y) \\
&\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (x + z') = y) \\
&\forall x\, (x + \mathsf{o}) = x \\
&\forall x\, \forall y\, (x + y') = (x + y)' \\
&\forall x\, (x \times \mathsf{o}) = \mathsf{o} \\
&\forall x\, \forall y\, (x \times y') = ((x \times y) + x)
\end{aligned}$$

plus all sentences of the form

$$(\varphi(\mathsf{o}) \land \forall x\, (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x\, \varphi(x)$$

The latter is a "schema," i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula $\varphi(x)$. We call this schema the (first-order) *axiom schema of induction.* In *second-order* Peano arithmetic $\mathbf{PA^2}$, induction can be stated as a single sentence. $\mathbf{PA^2}$ consists of the first eight axioms above plus the (second-order) *induction axiom*:

$$\forall X\, (X(\mathsf{o}) \land \forall x\, (X(x) \rightarrow X(x'))) \rightarrow \forall x\, X(x))$$

It says that if a subset X of the domain contains $\mathsf{o}^{\mathfrak{M}}$ and with any $x \in |\mathfrak{M}|$ also contains $\prime^{\mathfrak{M}}(x)$ (i.e., it is "closed under successor") it contains everything in the domain (i.e., $X = |\mathfrak{M}|$).

The induction axiom guarantees that any structure satisfying it contains only those elements of $|\mathfrak{M}|$ the axioms require to be there, i.e., the values of $\overline{n}$ for $n \in \mathbb{N}$. A model of $\mathbf{PA^2}$ contains no non-standard numbers.

sol:met:spa:thm:sol-pa-standard

Theorem met.1. *If* $\mathfrak{M} \vDash \mathbf{PA^2}$ *then* $|\mathfrak{M}| = \{\mathrm{Val}^n(M) : n \in \mathbb{N}\}$.

Proof. Let $N = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$, and suppose $\mathfrak{M} \vDash \mathbf{PA^2}$. Of course, for any $n \in \mathbb{N}$, $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \in |\mathfrak{M}|$, so $N \subseteq |\mathfrak{M}|$.

Now for inclusion in the other direction. Consider a variable assignment s with $s(X) = N$. By assumption,

$$\begin{aligned}
\mathfrak{M} &\vDash \forall X\, (X(\mathsf{o}) \land \forall x\, (X(x) \rightarrow X(x'))) \rightarrow \forall x\, X(x), \text{ thus} \\
\mathfrak{M}, s &\vDash (X(\mathsf{o}) \land \forall x\, (X(x) \rightarrow X(x'))) \rightarrow \forall x\, X(x).
\end{aligned}$$

Consider the antecedent of this conditional. $\mathrm{Val}^{\mathfrak{M}}(\mathsf{o}) \in N$, and so $\mathfrak{M}, s \vDash X(\mathsf{o})$. The second conjunct, $\forall x\, (X(x) \rightarrow X(x'))$ is also satisfied. For suppose $x \in N$. By definition of N, $x = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some n. That gives $\prime^{\mathfrak{M}}(x) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) \in N$. So, $\prime^{\mathfrak{M}}(x) \in N$.

We have that $\mathfrak{M}, s \vDash X(\mathsf{o}) \land \forall x\, (X(x) \rightarrow X(x'))$. Consequently, $\mathfrak{M}, s \vDash \forall x\, X(x)$. But that means that for every $x \in |\mathfrak{M}|$ we have $x \in s(X) = N$. So, $|\mathfrak{M}| \subseteq N$. □

Corollary met.2. *Any two models of* $\mathbf{PA^2}$ *are isomorphic.* *sol:met:spa: cor:sol-pa-categorical*

Proof. By Theorem met.1, the domain of any model of $\mathbf{PA^2}$ is exhausted by $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$. Any such model is also a model of $\mathbf{Q}$. By **??**, any such model is standard, i.e., isomorphic to $\mathfrak{N}$. □

Above we defined $\mathbf{PA^2}$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the *first two* of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

Proposition met.3. *Let* $\mathbf{PA^{2\dagger}}$ *be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then* $\leq$, $+$, *and* $\times$ *are definable in* $\mathbf{PA^{2\dagger}}$. *sol:met:spa: prop:sol-pa-definable*

Proof. To show that $\leq$ is definable, we have to find a formula $\varphi_{\leq}(x, y)$ such that $\mathfrak{N} \vDash \varphi(\overline{n}, \overline{m})$ iff $n < m$. Consider the formula

$$\psi(x, Y) \equiv Y(x) \land \forall y\, (Y(y) \to Y(y'))$$

Clearly, $\psi(\overline{n}, Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m\} \subseteq Y$, so we can take $\varphi_{\leq}(x, y) \equiv \forall Y\, (\psi(x, Y) \to Y(y))$. □

Problem met.1. Complete the proof of Proposition met.3.

Corollary met.4. $\mathfrak{M} \vDash \mathbf{PA^2}$ *iff* $\mathfrak{M} \vDash \mathbf{PA^{2\dagger}}$.

Proof. Immediate from Proposition met.3. □

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Bibliography