Chapter udf

Metatheory of Second-order Logic

met.1 Introduction

sol:met:int:sec First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a derivability relation $\vdash$ with the property that for any set of sentences Γ and sentence Q, $\Gamma \vDash \varphi$ iff $\Gamma \vdash \varphi$. This means in particular that the validities of first-order logic are computably enumerable. There is a computable function $f \colon \mathbb{N} \to \text{Sent}(\mathcal{L})$ such that the values of f are all and only the valid sentences of $\mathcal{L}$. This is so because derivations can be enumerated, and those that derive a single sentence are then mapped to that sentence. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we'll show that second-order logic is not only undecidable, but its validities are not even computably enumerable. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set Γ of sentences is satisfiable, Γ itself is satisfiable) and the Löwenheim–Skolem Theorem holds for it (if Γ has an infinite model it has a denumerable model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of domains that first-order logic cannot.

met.2 Second-order Arithmetic

sol:met:spa:sec

Recall that the theory **PA** of Peano arithmetic includes the eight axioms of **Q**,

$$\begin{aligned}
&\forall x\, x' \neq \mathsf{o}\\
&\forall x\, \forall y\, (x' = y' \rightarrow x = y)\\
&\forall x\, (x = \mathsf{o} \lor \exists y\, x = y')\\
&\forall x\, (x + \mathsf{o}) = x\\
&\forall x\, \forall y\, (x + y') = (x + y)'\\
&\forall x\, (x \times \mathsf{o}) = \mathsf{o}\\
&\forall x\, \forall y\, (x \times y') = ((x \times y) + x)\\
&\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (z' + x) = y)
\end{aligned}$$

plus all sentences of the form

$$(\varphi(\mathsf{o}) \land \forall x\, (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x\, \varphi(x).$$

The latter is a "schema," i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula $\varphi(x)$. We call this schema the (first-order) *axiom schema of induction.* In *second-order* Peano arithmetic $\mathbf{PA^2}$, induction can be stated as a single sentence. $\mathbf{PA^2}$ consists of the first eight axioms above plus the (second-order) *induction axiom*:

$$\forall X\, (X(\mathsf{o}) \land \forall x\, (X(x) \rightarrow X(x'))) \rightarrow \forall x\, X(x).$$

It says that if a subset X of the domain contains $\mathsf{o}^{\mathfrak{M}}$ and with any $x \in |\mathfrak{M}|$ also contains $\prime^{\mathfrak{M}}(x)$ (i.e., it is "closed under successor") it contains everything in the domain (i.e., $X = |\mathfrak{M}|$).

The induction axiom guarantees that any structure satisfying it contains only those elements of $|\mathfrak{M}|$ the axioms require to be there, i.e., the values of $\overline{n}$ for $n \in \mathbb{N}$. A model of $\mathbf{PA^2}$ contains no non-standard numbers.

Theorem met.1. *If* $\mathfrak{M} \vDash \mathbf{PA^2}$ *then* $|\mathfrak{M}| = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$. *sol:met:spa: thm:sol-pa-standard*

Proof. Let $N = \{\mathrm{Val}^{\mathfrak{M}}(\overline{n}) : n \in \mathbb{N}\}$, and suppose $\mathfrak{M} \vDash \mathbf{PA^2}$. Of course, for any $n \in \mathbb{N}$, $\mathrm{Val}^{\mathfrak{M}}(\overline{n}) \in |\mathfrak{M}|$, so $N \subseteq |\mathfrak{M}|$.

Now for inclusion in the other direction. Consider a variable assignment s with $s(X) = N$. By assumption,

$$\begin{aligned}
\mathfrak{M} &\vDash \forall X\, (X(\mathsf{o}) \land \forall x\, (X(x) \rightarrow X(x'))) \rightarrow \forall x\, X(x), \text{ thus}\\
\mathfrak{M}, s &\vDash (X(\mathsf{o}) \land \forall x\, (X(x) \rightarrow X(x'))) \rightarrow \forall x\, X(x).
\end{aligned}$$

Consider the antecedent of this conditional. $\mathrm{Val}^{\mathfrak{M}}(\mathsf{o}) \in N$, and so $\mathfrak{M}, s \vDash X(\mathsf{o})$. The second conjunct, $\forall x\, (X(x) \rightarrow X(x'))$ is also satisfied. For suppose $x \in N$. By definition of N, $x = \mathrm{Val}^{\mathfrak{M}}(\overline{n})$ for some n. That gives $\prime^{\mathfrak{M}}(x) = \mathrm{Val}^{\mathfrak{M}}(\overline{n+1}) \in N$. So, $\prime^{\mathfrak{M}}(x) \in N$.

We have that $\mathfrak{M}, s \vDash X(\mathsf{o}) \land \forall x\, (X(x) \rightarrow X(x'))$. Consequently, $\mathfrak{M}, s \vDash \forall x\, X(x)$. But that means that for every $x \in |\mathfrak{M}|$ we have $x \in s(X) = N$. So, $|\mathfrak{M}| \subseteq N$. □

sol:met:spa: cor:sol-pa-categorical

Corollary met.2. *Any two models of* $\mathbf{PA^2}$ *are isomorphic.*

Proof. By Theorem met.1, the domain of any model of $\mathbf{PA^2}$ is exhausted by $\mathrm{Val}^{\mathfrak{M}}(\overline{n})$. Any such model is also a model of $\mathbf{Q}$. By **??**, any such model is standard, i.e., isomorphic to $\mathfrak{N}$. □

Above we defined $\mathbf{PA^2}$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the *first two* of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

sol:met:spa: prop:sol-pa-definable

Proposition met.3. *Let* $\mathbf{PA^{2\dagger}}$ *be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then* $\leq$, $+$, *and* $\times$ *are definable in* $\mathbf{PA^{2\dagger}}$.

Proof. To show that $\leq$ is definable, we have to find a formula $\varphi_{\leq}(x, y)$ such that $\mathfrak{N} \vDash \varphi_{\leq}(\overline{n}, \overline{m})$ iff $n \leq m$. Consider the formula

$$\psi(x, Y) \equiv Y(x) \land \forall y\,(Y(y) \rightarrow Y(y'))$$

Clearly, $\psi(\overline{n}, Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m\} \subseteq Y$, so we can take $\varphi_{\leq}(x, y) \equiv \forall Y\,(\psi(x, Y) \rightarrow Y(y))$.

To see that addition is definable observe that $k + l = m$ iff there is a function u such that $u(0) = k$, $u(n') = u(n)'$ for all n, and $m = u(l)$. We can use this equivalence to define addition in $\mathbf{PA^{2\dagger}}$ by the following formula:

$$\varphi_{+}(x, y, z) \equiv \exists u\,(u(\mathsf{o}) = x \land \forall w\, u(x') = u(x)' \land u(y) = z)$$

It should be clear that $\mathfrak{N} \vDash \varphi_{+}(\overline{k}, \overline{l}, \overline{m})$ iff $k + l = m$. □

Problem met.1. Complete the proof of Proposition met.3.

met.3 Second-order Logic is not Axiomatizable

sol:met:nax: sec

sol:met:nax: thm:sol-undecidable

Theorem met.4. *Second-order logic is undecidable.*

Proof. A first-order sentence is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable. □

sol:met:nax: cor:sol-not-axiomatizable

Theorem met.5. *There is no sound and complete derivation system for second-order logic.*

Proof. Let φ be a sentence in the language of arithmetic. $\mathfrak{N} \vDash \varphi$ iff $\mathbf{PA^2} \vDash \varphi$. Let P be the conjunction of the nine axioms of $\mathbf{PA^2}$. $\mathbf{PA^2} \vDash \varphi$ iff $\vDash P \rightarrow \varphi$, i.e., $\mathfrak{M} \vDash P \rightarrow \varphi$. Now consider the sentence $\forall z\, \forall u\, \forall u'\, \forall u''\, \forall L\, (P' \rightarrow \varphi')$ resulting by replacing o by z, $\prime$ by the one-place function variable u, $+$ and $\times$ by the two-place function-variables u' and u'', respectively, and $<$ by the two-place relation variable L and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff $\mathbf{PA^2} \vDash \varphi$ iff $\mathfrak{N} \vDash \varphi$. Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration $f \colon \mathbb{N} \to \mathrm{Sent}(\mathcal{L}_A)$ of the sentences true in $\mathfrak{N}$. This function would be representable in $\mathbf{Q}$ by some first-order formula $\psi_f(x, y)$. Then the formula $\exists x \, \psi_f(x, y)$ would define the set of true first-order sentences of $\mathfrak{N}$, contradicting Tarski's Theorem. □

met.4 Second-order Logic is not Compact

explanation

sol:met:com:sec

Call a set of sentences Γ *finitely satisfiable* if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences Γ is finitely satisfiable, it is satisfiable. This property is called *compactness*. It has an equivalent version involving entailment: if $\Gamma \vDash \varphi$, then already $\Gamma_0 \vDash \varphi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \vDash \varphi$, by completeness $\Gamma \vdash \varphi$. But a derivation can only make use of finitely many sentences of Γ.

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some φ without a finite subset entailing φ.

Theorem met.6. *Second-order logic is not compact.*

sol:met:com:thm:sol-undecidable

Proof. Recall that

$$\mathrm{Inf} \equiv \exists u \, (\forall x \, \forall y \, (u(x) = u(y) \to x = y) \land \exists y \, \forall x \, y \neq u(x))$$

is satisfied in a structure iff its domain is infinite. Let $\varphi^{\geq n}$ be a sentence that asserts that the domain has at least n elements, e.g.,

$$\varphi^{\geq n} \equiv \exists x_1 \ldots \exists x_n \, (x_1 \neq x_2 \land x_1 \neq x_3 \land \cdots \land x_{n-1} \neq x_n).$$

Consider the set of sentences

$$\Gamma = \{\lnot \mathrm{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \ldots\}.$$

It is finitely satisfiable, since for any finite subset $\Gamma_0 \subseteq \Gamma$ there is some k so that $\varphi^{\geq k} \in \Gamma$ but no $\varphi^{\geq n} \in \Gamma$ for $n > k$. If $|\mathfrak{M}|$ has k elements, $\mathfrak{M} \vDash \Gamma_0$. But, Γ is not satisfiable: if $\mathfrak{M} \vDash \lnot \mathrm{Inf}$, $|\mathfrak{M}|$ must be finite, say, of size k. Then $\mathfrak{M} \nvDash \varphi^{\geq k+1}$. □

Problem met.2. Give an example of a set Γ and a sentence φ so that $\Gamma \vDash \varphi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \nvDash \varphi$.

met.5 The Löwenheim–Skolem Theorem Fails for Second-order Logic

sol:met:lst:sec

The (Downward) Löwenheim–Skolem Theorem states that every set of sentences with an infinite model has an enumerable model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of sentences, and that model is enumerable. There is also an Upward Löwenheim–Skolem Theorem, which guarantees that if a set of sentences has a denumerable model it also has a non-enumerable model. Both theorems fail in second-order logic. explanation

sol:met:lst:thm:sol-no-ls

Theorem met.7. *The Löwenheim–Skolem Theorem fails for second-order logic: There are sentences with infinite models but no enumerable models.*

Proof. Recall that

$$\mathrm{Count} \equiv \exists z\, \exists u\, \forall X\, ((X(z) \land \forall x\, (X(x) \to X(u(x)))) \to \forall x\, X(x))$$

is true in a structure $\mathfrak{M}$ iff $|\mathfrak{M}|$ is enumerable, so $\lnot\mathrm{Count}$ is true in $\mathfrak{M}$ iff $|\mathfrak{M}|$ is non-enumerable. There are such structures—take any non-enumerable set as the domain, e.g., $\wp(\mathbb{N})$ or $\mathbb{R}$. So $\lnot\mathrm{Count}$ has infinite models but no enumerable models. □

Theorem met.8. *There are sentences with denumerable but no non-enumerable models.*

Proof. $\mathrm{Count} \land \mathrm{Inf}$ is true in $\mathbb{N}$ but not in any structure $\mathfrak{M}$ with $|\mathfrak{M}|$ non-enumerable. □

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Bibliography