Chapter udf

Metatheory of Second-order Logic

met.1 Introduction

First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a derivability relation $\vdash$ with the property that for any set of sentences $\Gamma$ and sentence $Q$, $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$. This means in particular that the validities of first-order logic are computably enumerable. There is a computable function $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L})$ such that the values of $f$ are all and only the valid sentences of $\mathcal{L}$. This is so because derivations can be enumerated, and those that derive a single sentence are then mapped to that sentence. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we’ll show that second-order logic is not only undecidable, but its validities are not even computably enumerable. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set $\Gamma$ of sentences is satisfiable, $\Gamma$ itself is satisfiable) and the Löwenheim-Skolem Theorem holds for it (if $\Gamma$ has an infinite model it has a denumerable model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of domains that first-order logic cannot.

met.2 Second-order Arithmetic
Recall that the theory $\text{PA}$ of Peano arithmetic includes the eight axioms of $\mathbb{Q}$,

$$
\forall x \ x' \neq 0
\forall x \ \forall y \ (x' = y' \rightarrow x = y)
\forall x \ (x = 0 \lor \exists y \ x = y')
\forall x \ (x + 0) = x
\forall x \forall y \ (x + y') = (x + y)'
\forall x \forall y \ (x \times 0) = 0
\forall x \forall y \ (x \times y') = ((x \times y) + x)
\forall x \forall y \ (x < y \leftrightarrow \exists z \ (z' + x) = y)
$$

plus all sentences of the form

$$(\varphi(0) \land \forall x \ (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \varphi(x).$$

The latter is a “schema,” i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula $\varphi(x)$. We call this schema the (first-order) axiom schema of induction. In second-order Peano arithmetic $\text{PA}^2$, induction can be stated as a single sentence. $\text{PA}^2$ consists of the first eight axioms above plus the (second-order) induction axiom:

$$\forall X \ (X(0) \land \forall x \ (X(x) \rightarrow X(x'))) \rightarrow \forall x \ X(x).$$

It says that if a subset $X$ of the domain contains $0^\mathfrak{M}$ and with any $x \in [\mathfrak{M}]$ also contains $\rho^\mathfrak{M}(x)$ (i.e., it is “closed under successor”) it contains everything in the domain (i.e., $X = [\mathfrak{M}]$).

The induction axiom guarantees that any structure satisfying it contains only those elements of $[\mathfrak{M}]$ the axioms require to be there, i.e., the values of $\pi$ for $n \in \mathbb{N}$. A model of $\text{PA}^2$ contains no non-standard numbers.

**Theorem met.1.** If $\mathfrak{M} \models \text{PA}^2$ then $|\mathfrak{M}| = \{\text{Val}^\mathfrak{M}(\pi) : n \in \mathbb{N}\}$.

**Proof.** Let $N = \{\text{Val}^\mathfrak{M}(\pi) : n \in \mathbb{N}\}$, and suppose $\mathfrak{M} \models \text{PA}^2$. Of course, for any $n \in \mathbb{N}$, $\text{Val}^\mathfrak{M}(\pi) \in [\mathfrak{M}]$, so $N \subseteq [\mathfrak{M}]$.

Now for inclusion in the other direction. Consider a variable assignment $s$ with $s(X) = N$. By assumption,

$$
\mathfrak{M} \models \forall X \ (X(0) \land \forall x \ (X(x) \rightarrow X(x'))) \rightarrow \forall x \ X(x),
$$

$$
\mathfrak{M}, s \models (X(0) \land \forall x \ (X(x) \rightarrow X(x'))) \rightarrow \forall x \ X(x).
$$

Consider the antecedent of this conditional. $\text{Val}^\mathfrak{M}(0) \in N$, and so $\mathfrak{M}, s \models X(0)$. The second conjunct, $\forall x \ (X(x) \rightarrow X(x'))$ is also satisfied. For suppose $x \in N$. By definition of $N$, $x = \text{Val}^\mathfrak{M}(\pi)$ for some $n$. That gives $\rho^\mathfrak{M}(x) = \text{Val}^\mathfrak{M}(n + 1) \in N$. So, $\rho^\mathfrak{M}(x) \in N$.

We have that $\mathfrak{M}, s \models X(0) \land \forall x \ (X(x) \rightarrow X(x'))$. Consequently, $\mathfrak{M}, s \models \forall x \ X(x)$. But that means that for every $x \in [\mathfrak{M}]$ we have $x \in s(X) = N$. So, $[\mathfrak{M}] \subseteq N$. 

$$
\square
$$

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Corollary met.2. Any two models of $\text{PA}^2$ are isomorphic.

Proof. By Theorem met.1, the domain of any model of $\text{PA}^2$ is exhausted by $\text{Val}^\mathfrak{M}(\pi)$. Any such model is also a model of $Q$. By $?\text{?}$, any such model is standard, i.e., isomorphic to $\mathfrak{N}$. □

Above we defined $\text{PA}^2$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the first two of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

Proposition met.3. Let $\text{PA}^{2\dagger}$ be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then $\leq$, $+$, and $\times$ are definable in $\text{PA}^{2\dagger}$.

Proof. To show that $\leq$ is definable, we have to find a formula $\varphi_{\leq}(x,y)$ such that $\mathfrak{N} \models \varphi(n,m)$ iff $n < m$. Consider the formula

$$\psi(x,Y) \equiv Y(x) \land \forall y (Y(y) \rightarrow Y(y'))$$

Clearly, $\psi(\pi,Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m\} \subseteq Y$, so we can take $\varphi_{\leq}(x,y) \equiv \forall Y (\psi(x,Y) \rightarrow Y(y))$. □

Problem met.1. Complete the proof of Proposition met.3.

Corollary met.4. $\mathfrak{M} \models \text{PA}^2$ iff $\mathfrak{M} \models \text{PA}^{2\dagger}$.

Proof. Immediate from Proposition met.3. □

Theorem met.5. Second-order logic is undecidable.

Proof. A first-order sentence is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable. □

Theorem met.6. There is no sound and complete proof system for second-order logic.

Proof. Let $\varphi$ be a sentence in the language of arithmetic. $\mathfrak{N} \models \varphi$ iff $\text{PA}^2 \models \varphi$. Let $P$ be the conjunction of the nine axioms of $\text{PA}^2$. $\text{PA}^2 \models \varphi$ iff $\models P \rightarrow \varphi$, i.e., $\mathfrak{M} \models P \rightarrow \varphi$. Now consider the sentence $\forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi')$ resulting by replacing $0$ by $z$, $\rho$ by the one-place function variable $u$, $+$ and $\times$ by the two-place function-variables $u'$ and $u''$, respectively, and $<$ by the two-place relation variable $L$ and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff $\text{PA}^2 \models \varphi$ iff $\mathfrak{N} \models \varphi$. Thus if there were a sound and complete proof system for second-order logic,
we could use it to define a computable enumeration $f : \mathbb{N} \to \text{Sent}(L_A)$ of the sentences true in $\mathfrak{N}$. This function would be representable in $\mathbb{Q}$ by some first-order formula $\psi_f(x,y)$. Then the formula $\exists x \psi_f(x,y)$ would define the set of true first-order sentences of $\mathfrak{N}$, contradicting Tarski’s Theorem.

\[\square\]

**met.4 Second-order Logic is not Compact**

Call a set of sentences $\Gamma$ **finitely satisfiable** if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences $\Gamma$ is finitely satisfiable, it is satisfiable. This property is called **compactness**. It has an equivalent version involving entailment: if $\Gamma \models \varphi$, then already $\Gamma_0 \models \varphi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \models \varphi$, by completeness $\Gamma \vdash \varphi$. But a derivation can only make use of finitely many sentences of $\Gamma$.

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some $\varphi$ without a finite subset entailing $\varphi$.

**Theorem met.7.** Second-order logic is not compact.

**Proof.** Recall that

$$\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \land \exists y \forall x y \neq u(x))$$

is satisfied in a structure iff its domain is infinite. Let $\varphi^{2^m}$ be a sentence that asserts that the domain has at least $n$ elements, e.g.,

$$\varphi^{2^n} \equiv \exists x_1 \ldots \exists x_n (x_1 \neq x_2 \land x_1 \neq x_3 \land \cdots \land x_{n-1} \neq x_n).$$

Consider the set of sentences

$$\Gamma = \{ \neg \text{Inf}, \varphi^{2^1}, \varphi^{2^2}, \varphi^{2^3}, \ldots \}.$$ 

It is finitely satisfiable, since for any finite subset $\Gamma_0 \subseteq \Gamma$ there is some $k$ so that $\varphi^{2^k} \in \Gamma$ but no $\varphi^{2^n} \in \Gamma$ for $n > k$. If $\mathfrak{M}$ has $k$ elements, $\mathfrak{M} \models \Gamma_0$. But, $\Gamma$ is not satisfiable: if $\mathfrak{M} \models \neg \text{Inf}$, $|\mathfrak{M}|$ must be finite, say, of size $k$. Then $\mathfrak{M} \not\models \varphi^{2^{k+1}}$.

**Problem met.2.** Give an example of a set $\Gamma$ and a sentence $\varphi$ so that $\Gamma \models \varphi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \not\models \varphi$.

**met.5 The Löwenheim-Skolem Theorem Fails for Second-order Logic**

The (Downward) Löwenheim-Skolem Theorem states that every set of sentences with an infinite model has an **enumerable** model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a
model for any consistent set of sentences, and that model is enumerable. There is also an Upward Löwenheim-Skolem Theorem, which guarantees that if a set of sentences has a denumerable model it also has a non-enumerable model. Both theorems fail in second-order logic.

Theorem met.8. The Löwenheim-Skolem Theorem fails for second-order logic: There are sentences with infinite models but no enumerable models.

Proof. Recall that

\[
\text{Count} \equiv \exists z \exists u \forall X ((X(z) \land \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))
\]

is true in a structure $\mathfrak{M}$ iff $|\mathfrak{M}|$ is enumerable, so $\neg\text{Count}$ is true in $\mathfrak{M}$ iff $|\mathfrak{M}|$ is non-enumerable. There are such structures—take any non-enumerable set as the domain, e.g., $\wp(\mathbb{N})$ or $\mathbb{R}$. So $\neg\text{Count}$ has infinite models but no enumerable models. \qed

Theorem met.9. There are sentences with denumerable but no non-enumerable models.

Proof. $\text{Count} \land \text{Inf}$ is true in $\mathbb{N}$ but not in any structure $\mathfrak{M}$ with $|\mathfrak{M}|$ non-enumerable. \qed

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Bibliography