

Chapter udf

Metatheory of Second-order Logic

met.1 Introduction

sol:met:int:
sec

First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a **derivability** relation \vdash with the property that for any set of **sentences** Γ and **sentence** Q , $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$. This means in particular that the validities of first-order logic are **computably enumerable**. There is a computable function $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L})$ such that the values of f are all and only the valid **sentences** of \mathcal{L} . This is so because **derivations** can be enumerated, and those that **derive** a single **sentence** are then mapped to that **sentence**. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we'll show that second-order logic is not only undecidable, but its validities are not even **computably enumerable**. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set Γ of **sentences** is satisfiable, Γ itself is satisfiable) and the Löwenheim-Skolem Theorem holds for it (if Γ has an infinite model it has a **denumerable** model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of **domains** that first-order logic cannot.

met.2 Second-order Arithmetic

sol:met:spa:
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Recall that the theory **PA** of Peano arithmetic includes the eight axioms of **Q**,

$$\begin{aligned}
&\forall x x' \neq 0 \\
&\forall x \forall y (x' = y' \rightarrow x = y) \\
&\forall x \forall y (x < y \leftrightarrow \exists z (x + z') = y) \\
&\forall x (x + 0) = x \\
&\forall x \forall y (x + y') = (x + y)' \\
&\forall x (x \times 0) = 0 \\
&\forall x \forall y (x \times y') = ((x \times y) + x)
\end{aligned}$$

plus all sentences of the form

$$(\varphi(0) \wedge \forall x (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \varphi(x)$$

The latter is a “schema,” i.e., a pattern that generates infinitely many **sentences** of the language of arithmetic, one for each **formula** $\varphi(x)$. We call this schema the (first-order) *axiom schema of induction*. In *second-order* Peano arithmetic **PA²**, induction can be stated as a single sentence. **PA²** consists of the first eight axioms above plus the (second-order) *induction axiom*:

$$\forall X (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x)$$

It says that if a subset X of the **domain** contains $0^{\mathfrak{M}}$ and with any $x \in |\mathfrak{M}|$ also contains $r^{\mathfrak{M}}(x)$ (i.e., it is “closed under successor”) it contains everything in the **domain** (i.e., $X = |\mathfrak{M}|$).

The induction axiom guarantees that any **structure** satisfying it contains only those **elements** of $|\mathfrak{M}|$ the axioms require to be there, i.e., the values of \bar{n} for $n \in \mathbb{N}$. A model of **PA²** contains no non-standard numbers.

Theorem met.1. *If $\mathfrak{M} \models \mathbf{PA}^2$ then $|\mathfrak{M}| = \{\text{Val}^n(M) : n \in \mathbb{N}\}$.*

*sol:met:spa:
thm:sol-pa-standard*

Proof. Let $N = \{\text{Val}^{\mathfrak{M}}(\bar{n}) : n \in \mathbb{N}\}$, and suppose $\mathfrak{M} \models \mathbf{PA}^2$. Of course, for any $n \in \mathbb{N}$, $\text{Val}^{\mathfrak{M}}(\bar{n}) \in |\mathfrak{M}|$, so $N \subseteq |\mathfrak{M}|$.

Now for inclusion in the other direction. Consider a variable assignment s with $s(X) = N$. By assumption,

$$\begin{aligned}
\mathfrak{M} \models \forall X (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x), \text{ thus} \\
\mathfrak{M}, s \models (X(0) \wedge \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x).
\end{aligned}$$

Consider the antecedent of this conditional. $\text{Val}^{\mathfrak{M}}(0) \in N$, and so $\mathfrak{M}, s \models X(0)$. The second conjunct, $\forall x (X(x) \rightarrow X(x'))$ is also satisfied. For suppose $x \in N$. By definition of N , $x = \text{Val}^{\mathfrak{M}}(\bar{n})$ for some n . That gives $r^{\mathfrak{M}}(x) = \text{Val}^{\mathfrak{M}}(\overline{n+1}) \in N$. So, $r^{\mathfrak{M}}(x) \in N$.

We have that $\mathfrak{M}, s \models X(0) \wedge \forall x (X(x) \rightarrow X(x'))$. Consequently, $\mathfrak{M}, s \models \forall x X(x)$. But that means that for every $x \in |\mathfrak{M}|$ we have $x \in s(X) = N$. So, $|\mathfrak{M}| \subseteq N$. \square

sol:met:spa:
cor:sol-pa-categorical

Corollary met.2. *Any two models of \mathbf{PA}^2 are isomorphic.*

Proof. By [Theorem met.1](#), the domain of any model of \mathbf{PA}^2 is exhausted by $\text{Val}^{\mathfrak{M}}(\bar{n})$. Any such model is also a model of \mathbf{Q} . By [??](#), any such model is standard, i.e., isomorphic to \mathfrak{N} . \square

Above we defined \mathbf{PA}^2 as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the *first two* of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

sol:met:spa:
prop:sol-pa-definable

Proposition met.3. *Let $\mathbf{PA}^{2\ddagger}$ be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then \leq , $+$, and \times are definable in $\mathbf{PA}^{2\ddagger}$.*

Proof. To show that \leq is definable, we have to find a formula $\varphi_{\leq}(x, y)$ such that $\mathfrak{N} \models \varphi_{\leq}(\bar{n}, \bar{m})$ iff $n < m$. Consider the formula

$$\psi(x, Y) \equiv Y(x) \wedge \forall y (Y(y) \rightarrow Y(y'))$$

Clearly, $\psi(\bar{n}, Y)$ is satisfied by a set $Y \subseteq \mathbb{N}$ iff $\{m : n \leq m\} \subseteq Y$, so we can take $\varphi_{\leq}(x, y) \equiv \forall Y (\psi(x, Y) \rightarrow Y(y))$. \square

Problem met.1. Complete the proof of [Proposition met.3](#).

Corollary met.4. $\mathfrak{M} \models \mathbf{PA}^2$ iff $\mathfrak{M} \models \mathbf{PA}^{2\ddagger}$.

Proof. Immediate from [Proposition met.3](#). \square

met.3 Second-order Logic is not Axiomatizable

sol:met:max:
sec
sol:met:nax:
thm:sol-undecidable

Theorem met.5. *Second-order logic is undecidable.*

Proof. A first-order [sentence](#) is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable. \square

sol:met:nax:
cor:sol-not-axiomatizable

Theorem met.6. *There is no sound and complete proof system for second-order logic.*

Proof. Let φ be a [sentence](#) in the language of arithmetic. $\mathfrak{N} \models \varphi$ iff $\mathbf{PA}^2 \models \varphi$. Let P be the conjunction of the nine axioms of \mathbf{PA}^2 . $\mathbf{PA}^2 \models \varphi$ iff $\models P \rightarrow \varphi$, i.e., $\mathfrak{M} \models P \rightarrow \varphi$. Now consider the [sentence](#) $\forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi')$ resulting by replacing 0 by z , $'$ by the one-place function variable u , $+$ and \times by the two-place function-variables u' and u'' , respectively, and $<$ by the two-place relation variable L and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff $\mathbf{PA}^2 \models \varphi$ iff $\mathfrak{N} \models \varphi$. Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L}_A)$ of the

sentences true in \mathfrak{N} . This function would be representable in \mathbf{Q} by some first-order formula $\psi_f(x, y)$. Then the formula $\exists x \psi_f(x, y)$ would define the set of true first-order sentences of \mathfrak{N} , contradicting Tarski's Theorem. \square

met.4 Second-order Logic is not Compact

explanation Call a set of sentences Γ *finitely satisfiable* if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences Γ is finitely satisfiable, it is satisfiable. This property is called *compactness*. It has an equivalent version involving entailment: if $\Gamma \models \varphi$, then already $\Gamma_0 \models \varphi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \models \varphi$, by completeness $\Gamma \vdash \varphi$. But a *derivation* can only make use of finitely many sentences of Γ .

sol:met:com:
sec

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some φ without a finite subset entailing φ .

Theorem met.7. *Second-order logic is not compact.*

sol:met:com:
thm:sol-undecidable

Proof. Recall that

$$\text{Inf} \equiv \exists u \forall x \forall y (u(x) = u(y) \rightarrow x = y)$$

is satisfied in a structure iff its domain is infinite. Let $\varphi^{\geq n}$ be a sentence that asserts that the domain has at least n elements, e.g.,

$$\varphi^{\geq n} \equiv \exists x_1 \dots \exists x_n (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \dots \wedge x_{n-1} \neq x_n)$$

Consider

$$\Gamma = \{\neg \text{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \dots\}$$

It is finitely satisfiable, since for any finite subset Γ_0 there is some k so that $\varphi^{\geq k} \in \Gamma_0$ but no $\varphi^{\geq n} \in \Gamma_0$ for $n > k$. If $|\mathfrak{M}|$ has k elements, $\mathfrak{M} \models \Gamma_0$. But, Γ is not satisfiable: if $\mathfrak{M} \models \neg \text{Inf}$, $|\mathfrak{M}|$ must be finite, say, of size k . Then $\mathfrak{M} \not\models \varphi^{\geq k+1}$. \square

Problem met.2. Give an example of a set Γ and a sentence φ so that $\Gamma \models \varphi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \not\models \varphi$.

met.5 The Löwenheim-Skolem Theorem Fails for Second-order Logic

explanation The (Downward) Löwenheim-Skolem Theorem states that every set of sentences with an infinite model has an enumerable model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of sentences, and that model is enumerable. There is also an Upward Löwenheim-Skolem Theorem, which guarantees that if a set of sentences has a denumerable model it also has a non-enumerable model. Both theorems fail in second-order logic.

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sol:met:lst:
thm:sol-no-ls

Theorem met.8. *The Löwenheim-Skolem Theorem fails for second-order logic: There are **sentences** with infinite models but no **enumerable** models.*

Proof. Recall that

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

is true in a **structure** \mathfrak{M} iff $|\mathfrak{M}|$ is **enumerable**. So $\text{Inf} \wedge \neg\text{Count}$ is true in \mathfrak{M} iff $|\mathfrak{M}|$ is both infinite and not **enumerable**. There are such **structures**—take any **non-enumerable** set as the **domain**, e.g., $\wp(\mathbb{N})$ or \mathbb{R} . So $\text{Inf} \wedge \neg\text{Count}$ has infinite models but no **enumerable** models. \square

Theorem met.9. *There are **sentences** with **denumerable** but no **non-enumerable** models.*

Proof. $\text{Count} \wedge \text{Inf}$ is true in \mathbb{N} but not in any **structure** \mathfrak{M} with $|\mathfrak{M}|$ **non-enumerable**. \square

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Bibliography