Chapter udf

Metatheory of Second-order Logic

met.1 Introduction

First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a derivability relation \( \vdash \) with the property that for any set of sentences \( \Gamma \) and sentence \( Q \), \( \Gamma \models \varphi \) iff \( \Gamma \vdash \varphi \). This means in particular that the validities of first-order logic are computably enumerable. There is a computable function \( f : \mathbb{N} \to \text{Sent}(\mathcal{L}) \) such that the values of \( f \) are all and only the valid sentences of \( \mathcal{L} \). This is so because derivations can be enumerated, and those that derive a single sentence are then mapped to that sentence. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we’ll show that second-order logic is not only undecidable, but its validities are not even computably enumerable. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set \( \Gamma \) of sentences is satisfiable, \( \Gamma \) itself is satisfiable) and the Löwenheim-Skolem Theorem holds for it (if \( \Gamma \) has an infinite model it has a denumerable model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of domains that first-order logic cannot.

met.2 Second-order Arithmetic

Recall that the theory \( \text{PA} \) of Peano arithmetic includes the eight axioms
of $Q$.

$$
\forall x x' \neq 0 \\
\forall x \forall y (x' = y' \rightarrow x = y) \\
\forall x \forall y (x < y \leftrightarrow \exists z (x + z') = y) \\
\forall x (x + 0) = x \\
\forall x \forall y (x + y') = (x + y)' \\
\forall x (x \times 0) = 0 \\
\forall x \forall y (x \times y') = ((x \times y) + x)
$$

plus all sentences of the form

$$(\varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \varphi(x)$$

The latter is a “schema,” i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula $\varphi(x)$. We call this schema the (first-order) axiom schema of induction. In second-order Peano arithmetic $PA^2$, induction can be stated as a single sentence. $PA^2$ consists of the first eight axioms above plus the (second-order) induction axiom:

$$\forall X (X(0) \land \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x)$$

It says that if a subset $X$ of the domain contains $0^{\mathcal{M}}$ and with any $x \in [\mathcal{M}]$ also contains $\rho^\mathcal{M}(x)$ (i.e., it is “closed under successor”) it contains everything in the domain (i.e., $X = [\mathcal{M}]$).

The induction axiom guarantees that any structure satisfying it contains only those elements of $[\mathcal{M}]$ the axioms require to be there, i.e., the values of $\pi$ for $n \in \mathbb{N}$. A model of $PA^2$ contains no non-standard numbers.

**Theorem met.1.** If $\mathcal{M} \models PA^2$ then $[\mathcal{M}] = \{Val^{\mathcal{M}}(M) : n \in \mathbb{N}\}$.

**Proof.** Let $N = \{Val^{\mathcal{M}}(\pi) : n \in \mathbb{N}\}$, and suppose $\mathcal{M} \models PA^2$. Of course, for any $n \in \mathbb{N}$, $Val^{\mathcal{M}}(\pi) \in [\mathcal{M}]$, so $N \subseteq [\mathcal{M}]$.

Now for inclusion in the other direction. Consider a variable assignment $s$ with $s(X) = N$. By assumption,

$$\mathcal{M} \models \forall X (X(0) \land \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x), \text{ thus}\n
\mathcal{M}, s \models (X(0) \land \forall x (X(x) \rightarrow X(x'))) \rightarrow \forall x X(x).$$

Consider the antecedent of this conditional. $Val^{\mathcal{M}}(0) \in N$, and so $\mathcal{M}, s \models X(0)$. The second conjunct, $\forall x (X(x) \rightarrow X(x'))$ is also satisfied. For suppose $x \in N$. By definition of $N$, $x = Val^{\mathcal{M}}(\pi)$ for some $n$. That gives $\rho^\mathcal{M}(x) = Val^{\mathcal{M}}(n + 1) \in N$. So, $\rho^\mathcal{M}(x) \in N$.

We have that $\mathcal{M}, s \models X(0) \land \forall x (X(x) \rightarrow X(x'))$. Consequently, $\mathcal{M}, s \models \forall x X(x)$. But that means that for every $x \in [\mathcal{M}]$ we have $x \in s(X) = N$. So, $[\mathcal{M}] \subseteq N$. 

\[\square\]
**Corollary met.2.** Any two models of PA$^2$ are isomorphic.

*Proof.* By Theorem met.1, the domain of any model of PA$^2$ is exhausted by Val$_{r}(\mathcal{M})$. Any such model is also a model of Q. By ??, any such model is standard, i.e., isomorphic to N. □

Above we defined PA$^2$ as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the first two of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

**Proposition met.3.** Let PA$^{2\dagger}$ be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. $\geq$, $+$, and $\times$ are definable in PA$^{2\dagger}$.

*Proof.* Exercise. □

**Problem met.1.** Prove Proposition met.3.

**Corollary met.4.** $\mathcal{M} \models PA^2$ iff $\mathcal{M} \models PA^{2\dagger}$.

*Proof.* Immediate from Proposition met.3. □

### met.3 Second-order Logic is not Axiomatizable

**Theorem met.5.** Second-order logic is undecidable.

*Proof.* A first-order sentence is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable. □

**Theorem met.6.** There is no sound and complete proof system for second-order logic.

*Proof.* Let $\varphi$ be a sentence in the language of arithmetic. $\mathcal{M} \models \varphi$ iff PA$^2 \models \varphi$. Let $P$ be the conjunction of the nine axioms of PA$^2$. PA$^2 \models \varphi$ iff $P \rightarrow \varphi$, i.e., $\mathcal{M} \models P \rightarrow \varphi$. Now consider the sentence $\forall z \forall u \forall u' \forall u'' \forall L (P' \rightarrow \varphi')$ resulting by replacing $\circ$ by $z$, $\cdot$ by the one-place function variable $u$, $+$ and $\times$ by the two-place function-variables $u'$ and $u''$, respectively, and $<$ by the two-place relation variable $L$ and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff PA$^2 \models \varphi$ iff $\mathcal{M} \models \varphi$. Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration $f: \mathbb{N} \rightarrow \text{Sent}(L_A)$ of the sentences true in $\mathcal{M}$. This function would be representable in Q by some first-order formula $\psi_f(x,y)$. Then the formula $\exists x \psi_f(x,y)$ would define the set of true first-order sentences of $\mathcal{M}$, contradicting Tarski’s Theorem. □
Second-order Logic is not Compact

Call a set of sentences $\Gamma$ finitely satisfiable if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences $\Gamma$ is finitely satisfiable, it is satisfiable. This property is called compactness. It has an equivalent version involving entailment: if $\Gamma \vdash \varphi$, then already $\Gamma_0 \vdash \varphi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \vdash \varphi$, by completeness $\Gamma \vdash \varphi$. But a derivation can only make use of finitely many sentences of $\Gamma$.

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some $\varphi$ without a finite subset entailing $\varphi$.

Theorem met.7. Second-order logic is not compact.

Proof. Recall that

$$\text{Inf} \equiv \exists u \forall x \forall y (u(x) = u(y) \rightarrow x = y)$$

is satisfied in a structure iff its domain is infinite. Let $\varphi^{\geq n}$ be a sentence that asserts that the domain has at least $n$ elements, e.g.,

$$\varphi^{\geq n} \equiv \exists x_1 \ldots \exists x_n (x_1 \neq x_2 \land x_1 \neq x_3 \land \cdots \land x_{n-1} \neq x_n)$$

Consider

$$\Gamma = \{ \neg \text{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \ldots \}$$

It is finitely satisfiable, since for any finite subset $\Gamma_0$ there is some $k$ so that $\varphi^{\geq k} \in \Gamma$ but no $\varphi^{\geq n} \in \Gamma$ for $n > k$. If $|\mathfrak{M}|$ has $k$ elements, $\mathfrak{M} \models \Gamma_0$. But, $\Gamma$ is not satisfiable: if $\mathfrak{M} \models \neg \text{Inf}$, $|\mathfrak{M}|$ must be finite, say, of size $k$. Then $\mathfrak{M} \not\models \varphi^{\geq k+1}$.

Problem met.2. Give an example of a set $\Gamma$ and a sentence $\varphi$ so that $\Gamma \vdash \varphi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \not\vdash \varphi$.

The Löwenheim-Skolem Theorem Fails for Second-order Logic

The (Downward) Löwenheim-Skolem Theorem states that every set of sentences with an infinite model has an enumerable model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of sentences, and that model is enumerable. There is also an Upward Löwenheim-Skolem Theorem, which guarantees that if a set of sentences has a denumerable model it also has a non-enumerable model. Both theorems fail in second-order logic.

Theorem met.8. The Löwenheim-Skolem Theorem fails for second-order logic: There are sentences with infinite models but no enumerable models.
Proof. Recall that

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \land \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

is true in a structure $\mathcal{M}$ iff $|\mathcal{M}|$ is enumerable. So $\text{Inf} \land \neg \text{Count}$ is true in $\mathcal{M}$ iff $|\mathcal{M}|$ is both infinite and not enumerable. There are such structures—take any non-enumerable set as the domain, e.g., $\wp(\mathbb{N})$ or $\mathbb{R}$. So $\text{Inf} \land \text{Count}$ has infinite models but no enumerable models. □

Theorem met.9. There are sentences with denumerable but not with non-enumerable models.

Proof. $\text{Count} \land \text{Inf}$ is true in $\mathbb{N}$ but not in any structure $\mathcal{M}$ with $|\mathcal{M}|$ non-enumerable. □

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Bibliography