Metatheory of Second-order Logic

met.1 Introduction

First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a derivability relation $\vdash$ with the property that for any set of sentences $\Gamma$ and sentence $Q$, $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$. This means in particular that the validities of first-order logic are computably enumerable. There is a computable function $f : \mathbb{N} \to \text{Sent}(\mathcal{L})$ such that the values of $f$ are all and only the valid sentences of $\mathcal{L}$. This is so because derivations can be enumerated, and those that derive a single sentence are then mapped to that sentence. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we’ll show that second-order logic is not only undecidable, but its validities are not even computably enumerable. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set $\Gamma$ of sentences is satisfiable, $\Gamma$ itself is satisfiable) and the Löwenheim–Skolem Theorem holds for it (if $\Gamma$ has an infinite model it has a denumerable model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of domains that first-order logic cannot.

met.2 Second-order Arithmetic
Recall that the theory $\text{PA}$ of Peano arithmetic includes the eight axioms of $\mathbb{Q}$,

\[
\forall x \, x' \neq 0 \\
\forall x \, \forall y \, (x' = y' \rightarrow x = y) \\
\forall x \, (x = 0 \lor \exists y \, x = y') \\
\forall x \, (x + 0) = x \\
\forall x \, \forall y \, (x + y') = (x + y)' \\
\forall x \, (x \times 0) = 0 \\
\forall x \, \forall y \, (x \times y') = ((x \times y) + x) \\
\forall x \, \forall y \, (x < y \leftrightarrow \exists z \, (z' + x) = y)
\]

plus all sentences of the form

\[
(\varphi(0) \land \forall x \, (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x \, \varphi(x).
\]

The latter is a “schema,” i.e., a pattern that generates infinitely many sentences of the language of arithmetic, one for each formula $\varphi(x)$. We call this schema the (first-order) axiom schema of induction. In second-order Peano arithmetic $\text{PA}^2$, induction can be stated as a single sentence. $\text{PA}^2$ consists of the first eight axioms above plus the (second-order) induction axiom:

\[
\forall X \, (X(0) \land \forall x \, (X(x) \rightarrow X(x'))) \rightarrow \forall x \, X(x).
\]

It says that if a subset $X$ of the domain contains $0^M$ and with any $x \in [M]$ also contains $\rho^M(x)$ (i.e., it is “closed under successor”) it contains everything in the domain (i.e., $X = |M|$).

The induction axiom guarantees that any structure satisfying it contains only those elements of $[M]$ the axioms require to be there, i.e., the values of $\pi$ for $n \in \mathbb{N}$. A model of $\text{PA}^2$ contains no non-standard numbers.

**Theorem met.1.** If $M \models \text{PA}^2$ then $|M| = \{\text{Val}^M(\pi) : n \in \mathbb{N}\}$.

*Proof.* Let $N = \{\text{Val}^M(\pi) : n \in \mathbb{N}\}$, and suppose $M \models \text{PA}^2$. Of course, for any $n \in \mathbb{N}$, $\text{Val}^M(\pi) \in [M]$, so $N \subseteq [M]$.

Now for inclusion in the other direction. Consider a variable assignment $s$ with $s(X) = N$. By assumption,

\[
\begin{align*}
M & \models \forall X \, (X(0) \land \forall x \, (X(x) \rightarrow X(x'))) \rightarrow \forall x \, X(x), \\
M, s & \models (X(0) \land \forall x \, (X(x) \rightarrow X(x'))) \rightarrow \forall x \, X(x).
\end{align*}
\]

Consider the antecedent of this conditional. $\text{Val}^M(0) \in N$, and so $M, s \models X(0)$. The second conjunct, $\forall x \, (X(x) \rightarrow X(x'))$ is also satisfied. For suppose $x \in N$. By definition of $N$, $x = \text{Val}^M(\pi)$ for some $n$. That gives $\rho^M(x) = \text{Val}^M(n + 1) \in N$. So, $\rho^M(x) \in N$.

We have that $M, s \models X(0) \land \forall x \, (X(x) \rightarrow X(x'))$. Consequently, $M, s \models \forall x \, X(x)$. But that means that for every $x \in [M]$ we have $x \in s(X) = N$. So, $|M| \subseteq N$. 

\[\square\]
Corollary met.2. Any two models of PA are isomorphic.

Proof. By Theorem met.1, the domain of any model of PA is exhausted by Val(\mathbb{N}). Any such model is also a model of Q. By ??, any such model is standard, i.e., isomorphic to \mathbb{N}.

Above we defined PA as the theory that contains the first eight arithmetical axioms plus the second-order induction axiom. In fact, thanks to the expressive power of second-order logic, only the first two of the arithmetical axioms plus induction are needed for second-order Peano arithmetic.

Proposition met.3. Let PA be the second-order theory containing the first two arithmetical axioms (the successor axioms) and the second-order induction axiom. Then ≤, +, and × are definable in PA.

Proof. To show that ≤ is definable, we have to find a formula φ≤(x, y) such that \mathbb{N} ⊨ φ≤(n, m) iff n ≤ m. Consider the formula

φ(x, Y) ≡ Y(x) ∧ ∀y (Y(y) → Y(y'))

Clearly, φ(n, Y) is satisfied by a set Y ⊆ \mathbb{N} iff \{m : n ≤ m\} ⊆ Y, so we can take φ≤(x, y) ≡ ∀Y (φ(x, Y) → Y(y)).

To see that addition is definable observe that k+l = m iff there is a function u such that u(0) = k, u(n') = u(u(n)) for all n, and m = u(l). We can use this equivalence to define addition in PA by the following formula:

φ+(x, y, z) ≡ ∃u (u(0) = x ∧ ∀w u(x') = u(x') ∧ u(y) = z)

It should be clear that \mathbb{N} ⊨ φ+(k, l, m) iff k + l = m.

Problem met.1. Complete the proof of Proposition met.3.

met.3 Second-order Logic is not Axiomatizable

Theorem met.4. Second-order logic is undecidable.

Proof. A first-order sentence is valid in first-order logic iff it is valid in second-order logic, and first-order logic is undecidable.

Theorem met.5. There is no sound and complete derivation system for second-order logic.

Proof. Let \varphi be a sentence in the language of arithmetic. \mathfrak{M} ⊨ \varphi iff PA ⊨ \varphi. Let P be the conjunction of the nine axioms of PA. PA ⊨ P iff \mathfrak{M} ⊨ P → \varphi. Now consider the sentence ∀z ∀u ∀u' ∀u'' ∀L (P' → \varphi') resulting by replacing 0 by z, ′ by the one-place function variable u, + and × by the two-place function-variables u' and u'', respectively, and < by the two-place
relation variable $L$ and universally quantifying. It is a valid sentence of pure second-order logic iff the original sentence was valid iff $\text{PA}^2 \models \varphi$ iff $\mathfrak{M} \models \varphi$. Thus if there were a sound and complete proof system for second-order logic, we could use it to define a computable enumeration $f : \mathbb{N} \to \text{Sent}(L_A)$ of the sentences true in $\mathfrak{M}$. This function would be representable in $\text{Q}$ by some first-order formula $\psi_f(x, y)$. Then the formula $\exists x \psi_f(x, y)$ would define the set of true first-order sentences of $\mathfrak{M}$, contradicting Tarski’s Theorem.

### met.4 Second-order Logic is not Compact

Call a set of sentences $\Gamma$ finitely satisfiable if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences $\Gamma$ is finitely satisfiable, it is satisfiable. This property is called compactness. It has an equivalent version involving entailment: if $\Gamma \models \varphi$, then already $\Gamma_0 \models \varphi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \models \varphi$, by completeness $\Gamma \vdash \varphi$. But a derivation can only make use of finitely many sentences of $\Gamma$.

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some $\varphi$ without a finite subset entailing $\varphi$.

**Theorem met.6.** Second-order logic is not compact.

**Proof.** Recall that

$$\text{Inf} \equiv \exists u \left( \forall x \forall y \left( u(x) = u(y) \rightarrow x = y \right) \land \exists y \forall x \, x \neq u(x) \right)$$

is satisfied in a structure iff its domain is infinite. Let $\varphi^{\geq n}$ be a sentence that asserts that the domain has at least $n$ elements, e.g.,

$$\varphi^{\geq n} \equiv \exists x_1 \ldots \exists x_n \left( x_1 \neq x_2 \land x_2 \neq x_3 \land \cdots \land x_{n-1} \neq x_n \right).$$

Consider the set of sentences

$$\Gamma = \left\{ \neg \text{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \ldots \right\}.$$ 

It is finitely satisfiable, since for any finite subset $\Gamma_0 \subseteq \Gamma$ there is some $k$ so that $\varphi^{\geq k} \in \Gamma$ but no $\varphi^{\geq n} \in \Gamma$ for $n > k$. If $\mathfrak{M}$ has $k$ elements, $\mathfrak{M} \models \Gamma_0$. But, $\Gamma$ is not satisfiable: if $\mathfrak{M} \models \neg \text{Inf}$, $|\mathfrak{M}|$ must be finite, say, of size $k$. Then $\mathfrak{M} \not\models \varphi^{\geq k+1}$.

**Problem met.2.** Give an example of a set $\Gamma$ and a sentence $\varphi$ so that $\Gamma \models \varphi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \not\models \varphi$. 

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The (Downward) Löwenheim–Skolem Theorem states that every set of sentences with an infinite model has an enumerable model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of sentences, and that model is enumerable. There is also an Upward Löwenheim–Skolem Theorem, which guarantees that if a set of sentences has a denumerable model it also has a non-enumerable model. Both theorems fail in second-order logic.

Theorem met.7. The Löwenheim–Skolem Theorem fails for second-order logic: There are sentences with infinite models but no enumerable models.

Proof. Recall that

\[\text{Count} \equiv \exists z \exists u \forall X ((X(z) \land \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))\]

is true in a structure \(\mathfrak{M}\) iff \(|\mathfrak{M}|\) is enumerable, so \(\neg\text{Count}\) is true in \(\mathfrak{M}\) iff \(|\mathfrak{M}|\) is non-enumerable. There are such structures—take any non-enumerable set as the domain, e.g., \(\mathcal{P}(\mathbb{N})\) or \(\mathbb{R}\). So \(\neg\text{Count}\) has infinite models but no enumerable models.

\[\square\]

Theorem met.8. There are sentences with denumerable but no non-enumerable models.

Proof. \(\text{Count} \land \text{Inf}\) is true in \(\mathbb{N}\) but not in any structure \(\mathfrak{M}\) with \(|\mathfrak{M}|\) non-enumerable.

\[\square\]
Bibliography