

## met.1 The Löwenheim-Skolem Theorem Fails for Second-order Logic

sol:met:lst: sec The (Downward) Löwenheim-Skolem Theorem states that every set of sentences with an infinite model has an enumerable model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of sentences, and that model is enumerable. There is also an Upward Löwenheim-Skolem Theorem, which guarantees that if a set of sentences has a denumerable model it also has a non-enumerable model. Both theorems fail in second-order logic. explanation

sol:met:lst: thm:sol-no-ls **Theorem met.1.** *The Löwenheim-Skolem Theorem fails for second-order logic: There are sentences with infinite models but no enumerable models.*

*Proof.* Recall that

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

is true in a structure  $\mathfrak{M}$  iff  $|\mathfrak{M}|$  is enumerable. So  $\text{Inf} \wedge \neg\text{Count}$  is true in  $\mathfrak{M}$  iff  $|\mathfrak{M}|$  is both infinite and not enumerable. There are such structures—take any non-enumerable set as the domain, e.g.,  $\wp(\mathbb{N})$  or  $\mathbb{R}$ . So  $\text{Inf} \wedge \neg\text{Count}$  has infinite models but no enumerable models.  $\square$

**Theorem met.2.** *There are sentences with denumerable but no non-enumerable models.*

*Proof.*  $\text{Count} \wedge \text{Inf}$  is true in  $\mathbb{N}$  but not in any structure  $\mathfrak{M}$  with  $|\mathfrak{M}|$  non-enumerable.  $\square$

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## Bibliography