

met.1 The Löwenheim-Skolem Theorem Fails for Second-order Logic

sol:met:lst:
sec The (Downward) Löwenheim-Skolem Theorem states that every set of sentences with an infinite model has an enumerable model. It, too, is a consequence of the completeness theorem: the proof of completeness generates a model for any consistent set of sentences, and that model is enumerable. There is also an Upward Löwenheim-Skolem Theorem, which guarantees that if a set of sentences has a denumerable model it also has a non-enumerable model. Both theorems fail in second-order logic. explanation

sol:met:lst:
thm:sol-no-ls **Theorem met.1.** *The Löwenheim-Skolem Theorem fails for second-order logic: There are sentences with infinite models but no enumerable models.*

Proof. Recall that

$$\text{Count} \equiv \exists z \exists u \forall X ((X(z) \wedge \forall x (X(x) \rightarrow X(u(x)))) \rightarrow \forall x X(x))$$

is true in a structure \mathfrak{M} iff $|\mathfrak{M}|$ is enumerable. So $\text{Inf} \wedge \neg\text{Count}$ is true in \mathfrak{M} iff $|\mathfrak{M}|$ is both infinite and not enumerable. There are such structures—take any non-enumerable set as the domain, e.g., $\wp(\mathbb{N})$ or \mathbb{R} . So $\text{Inf} \wedge \text{Count}$ has infinite models but no enumerable models. \square

Theorem met.2. *There are sentences with denumerable but not with non-enumerable models.*

Proof. $\text{Count} \wedge \text{Inf}$ is true in \mathbb{N} but not in any structure \mathfrak{M} with $|\mathfrak{M}|$ non-enumerable. \square

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Bibliography