

## met.1 Introduction

sol:met:int:  
sec First-order logic has a number of nice properties. We know it is not decidable, but at least it is axiomatizable. That is, there are proof systems for first-order logic which are sound and complete, i.e., they give rise to a **derivability** relation  $\vdash$  with the property that for any set of **sentences**  $\Gamma$  and **sentence**  $Q$ ,  $\Gamma \models \varphi$  iff  $\Gamma \vdash \varphi$ . This means in particular that the validities of first-order logic are **computably enumerable**. There is a computable function  $f: \mathbb{N} \rightarrow \text{Sent}(\mathcal{L})$  such that the values of  $f$  are all and only the valid **sentences** of  $\mathcal{L}$ . This is so because **derivations** can be enumerated, and those that **derive** a single **sentence** are then mapped to that **sentence**. Second-order logic is more expressive than first-order logic, and so it is in general more complicated to capture its validities. In fact, we'll show that second-order logic is not only undecidable, but its validities are not even **computably enumerable**. This means there can be no sound and complete proof system for second-order logic (although sound, but incomplete proof systems are available and in fact are important objects of research).

First-order logic also has two more properties: it is compact (if every finite subset of a set  $\Gamma$  of **sentences** is satisfiable,  $\Gamma$  itself is satisfiable) and the Löwenheim-Skolem Theorem holds for it (if  $\Gamma$  has an infinite model it has a **denumerable** model). Both of these results fail for second-order logic. Again, the reason is that second-order logic can express facts about the size of **domains** that first-order logic cannot.

## Photo Credits

## Bibliography