Second-order Logic is not Compact

Call a set of sentences $\Gamma$ finitely satisfiable if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences $\Gamma$ is finitely satisfiable, it is satisfiable. This property is called compactness. It has an equivalent version involving entailment: if $\Gamma \models \phi$, then already $\Gamma_0 \models \phi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \models \phi$, by completeness $\Gamma \vdash \phi$. But a derivation can only make use of finitely many sentences of $\Gamma$.

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some $\phi$ without a finite subset entailing $\phi$.

**Theorem met.1.** Second-order logic is not compact.

**Proof.** Recall that
\[
\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \land \exists y \forall x y \neq u(x))
\]
is satisfied in a structure iff its domain is infinite. Let $\varphi^{\geq n}$ be a sentence that asserts that the domain has at least $n$ elements, e.g.,
\[
\varphi^{\geq n} \equiv \exists x_1 \ldots \exists x_n (x_1 \neq x_2 \land x_1 \neq x_3 \land \ldots \land x_{n-1} \neq x_n).
\]
Consider the set of sentences
\[
\Gamma = \{-\text{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \ldots \}.
\]
It is finitely satisfiable, since for any finite subset $\Gamma_0 \subseteq \Gamma$ there is some $k$ so that $\varphi^{\geq k} \in \Gamma$ but no $\varphi^{\geq n} \in \Gamma$ for $n > k$. If $|\mathcal{M}|$ has $k$ elements, $\mathcal{M} \models \Gamma_0$. But, $\Gamma$ is not satisfiable: if $\mathcal{M} \models -\text{Inf}$, $|\mathcal{M}|$ must be finite, say, of size $k$. Then $\mathcal{M} \not\models \varphi^{\geq k+1}$.

**Problem met.1.** Give an example of a set $\Gamma$ and a sentence $\phi$ so that $\Gamma \models \phi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \not\models \phi$. 

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Bibliography