Second-order Logic is not Compact

Call a set of sentences $\Gamma$ \textit{finitely satisfiable} if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences $\Gamma$ is finitely satisfiable, it is satisfiable. This property is called \textit{compactness}. It has an equivalent version involving entailment: if $\Gamma \models \varphi$, then already $\Gamma_0 \models \varphi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \models \varphi$, by completeness $\Gamma \vdash \varphi$. But a derivation can only make use of finitely many \textit{sentences} of $\Gamma$.

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some $\varphi$ without a finite subset entailing $\varphi$.

**Theorem met.1.** Second-order logic is not compact.

**Proof.** Recall that

$$\text{Inf} \equiv \exists u (\forall x \forall y (u(x) = u(y) \rightarrow x = y) \land \exists y \forall x y \neq u(x))$$

is satisfied in a structure iff its domain is infinite. Let $\varphi_{\geq n}$ be a sentence that asserts that the domain has at least $n$ elements, e.g.,

$$\varphi_{\geq n} \equiv \exists x_1 \ldots \exists x_n (x_1 \neq x_2 \land x_1 \neq x_3 \land \cdots \land x_{n-1} \neq x_n).$$

Consider the set of sentences

$$\Gamma = \{ \neg \text{Inf}, \varphi_{\geq 1}, \varphi_{\geq 2}, \varphi_{\geq 3}, \ldots \}.$$

It is finitely satisfiable, since for any finite subset $\Gamma_0 \subseteq \Gamma$ there is some $k$ so that $\varphi_{\geq k} \in \Gamma$ but no $\varphi_{\geq n} \in \Gamma$ for $n > k$. If $|\mathcal{M}|$ has $k$ elements, $\mathcal{M} \models \Gamma_0$. But, $\Gamma$ is not satisfiable: if $\mathcal{M} \models \neg \text{Inf}$, $|\mathcal{M}|$ must be finite, say, of size $k$. Then $\mathcal{M} \not\models \varphi_{\geq k+1}$. \hfill $\Box$

**Problem met.1.** Give an example of a set $\Gamma$ and a sentence $\varphi$ so that $\Gamma \models \varphi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \not\models \varphi$.

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Bibliography