

met.1 Second-order Logic is not Compact

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sec

Call a set of sentences Γ *finitely satisfiable* if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences Γ is finitely satisfiable, it is satisfiable. This property is called *compactness*. It has an equivalent version involving entailment: if $\Gamma \models \varphi$, then already $\Gamma_0 \models \varphi$ for some finite subset $\Gamma_0 \subseteq \Gamma$. In this version it is an immediate corollary of the completeness theorem: for if $\Gamma \models \varphi$, by completeness $\Gamma \vdash \varphi$. But a *derivation* can only make use of finitely many sentences of Γ .

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some φ without a finite subset entailing φ .

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thm:sol-undecidable

Theorem met.1. *Second-order logic is not compact.*

Proof. Recall that

$$\text{Inf} \equiv \exists u \forall x \forall y (u(x) = u(y) \rightarrow x = y)$$

is satisfied in a structure iff its domain is infinite. Let $\varphi^{\geq n}$ be a sentence that asserts that the domain has at least n elements, e.g.,

$$\varphi^{\geq n} \equiv \exists x_1 \dots \exists x_n (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \dots \wedge x_{n-1} \neq x_n)$$

Consider

$$\Gamma = \{\neg \text{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \dots\}$$

It is finitely satisfiable, since for any finite subset Γ_0 there is some k so that $\varphi^{\geq k} \in \Gamma_0$ but no $\varphi^{\geq n} \in \Gamma_0$ for $n > k$. If \mathfrak{M} has k elements, $\mathfrak{M} \models \Gamma_0$. But, Γ is not satisfiable: if $\mathfrak{M} \models \neg \text{Inf}$, $|\mathfrak{M}|$ must be finite, say, of size k . Then $\mathfrak{M} \not\models \varphi^{\geq k+1}$. \square

Problem met.1. Give an example of a set Γ and a sentence φ so that $\Gamma \models \varphi$ but for every finite subset $\Gamma_0 \subseteq \Gamma$, $\Gamma_0 \not\models \varphi$.

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Bibliography