

## met.1 Second-order Logic is not Compact

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sec

Call a set of sentences  $\Gamma$  *finitely satisfiable* if every one of its finite subsets is satisfiable. First-order logic has the property that if a set of sentences  $\Gamma$  is finitely satisfiable, it is satisfiable. This property is called *compactness*. It has an equivalent version involving entailment: if  $\Gamma \models \varphi$ , then already  $\Gamma_0 \models \varphi$  for some finite subset  $\Gamma_0 \subseteq \Gamma$ . In this version it is an immediate corollary of the completeness theorem: for if  $\Gamma \models \varphi$ , by completeness  $\Gamma \vdash \varphi$ . But a *derivation* can only make use of finitely many sentences of  $\Gamma$ .

explanation

Compactness is not true for second-order logic. There are sets of second-order sentences that are finitely satisfiable but not satisfiable, and that entail some  $\varphi$  without a finite subset entailing  $\varphi$ .

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thm:sol-undecidable

**Theorem met.1.** *Second-order logic is not compact.*

*Proof.* Recall that

$$\text{Inf} \equiv \exists u \forall x \forall y (u(x) = u(y) \rightarrow x = y)$$

is satisfied in a structure iff its domain is infinite. Let  $\varphi^{\geq n}$  be a sentence that asserts that the domain has at least  $n$  elements, e.g.,

$$\varphi^{\geq n} \equiv \exists x_1 \dots \exists x_n (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \dots \wedge x_{n-1} \neq x_n)$$

Consider

$$\Gamma = \{\neg \text{Inf}, \varphi^{\geq 1}, \varphi^{\geq 2}, \varphi^{\geq 3}, \dots\}$$

It is finitely satisfiable, since for any finite subset  $\Gamma_0$  there is some  $k$  so that  $\varphi^{\geq k} \in \Gamma_0$  but no  $\varphi^{\geq n} \in \Gamma_0$  for  $n > k$ . If  $\mathfrak{M}$  has  $k$  elements,  $\mathfrak{M} \models \Gamma_0$ . But,  $\Gamma$  is not satisfiable: if  $\mathfrak{M} \models \neg \text{Inf}$ ,  $|\mathfrak{M}|$  must be finite, say, of size  $k$ . Then  $\mathfrak{M} \not\models \varphi^{\geq k+1}$ .  $\square$

**Problem met.1.** Give an example of a set  $\Gamma$  and a sentence  $\varphi$  so that  $\Gamma \models \varphi$  but for every finite subset  $\Gamma_0 \subseteq \Gamma$ ,  $\Gamma_0 \not\models \varphi$ .

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## Bibliography