syn.1 Valuations and Satisfaction

pl:syn:val:sec

Definition syn.1 (Valuations). Let $\{\mathbb{T}, \mathbb{F}\}$ be the set of the two truth values, "true" and "false." A *valuation* for $\mathcal{L}_0$ is a function $\mathfrak{v}$ assigning either $\mathbb{T}$ or $\mathbb{F}$ to the propositional variables of the language, i.e., $\mathfrak{v}\colon \mathrm{At}_0 \to \{\mathbb{T}, \mathbb{F}\}$.

Definition syn.2. Given a valuation $\mathfrak{v}$, define the evaluation function $\overline{\mathfrak{v}}\colon \mathrm{Frm}(\mathcal{L}_0) \to \{\mathbb{T}, \mathbb{F}\}$ inductively by:

$$\overline{\mathfrak{v}}(\bot) = \mathbb{F};$$
$$\overline{\mathfrak{v}}(\top) = \mathbb{T};$$
$$\overline{\mathfrak{v}}(p_n) = \mathfrak{v}(p_n);$$
$$\overline{\mathfrak{v}}(\neg\varphi) = \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F}; \\ \mathbb{F} & \text{otherwise.} \end{cases}$$
$$\overline{\mathfrak{v}}(\varphi \land \psi) = \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T} \text{ and } \overline{\mathfrak{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F} \text{ or } \overline{\mathfrak{v}}(\psi) = \mathbb{F}. \end{cases}$$
$$\overline{\mathfrak{v}}(\varphi \lor \psi) = \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T} \text{ or } \overline{\mathfrak{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F} \text{ and } \overline{\mathfrak{v}}(\psi) = \mathbb{F}. \end{cases}$$
$$\overline{\mathfrak{v}}(\varphi \to \psi) = \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F} \text{ or } \overline{\mathfrak{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T} \text{ and } \overline{\mathfrak{v}}(\psi) = \mathbb{F}. \end{cases}$$
$$\overline{\mathfrak{v}}(\varphi \leftrightarrow \psi) = \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \overline{\mathfrak{v}}(\psi); \\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) \neq \overline{\mathfrak{v}}(\psi). \end{cases}$$

The clauses correspond to the following truth tables: explanation

φ	$\neg\varphi$
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$

φ	ψ	$\varphi \land \psi$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

φ	ψ	$\varphi \lor \psi$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

φ	ψ	$\varphi \to \psi$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$

φ	ψ	$\varphi \leftrightarrow \psi$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$

Problem syn.1. Consider adding to $\mathcal{L}_0$ a ternary connective $\diamondsuit$ with evaluation given by

$$\overline{\mathfrak{v}}(\diamondsuit(\varphi, \psi, \chi)) = \begin{cases} \overline{\mathfrak{v}}(\psi) & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T}; \\ \overline{\mathfrak{v}}(\chi) & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F}. \end{cases}$$

Write down the truth table for this connective.

Theorem syn.3 (Local Determination). *Suppose that $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are valuations that agree on the propositional letters occurring in φ, i.e., $\mathfrak{v}_1(p_n) = \mathfrak{v}_2(p_n)$ whenever p_n occurs in some formula φ. Then $\overline{\mathfrak{v}_1}$ and $\overline{\mathfrak{v}_2}$ also agree on φ, i.e., $\overline{\mathfrak{v}_1}(\varphi) = \overline{\mathfrak{v}_2}(\varphi)$.* pl:syn:val: thm:LocalDetermination

Proof. By induction on φ. □

Definition syn.4 (Satisfaction). We can inductively define the notion of *satisfaction of a formula φ by a valuation* $\mathfrak{v}$, $\mathfrak{v} \vDash \varphi$, as follows. (We write $\mathfrak{v} \nvDash \varphi$ to mean "not $\mathfrak{v} \vDash \varphi$.") pl:syn:val: defn:satisfaction

1. $\varphi \equiv \bot$: $\mathfrak{v} \nvDash \varphi$.
2. $\varphi \equiv \top$: $\mathfrak{v} \vDash \varphi$.
3. $\varphi \equiv p_i$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v}(p_i) = \mathbb{T}$.
4. $\varphi \equiv \neg\psi$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \nvDash \psi$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \vDash \psi$ and $\mathfrak{v} \vDash \chi$.
6. $\varphi \equiv (\psi \vee \chi)$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \vDash \psi$ or $\mathfrak{v} \vDash \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \nvDash \psi$ or $\mathfrak{v} \vDash \chi$ (or both).
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{v} \vDash \varphi$ iff either both $\mathfrak{v} \vDash \psi$ and $\mathfrak{v} \vDash \chi$, or neither $\mathfrak{v} \vDash \psi$ nor $\mathfrak{v} \vDash \chi$.

If Γ is a set of formulas, $\mathfrak{v} \vDash \Gamma$ iff $\mathfrak{v} \vDash \varphi$ for every $\varphi \in \Gamma$.

Proposition syn.5. $\mathfrak{v} \vDash \varphi$ *iff* $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$. pl:syn:val: prop:sat-value

Proof. By induction on φ. □

Problem syn.2. Prove Proposition syn.5

Photo Credits

Bibliography