

Chapter udf

Syntax and Semantics

This is a very quick summary of definitions only. It should be expanded to provide a gentle intro to proofs by induction on formulas, with lots more examples.

syn.1 Introduction

pl:syn:int:
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Propositional logic deals with **formulas** that are built from **propositional variables** using the propositional connectives \neg , \wedge , \vee , \rightarrow , and \leftrightarrow . Intuitively, a **propositional variable** p stands for a sentence or proposition that is true or false. Whenever the “truth value” of the **propositional variable** in a **formula** are determined, so is the truth value of any **formulas** formed from them using propositional connectives. We say that propositional logic is *truth functional*, because its semantics is given by functions of truth values. In particular, in propositional logic we leave out of consideration any further determination of truth and falsity, e.g., whether something is necessarily true rather than just contingently true, or whether something is known to be true, or whether something is true now rather than was true or will be true. We only consider two truth values true (\mathbb{T}) and false (\mathbb{F}), and so exclude from discussion the possibility that a statement may be neither true nor false, or only half true. We also concentrate only on connectives where the truth value of a **formula** built from them is completely determined by the truth values of its parts (and not, say, on its meaning). In particular, whether the truth value of conditionals in English is truth functional in this sense is contentious. The material conditional \rightarrow is; other logics deal with conditionals that are not truth functional.

In order to develop the theory and metatheory of truth-functional propositional logic, we must first define the syntax and semantics of its expressions. We will describe one way of constructing **formulas** from **propositional variables** using the connectives. Alternative definitions are possible. Other systems will chose different symbols, will select different sets of connectives as primitive, will use parentheses differently (or even not at all, as in the case of so-called Polish

notation). What all approaches have in common, though, is that the formation rules define the set of **formulas** *inductively*. If done properly, every expression can result essentially in only one way according to the formation rules. The inductive definition resulting in expressions that are *uniquely readable* means we can give meanings to these expressions using the same method—inductive definition.

Giving the meaning of expressions is the domain of semantics. The central concept in semantics for propositional logic is that of satisfaction in a **valuation**. A **valuation** \mathbf{v} assigns truth values \mathbb{T} , \mathbb{F} to the **propositional variables**. Any **valuation** determines a truth value $\bar{\mathbf{v}}(\varphi)$ for any **formula** φ . A **formula** is satisfied in a **valuation** \mathbf{v} iff $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ —we write this as $\mathbf{v} \models \varphi$. This relation can also be defined by induction on the structure of φ , using the truth functions for the logical connectives to define, say, satisfaction of $\varphi \wedge \psi$ in terms of satisfaction (or not) of φ and ψ .

On the basis of the satisfaction relation $\mathbf{v} \models \varphi$ for sentences we can then define the basic semantic notions of tautology, entailment, and satisfiability. A **formula** is a tautology, $\models \varphi$, if every **valuation** satisfies it, i.e., $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ for any \mathbf{v} . It is entailed by a set of **formulas**, $\Gamma \models \varphi$, if every **valuation** that satisfies all the **formulas** in Γ also satisfies φ . And a set of **formulas** is satisfiable if some **valuation** satisfies all **formulas** in it at the same time. Because **formulas** are inductively defined, and satisfaction is in turn defined by induction on the structure of **formulas**, we can use induction to prove properties of our semantics and to relate the semantic notions defined.

syn.2 Propositional Formulas

Formulas of propositional logic are built up from *propositional variables*, the propositional constant \perp and the propositional constant \top using *logical connectives*. pl:syn:fml:
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1. A denumerable set At_0 of **propositional variables** p_0, p_1, \dots
2. The propositional constant for **falsity** \perp .
3. The propositional constant for **truth** \top .
4. The logical connectives: \neg (negation), \wedge (conjunction), \vee (disjunction), \rightarrow (**conditional**), \leftrightarrow (**biconditional**)
5. Punctuation marks: $(,)$, and the comma.

intro You may be familiar with different terminology and symbols than the ones we use above. Logic texts (and teachers) commonly use either \sim , \neg , and $!$ for “negation”, \wedge , \cdot , and $\&$ for “conjunction”. Commonly used symbols for the “conditional” or “implication” are \rightarrow , \Rightarrow , and \supset . Symbols for “biconditional,” “bi-implication,” or “(material) equivalence” are \leftrightarrow , \Leftrightarrow , and \equiv . The \perp symbol is variously called “falsity,” “falsum,” “absurdity,” or “bottom.” The \top symbol is variously called “truth,” “verum,” or “top.”

pl:syn:fml:
defn:formulas

Definition syn.1 (Formula). The set $\text{Frm}(\mathcal{L}_0)$ of *formulas* of propositional logic is defined inductively as follows:

1. \perp is an atomic *formula*.
2. \top is an atomic *formula*.
3. Every *propositional variable* p_i is an atomic *formula*.
4. If φ is a *formula*, then $\neg\varphi$ is *formula*.
5. If φ and ψ are *formulas*, then $(\varphi \wedge \psi)$ is a *formula*.
6. If φ and ψ are *formulas*, then $(\varphi \vee \psi)$ is a *formula*.
7. If φ and ψ are *formulas*, then $(\varphi \rightarrow \psi)$ is a *formula*.
8. If φ and ψ are *formulas*, then $(\varphi \leftrightarrow \psi)$ is a *formula*.
9. If φ is a *formula* and x is a *variable*, then $\forall x \varphi$ is a *formula*.
10. If φ is a *formula* and x is a *variable*, then $\exists x \varphi$ is a *formula*.
11. Nothing else is a *formula*.

The definitions of the set of terms and that of *formulas* are *inductive definitions*. Essentially, we construct the set of *formulas* in infinitely many stages. In the initial stage, we pronounce all atomic formulas to be formulas; this corresponds to the first few cases of the definition, i.e., the cases for \top , \perp , p_i . “Atomic *formula*” thus means any *formula* of this form. explanation

The other cases of the definition give rules for constructing new *formulas* out of *formulas* already constructed. At the second stage, we can use them to construct *formulas* out of atomic *formulas*. At the third stage, we construct new formulas from the atomic formulas and those obtained in the second stage, and so on. A *formula* is anything that is eventually constructed at such a stage, and nothing else.

Definition syn.2 (Syntactic identity). The symbol \equiv expresses syntactic identity between strings of symbols, i.e., $\varphi \equiv \psi$ iff φ and ψ are strings of symbols of the same length and which contain the same symbol in each place.

The \equiv symbol may be flanked by strings obtained by concatenation, e.g., $\varphi \equiv (\psi \vee \chi)$ means: the string of symbols φ is the same string as the one obtained by concatenating an opening parenthesis, the string ψ , the \vee symbol, the string χ , and a closing parenthesis, in this order. If this is the case, then we know that the first symbol of φ is an opening parenthesis, φ contains ψ as a substring (starting at the second symbol), that substring is followed by \vee , etc.

syn.3 Preliminaries

Theorem syn.3 (*Principle of induction on formulas*). If some property P holds for all the atomic *formulas* and is such that

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pl:syn:pre:
thm:induction

1. it holds for $\neg\varphi$ whenever it holds for φ ;
2. it holds for $(\varphi \wedge \psi)$ whenever it holds for φ and ψ ;
3. it holds for $(\varphi \vee \psi)$ whenever it holds for φ and ψ ;
4. it holds for $(\varphi \rightarrow \psi)$ whenever it holds for φ and ψ ;
5. it holds for $(\varphi \leftrightarrow \psi)$ whenever it holds for φ and ψ ;

then P holds for all *formulas*.

Proof. Let S be the collection of all *formulas* with property P . Clearly $S \subseteq \text{Frm}(\mathcal{L}_0)$. S satisfies all the conditions of [Definition syn.1](#): it contains all atomic *formulas* and is closed under the *logical operators*. $\text{Frm}(\mathcal{L}_0)$ is the smallest such class, so $\text{Frm}(\mathcal{L}_0) \subseteq S$. So $\text{Frm}(\mathcal{L}_0) = S$, and every formula has property P . \square

Proposition syn.4. Any *formula* in $\text{Frm}(\mathcal{L}_0)$ is balanced, in that it has as many left parentheses as right ones.

pl:syn:pre:
prop:balanced

Problem syn.1. Prove [Proposition syn.4](#)

Proposition syn.5. No proper initial segment of a *formula* is a *formula*.

pl:syn:pre:
prop:noinit

Problem syn.2. Prove [Proposition syn.5](#)

Proposition syn.6 (Unique Readability). Any *formula* φ in $\text{Frm}(\mathcal{L}_0)$ has exactly one parsing as one of the following

1. \perp .
2. \top .
3. p_n for some $p_n \in \text{At}_0$.
4. $\neg\psi$ for some *formula* ψ .
5. $(\psi \wedge \chi)$ for some *formulas* ψ and χ .
6. $(\psi \vee \chi)$ for some *formulas* ψ and χ .
7. $(\psi \rightarrow \chi)$ for some *formulas* ψ and χ .
8. $(\psi \leftrightarrow \chi)$ for some *formulas* ψ and χ .

Moreover, this parsing is unique.

Proof. By induction on φ . For instance, suppose that φ has two distinct readings as $(\psi \rightarrow \chi)$ and $(\psi' \rightarrow \chi')$. Then ψ and ψ' must be the same (or else one would be a proper initial segment of the other); so if the two readings of φ are distinct it must be because χ and χ' are distinct readings of the same sequence of symbols, which is impossible by the inductive hypothesis. \square

Definition syn.7 (Uniform Substitution). If φ and ψ are **formulas**, and p_i is a propositional **variable**, then $\varphi[\psi/p_i]$ denotes the result of replacing each occurrence of p_i by an occurrence of ψ in φ ; similarly, the simultaneous substitution of p_1, \dots, p_n by **formulas** ψ_1, \dots, ψ_n is denoted by $\varphi[\psi_1/p_1, \dots, \psi_n/p_n]$.

Problem syn.3. Give a mathematically rigorous definition of $\varphi[\psi/p]$ by induction.

syn.4 Valuations and Satisfaction

pl:syn:val:
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Definition syn.8 (**Valuations**). Let $\{\mathbb{T}, \mathbb{F}\}$ be the set of the two truth values, “true” and “false.” A **valuation** for \mathcal{L}_0 is a function \mathbf{v} assigning either \mathbb{T} or \mathbb{F} to the **propositional variables** of the language, i.e., $\mathbf{v}: \text{At}_0 \rightarrow \{\mathbb{T}, \mathbb{F}\}$.

Definition syn.9. Given a **valuation** v , define the evaluation function $\bar{\mathbf{v}}(\cdot) : \text{Frm}(\mathcal{L}_0) \rightarrow \{\mathbb{T}, \mathbb{F}\}$ inductively by:

$$\begin{aligned} \bar{\mathbf{v}}(\perp) &= \mathbb{F}; \\ \bar{\mathbf{v}}(\top) &= \mathbb{T}; \\ \bar{\mathbf{v}}(p_n) &= \mathbf{v}(p_n); \\ \bar{\mathbf{v}}(\neg\varphi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F}; \\ \mathbb{F} & \text{otherwise.} \end{cases} \\ \bar{\mathbf{v}}(\varphi \wedge \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{T} \text{ and } \bar{\mathbf{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F} \text{ or } \bar{\mathbf{v}}(\psi) = \mathbb{F}. \end{cases} \\ \bar{\mathbf{v}}(\varphi \vee \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{T} \text{ or } \bar{\mathbf{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F} \text{ and } \bar{\mathbf{v}}(\psi) = \mathbb{F}. \end{cases} \\ \bar{\mathbf{v}}(\varphi \rightarrow \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F} \text{ or } \bar{\mathbf{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{T} \text{ and } \bar{\mathbf{v}}(\psi) = \mathbb{F}. \end{cases} \\ \bar{\mathbf{v}}(\varphi \leftrightarrow \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \bar{\mathbf{v}}(\psi); \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) \neq \bar{\mathbf{v}}(\psi). \end{cases} \end{aligned}$$

The **valuation** clauses correspond to the following truth tables:

[explanation](#)

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}
\mathbb{T}	\mathbb{F}	\mathbb{F}	\mathbb{T}	\mathbb{F}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{F}
\mathbb{F}	\mathbb{F}	\mathbb{F}	\mathbb{F}	\mathbb{T}	\mathbb{T}

Theorem syn.10 (Local Determination). Suppose that \mathbf{v}_1 and \mathbf{v}_2 are *valuations* that agree on the propositional letters occurring in φ , i.e., $\overline{\mathbf{v}_1}(p_n) = \overline{\mathbf{v}_2}(p_n)$ whenever p_n occurs in φ . Then they also agree on any φ , i.e., $\overline{\mathbf{v}_1}(\varphi) = \overline{\mathbf{v}_2}(\varphi)$. pl:syn:val:
thm:LocalDetermination

Proof. By induction on φ . □

Definition syn.11 (Satisfaction). Using the evaluation function, we can define the notion of *satisfaction of a formula φ by a valuation \mathbf{v}* , $\mathbf{v} \models \varphi$, inductively as follows. (We write $\mathbf{v} \not\models \varphi$ to mean “not $\mathbf{v} \models \varphi$.”) pl:syn:val:
defn:satisfaction

1. $\varphi \equiv \perp$: $\mathbf{v} \not\models \varphi$.
2. $\varphi \equiv \top$: $\mathbf{v} \models \varphi$.
3. $\varphi \equiv p_i$: $\mathfrak{M} \models \varphi$ iff $\overline{\mathbf{v}}(p_i) = \mathbb{T}$.
4. $\varphi \equiv \neg\psi$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \not\models \psi$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \models \psi$ and $\mathbf{v} \models \chi$.
6. $\varphi \equiv (\psi \vee \chi)$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \models \psi$ or $\mathbf{v} \models \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \not\models \psi$ or $\mathbf{v} \models \chi$ (or both).
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathbf{v} \models \varphi$ iff either both $\mathbf{v} \models \psi$ and $\mathbf{v} \models \chi$, or neither $\mathbf{v} \models \psi$ nor $\mathbf{v} \models \chi$.

If Γ is a set of *formulas*, $\mathbf{v} \models \Gamma$ iff $\mathbf{v} \models \varphi$ for every $\varphi \in \Gamma$.

Proposition syn.12. $\mathbf{v} \models \varphi$ iff $\overline{\mathbf{v}}(\varphi) = \mathbb{T}$. pl:syn:val:
prop:sat-value

Proof. By induction on φ . □

Problem syn.4. Prove [Proposition syn.12](#)

syn.5 Semantic Notions

We define the following semantic notions: pl:syn:sem:
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Definition syn.13. 1. A *formula* φ is *satisfiable* if for some \mathbf{v} , $\mathbf{v} \models \varphi$; it is *unsatisfiable* if for no \mathbf{v} , $\mathbf{v} \models \varphi$;

2. A *formula* φ is a *tautology* if $\mathbf{v} \models \varphi$ for all *valuations* \mathbf{v} ;

3. A *formula* φ is *contingent* if it is satisfiable but not a tautology;

4. If Γ is a set of formulas, $\Gamma \models \varphi$ (“ Γ entails φ ”) if and only if $\mathfrak{v} \models \varphi$ for every valuation \mathfrak{v} for which $\mathfrak{v} \models \Gamma$.
5. If Γ is a set of formulas, Γ is *satisfiable* if there is a valuation \mathfrak{v} for which $\mathfrak{v} \models \Gamma$, and Γ is *unsatisfiable* otherwise.

pl:syn:sem:
prop:semanticalfacts

Proposition syn.14.

1. φ is a tautology if and only if $\emptyset \models \varphi$;
2. If $\Gamma \models \varphi$ and $\Gamma \models \varphi \rightarrow \psi$ then $\Gamma \models \psi$;
3. If Γ is satisfiable then every finite subset of Γ is also satisfiable;

pl:syn:sem:
def:Monotony

4. *Monotony:* if $\Gamma \subseteq \Delta$ and $\Gamma \models \varphi$ then also $\Delta \models \varphi$;

pl:syn:sem:
def:Cut

5. *Transitivity:* if $\Gamma \models \varphi$ and $\Delta \cup \{\varphi\} \models \psi$ then $\Gamma \cup \Delta \models \psi$;

Proof. Exercise. □

Problem syn.5. Prove Proposition syn.14

pl:syn:sem:
prop:entails-unsat

Proposition syn.15. $\Gamma \models \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable;

Proof. Exercise. □

Problem syn.6. Prove Proposition syn.15

pl:syn:sem:
thm:sem-deduction

Theorem syn.16 (Semantic Deduction Theorem). $\Gamma \models \varphi \rightarrow \psi$ if and only if $\Gamma \cup \{\varphi\} \models \psi$.

Proof. Exercise. □

Problem syn.7. Prove Theorem syn.16

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Bibliography