Theorem syn.1 (Principle of induction on formulas). If some property $P$ holds for all the atomic formulas and is such that

1. it holds for $\neg \varphi$ whenever it holds for $\varphi$;
2. it holds for $(\varphi \land \psi)$ whenever it holds for $\varphi$ and $\psi$;
3. it holds for $(\varphi \lor \psi)$ whenever it holds for $\varphi$ and $\psi$;
4. it holds for $(\varphi \rightarrow \psi)$ whenever it holds for $\varphi$ and $\psi$;
5. it holds for $(\varphi \leftrightarrow \psi)$ whenever it holds for $\varphi$ and $\psi$;

then $P$ holds for all formulas.

Proof. Let $S$ be the collection of all formulas with property $P$. Clearly $S \subseteq \text{Frm}(L_0)$. $S$ satisfies all the conditions of ??; it contains all atomic formulas and is closed under the logical operators. $\text{Frm}(L_0)$ is the smallest such class, so $\text{Frm}(L_0) \subseteq S$. So $\text{Frm}(L_0) = S$, and every formula has property $P$. $\square$

Proposition syn.2. Any formula in $\text{Frm}(L_0)$ is balanced, in that it has as many left parentheses as right ones.

Problem syn.1. Prove Proposition syn.2

Proposition syn.3. No proper initial segment of a formula is a formula.

Problem syn.2. Prove Proposition syn.3

Proposition syn.4 (Unique Readability). Any formula $\varphi$ in $\text{Frm}(L_0)$ has exactly one parsing as one of the following

1. $\bot$.
2. $\top$.
3. $p_n$ for some $p_n \in \text{At}_0$.
4. $\neg \psi$ for some formula $\psi$.
5. $(\psi \land \chi)$ for some formulas $\psi$ and $\chi$.
6. $(\psi \lor \chi)$ for some formulas $\psi$ and $\chi$.
7. $(\psi \rightarrow \chi)$ for some formulas $\psi$ and $\chi$.
8. $(\psi \leftrightarrow \chi)$ for some formulas $\psi$ and $\chi$.

Moreover, this parsing is unique.
Proof. By induction on $\varphi$. For instance, suppose that $\varphi$ has two distinct readings as $(\psi \to \chi)$ and $(\psi' \to \chi')$. Then $\psi$ and $\psi'$ must be the same (or else one would be a proper initial segment of the other); so if the two readings of $\varphi$ are distinct it must be because $\chi$ and $\chi'$ are distinct readings of the same sequence of symbols, which is impossible by the inductive hypothesis.

Definition syn.5 (Uniform Substitution). If $\varphi$ and $\psi$ are formulas, and $p_i$ is a propositional variable, then $\varphi[\psi/p_i]$ denotes the result of replacing each occurrence of $p_i$ by an occurrence of $\psi$ in $\varphi$; similarly, the simultaneous substitution of $p_1, \ldots, p_n$ by formulas $\psi_1, \ldots, \psi_n$ is denoted by $\varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$.

Problem syn.3. Give a mathematically rigorous definition of $\varphi[\psi/p]$ by induction.

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Bibliography