

syn.1 Preliminaries

pl:syn:pre:
sec
pl:syn:pre:
thm:induction

Theorem syn.1 (Principle of induction on formulas). *If some property P holds for all the atomic formulas and is such that*

1. *it holds for $\neg\varphi$ whenever it holds for φ ;*
2. *it holds for $(\varphi \wedge \psi)$ whenever it holds for φ and ψ ;*
3. *it holds for $(\varphi \vee \psi)$ whenever it holds for φ and ψ ;*
4. *it holds for $(\varphi \rightarrow \psi)$ whenever it holds for φ and ψ ;*
5. *it holds for $(\varphi \leftrightarrow \psi)$ whenever it holds for φ and ψ ;*

then P holds for all formulas.

Proof. Let S be the collection of all formulas with property P . Clearly $S \subseteq \text{Frm}(\mathcal{L}_0)$. S satisfies all the conditions of ???: it contains all atomic formulas and is closed under the logical operators. $\text{Frm}(\mathcal{L}_0)$ is the smallest such class, so $\text{Frm}(\mathcal{L}_0) \subseteq S$. So $\text{Frm}(\mathcal{L}_0) = S$, and every formula has property P . \square

pl:syn:pre:
prop:balanced

Proposition syn.2. *Any formula in $\text{Frm}(\mathcal{L}_0)$ is balanced, in that it has as many left parentheses as right ones.*

Problem syn.1. Prove Proposition syn.2

pl:syn:pre:
prop:noinit

Proposition syn.3. *No proper initial segment of a formula is a formula.*

Problem syn.2. Prove Proposition syn.3

Proposition syn.4 (Unique Readability). *Any formula φ in $\text{Frm}(\mathcal{L}_0)$ has exactly one parsing as one of the following*

1. \perp .
2. \top .
3. p_n for some $p_n \in \text{At}_0$.
4. $\neg\psi$ for some formula ψ .
5. $(\psi \wedge \chi)$ for some formulas ψ and χ .
6. $(\psi \vee \chi)$ for some formulas ψ and χ .
7. $(\psi \rightarrow \chi)$ for some formulas ψ and χ .
8. $(\psi \leftrightarrow \chi)$ for some formulas ψ and χ .

Moreover, this parsing is unique.

Proof. By induction on φ . For instance, suppose that φ has two distinct readings as $(\psi \rightarrow \chi)$ and $(\psi' \rightarrow \chi')$. Then ψ and ψ' must be the same (or else one would be a proper initial segment of the other); so if the two readings of φ are distinct it must be because χ and χ' are distinct readings of the same sequence of symbols, which is impossible by the inductive hypothesis. \square

Definition syn.5 (Uniform Substitution). If φ and ψ are **formulas**, and p_i is a propositional **variable**, then $\varphi[\psi/p_i]$ denotes the result of replacing each occurrence of p_i by an occurrence of ψ in φ ; similarly, the simultaneous substitution of p_1, \dots, p_n by **formulas** ψ_1, \dots, ψ_n is denoted by $\varphi[\psi_1/p_1, \dots, \psi_n/p_n]$.

Problem syn.3. For each of the five **formulas** below determine whether the **formula** can be expressed as a substitution $\varphi[\psi/p_i]$ where φ is (i) p_0 ; (ii) $(\neg p_0 \wedge p_1)$; and (iii) $((\neg p_0 \rightarrow p_1) \wedge p_2)$. In each case specify the relevant substitution.

1. p_1
2. $(\neg p_0 \wedge p_0)$
3. $((p_0 \vee p_1) \wedge p_2)$
4. $\neg((p_0 \rightarrow p_1) \wedge p_2)$
5. $((\neg(p_0 \rightarrow p_1) \rightarrow (p_0 \vee p_1)) \wedge \neg(p_0 \wedge p_1))$

Problem syn.4. Give a mathematically rigorous definition of $\varphi[\psi/p]$ by induction.

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Bibliography