Theorem syn.1 (Principle of induction on formulas). If some property \( P \) holds for all the atomic formulas and is such that

1. it holds for \( \neg \phi \) whenever it holds for \( \phi \);
2. it holds for \( (\phi \land \psi) \) whenever it holds for \( \phi \) and \( \psi \);
3. it holds for \( (\phi \lor \psi) \) whenever it holds for \( \phi \) and \( \psi \);
4. it holds for \( (\phi \rightarrow \psi) \) whenever it holds for \( \phi \) and \( \psi \);
5. it holds for \( (\phi \leftrightarrow \psi) \) whenever it holds for \( \phi \) and \( \psi \);

then \( P \) holds for all formulas.

Proof. Let \( S \) be the collection of all formulas with property \( P \). Clearly \( S \subseteq \text{Frm}(L_0) \). \( S \) satisfies all the conditions of ??: it contains all atomic formulas and is closed under the logical operators. \( \text{Frm}(L_0) \) is the smallest such class, so \( \text{Frm}(L_0) \subseteq S \). So \( \text{Frm}(L_0) = S \), and every formula has property \( P \). \( \square \)

Proposition syn.2. Any formula in \( \text{Frm}(L_0) \) is balanced, in that it has as many left parentheses as right ones.

Problem syn.1. Prove Proposition syn.2

Proposition syn.3. No proper initial segment of a formula is a formula.

Problem syn.2. Prove Proposition syn.3

Proposition syn.4 (Unique Readability). Any formula \( \phi \) in \( \text{Frm}(L_0) \) has exactly one parsing as one of the following

1. \( \bot \).
2. \( \top \).
3. \( p_n \) for some \( p_n \in \text{At}_0 \).
4. \( \neg \psi \) for some formula \( \psi \).
5. \( (\psi \land \chi) \) for some formulas \( \psi \) and \( \chi \).
6. \( (\psi \lor \chi) \) for some formulas \( \psi \) and \( \chi \).
7. \( (\psi \rightarrow \chi) \) for some formulas \( \psi \) and \( \chi \).
8. \( (\psi \leftrightarrow \chi) \) for some formulas \( \psi \) and \( \chi \).

Moreover, this parsing is unique.
Proof. By induction on $\varphi$. For instance, suppose that $\varphi$ has two distinct readings as $(\psi \rightarrow \chi)$ and $(\psi' \rightarrow \chi')$. Then $\psi$ and $\psi'$ must be the same (or else one would be a proper initial segment of the other); so if the two readings of $\varphi$ are distinct it must be because $\chi$ and $\chi'$ are distinct readings of the same sequence of symbols, which is impossible by the inductive hypothesis.

Definition syn.5 (Uniform Substitution). If $\varphi$ and $\psi$ are formulas, and $p_i$ is a propositional variable, then $\varphi[\psi/p_i]$ denotes the result of replacing each occurrence of $p_i$ by an occurrence of $\psi$ in $\varphi$; similarly, the simultaneous substitution of $p_1, \ldots, p_n$ by formulas $\psi_1, \ldots, \psi_n$ is denoted by $\varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$.

Problem syn.3. For each of the five formulas below determine whether the formula can be expressed as a substitution $\varphi[\psi/p_i]$ where $\varphi$ is (i) $p_0$; (ii) $(\neg p_0 \land p_1)$; and (iii) $((\neg p_0 \rightarrow p_1) \land p_2)$. In each case specify the relevant substitution.

1. $p_1$
2. $(\neg p_0 \land p_0)$
3. $((p_0 \lor p_1) \land p_2)$
4. $\neg((p_0 \rightarrow p_1) \land p_2)$
5. $((\neg(p_0 \rightarrow p_1) \rightarrow (p_0 \lor p_1)) \land \neg(p_0 \land p_1))$

Problem syn.4. Give a mathematically rigorous definition of $\varphi[\psi/p]$ by induction.

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Bibliography