

## syn.1 Preliminaries

pl:syn:pre:  
sec  
pl:syn:pre:  
thm:induction

**Theorem syn.1** (*Principle of induction on formulas*). *If some property  $P$  holds for all the atomic formulas and is such that*

1. *it holds for  $\neg\varphi$  whenever it holds for  $\varphi$ ;*
2. *it holds for  $(\varphi \wedge \psi)$  whenever it holds for  $\varphi$  and  $\psi$ ;*
3. *it holds for  $(\varphi \vee \psi)$  whenever it holds for  $\varphi$  and  $\psi$ ;*
4. *it holds for  $(\varphi \rightarrow \psi)$  whenever it holds for  $\varphi$  and  $\psi$ ;*
5. *it holds for  $(\varphi \leftrightarrow \psi)$  whenever it holds for  $\varphi$  and  $\psi$ ;*

*then  $P$  holds for all formulas.*

*Proof.* Let  $S$  be the collection of all formulas with property  $P$ . Clearly  $S \subseteq \text{Frm}(\mathcal{L}_0)$ .  $S$  satisfies all the conditions of ???: it contains all atomic formulas and is closed under the logical operators.  $\text{Frm}(\mathcal{L}_0)$  is the smallest such class, so  $\text{Frm}(\mathcal{L}_0) \subseteq S$ . So  $\text{Frm}(\mathcal{L}_0) = S$ , and every formula has property  $P$ .  $\square$

pl:syn:pre:  
prop:balanced

**Proposition syn.2.** *Any formula in  $\text{Frm}(\mathcal{L}_0)$  is balanced, in that it has as many left parentheses as right ones.*

**Problem syn.1.** Prove Proposition syn.2

pl:syn:pre:  
prop:moinit

**Proposition syn.3.** *No proper initial segment of a formula is a formula.*

**Problem syn.2.** Prove Proposition syn.3

**Proposition syn.4** (Unique Readability). *Any formula  $\varphi$  in  $\text{Frm}(\mathcal{L}_0)$  has exactly one parsing as one of the following*

1.  $\perp$ .
2.  $\top$ .
3.  $p_n$  for some  $p_n \in \text{At}_0$ .
4.  $\neg\psi$  for some formula  $\psi$ .
5.  $(\psi \wedge \chi)$  for some formulas  $\psi$  and  $\chi$ .
6.  $(\psi \vee \chi)$  for some formulas  $\psi$  and  $\chi$ .
7.  $(\psi \rightarrow \chi)$  for some formulas  $\psi$  and  $\chi$ .
8.  $(\psi \leftrightarrow \chi)$  for some formulas  $\psi$  and  $\chi$ .

*Moreover, this parsing is unique.*

*Proof.* By induction on  $\varphi$ . For instance, suppose that  $\varphi$  has two distinct readings as  $(\psi \rightarrow \chi)$  and  $(\psi' \rightarrow \chi')$ . Then  $\psi$  and  $\psi'$  must be the same (or else one would be a proper initial segment of the other); so if the two readings of  $\varphi$  are distinct it must be because  $\chi$  and  $\chi'$  are distinct readings of the same sequence of symbols, which is impossible by the inductive hypothesis.  $\square$

**Definition syn.5** (Uniform Substitution). If  $\varphi$  and  $\psi$  are **formulas**, and  $p_i$  is a propositional **variable**, then  $\varphi[\psi/p_i]$  denotes the result of replacing each occurrence of  $p_i$  by an occurrence of  $\psi$  in  $\varphi$ ; similarly, the simultaneous substitution of  $p_1, \dots, p_n$  by **formulas**  $\psi_1, \dots, \psi_n$  is denoted by  $\varphi[\psi_1/p_1, \dots, \psi_n/p_n]$ .

**Problem syn.3.** Give a mathematically rigorous definition of  $\varphi[\psi/p]$  by induction.

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## Bibliography