syn.1 Formation Sequences

pl:syn:fseq:sec

Defining formulas via an inductive definition, and the complementary technique of proving properties of formulas via induction, is an elegant and efficient approach. However, it can also be useful to consider a more bottom-up, step-by-step approach to the construction of formulas, which we do here using the notion of a *formation sequence*.

pl:syn:fseq:defn:fseq-frm

Definition syn.1 (Formation sequences for formulas). A finite sequence $\langle \varphi_0, \dots, \varphi_n \rangle$ of strings of symbols from the language $\mathcal{L}_0$ is a *formation sequence* for φ if $\varphi \equiv \varphi_n$ and for all $i \leq n$, either φ_i is an atomic formula or there exist $j, k < i$ such that one of the following holds:

1. $\varphi_i \equiv \lnot \varphi_j$.
2. $\varphi_i \equiv (\varphi_j \land \varphi_k)$.
3. $\varphi_i \equiv (\varphi_j \lor \varphi_k)$.
4. $\varphi_i \equiv (\varphi_j \to \varphi_k)$.
5. $\varphi_i \equiv (\varphi_j \leftrightarrow \varphi_k)$.

Example syn.2.

$$\langle p_0, p_1, (p_1 \land p_0), \lnot(p_1 \land p_0) \rangle$$

is a formation sequence of $\lnot(p_1 \land p_0)$, as is

$$\langle p_0, p_1, p_0, (p_1 \land p_0), (p_0 \to p_1), \lnot(p_1 \land p_0) \rangle.$$

As can be seen from the second example, formation sequences may contain 'junk': formulas which are redundant or do not contribute to the construction.

pl:syn:fseq:prop:formed

Proposition syn.3. *Every formula φ in $\mathrm{Frm}(\mathcal{L}_0)$ has a formation sequence.*

Proof. Suppose φ is atomic. Then the sequence $\langle \varphi \rangle$ is a formation sequence for φ. Now suppose that ψ and χ have formation sequences $\langle \psi_0, \dots, \psi_n \rangle$ and $\langle \chi_0, \dots, \chi_m \rangle$ respectively.

1. If $\varphi \equiv \lnot\psi$, then $\langle \psi_0, \dots, \psi_n, \lnot\psi_n \rangle$ is a formation sequence for φ.
2. If $\varphi \equiv (\psi \land \chi)$, then $\langle \psi_0, \dots, \psi_n, \chi_0, \dots, \chi_m, (\psi_n \land \chi_m) \rangle$ is a formation sequence for φ.
3. If $\varphi \equiv (\psi \lor \chi)$, then $\langle \psi_0, \dots, \psi_n, \chi_0, \dots, \chi_m, (\psi_n \lor \chi_m) \rangle$ is a formation sequence for φ.
4. If $\varphi \equiv (\psi \to \chi)$, then $\langle \psi_0, \dots, \psi_n, \chi_0, \dots, \chi_m, (\psi_n \to \chi_m) \rangle$ is a formation sequence for φ.

5. If $\varphi \equiv (\psi \leftrightarrow \chi)$, then $\langle \psi_0, \ldots, \psi_n, \chi_0, \ldots, \chi_m, (\psi_n \leftrightarrow \chi_m) \rangle$ is a formation sequence for φ.

By the principle of induction on formulas, every formula has a formation sequence. □

We can also prove the converse. This is important because it shows that our two ways of defining formulas are equivalent: they give the same results. It also means that we can prove theorems about formulas by using ordinary induction on the length of formation sequences.

Lemma syn.4. *Suppose that $\langle \varphi_0, \ldots, \varphi_n \rangle$ is a formation sequence for φ_n, and that $k \le n$. Then $\langle \varphi_0, \ldots, \varphi_k \rangle$ is a formation sequence for φ_k.* pl:syn:fseq: lem:fseq-init

Theorem syn.5. $\mathrm{Frm}(\mathcal{L}_0)$ *is the set of all expressions (strings of symbols) in the language $\mathcal{L}_0$ with a formation sequence.* pl:syn:fseq: thm:fseq-frm-equiv

Proof. Let F be the set of all strings of symbols in the language $\mathcal{L}_0$ that have a formation sequence. We have seen in Proposition syn.3 that $\mathrm{Frm}(\mathcal{L}_0) \subseteq F$, so now we prove the converse.

Suppose φ has a formation sequence $\langle \varphi_0, \ldots, \varphi_n \rangle$. We prove that $\varphi \in \mathrm{Frm}(\mathcal{L}_0)$ by strong induction on n. Our induction hypothesis is that every string of symbols with a formation sequence of length $m < n$ is in $\mathrm{Frm}(\mathcal{L}_0)$. By the definition of a formation sequence, either φ_n is atomic or there must exist $j, k < n$ such that one of the following is the case:

1. $\varphi_i \equiv \neg\varphi_j$.
2. $\varphi_i \equiv (\varphi_j \land \varphi_k)$.
3. $\varphi_i \equiv (\varphi_j \lor \varphi_k)$.
4. $\varphi_i \equiv (\varphi_j \rightarrow \varphi_k)$.
5. $\varphi_i \equiv (\varphi_j \leftrightarrow \varphi_k)$.

Now we reason by cases. If φ_n is atomic then $\varphi_n \in \mathrm{Frm}(\mathcal{L}_0)$. Suppose instead that $\varphi \equiv (\varphi_j \land \varphi_k)$. By Lemma syn.4, $\langle \varphi_0, \ldots, \varphi_j \rangle$ and $\langle \varphi_0, \ldots, \varphi_k \rangle$ are formation sequences for φ_j and φ_k respectively. Since these are proper initial subsequences of the formation sequence for φ, they both have length less than n. Therefore by the induction hypothesis, φ_j and φ_k are in $\mathrm{Frm}(\mathcal{L}_0)$, and so by the definition of a formula, so is $(\varphi_j \land \varphi_k)$. The other cases follow by parallel reasoning. □

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Bibliography