

Part I

Propositional Logic

This part contains material on classical propositional logic. The first chapter is relatively rudimentary and just lists definitions and results, many proofs are not carried out but are left as exercises. The material on proof systems and the completeness theorem is included from the part on first-order logic, with the “FOL” tag set to false. This leaves out everything related to predicates, terms, and quantifiers, and replaces talk of **structures** \mathfrak{M} with talk about **valuations** \mathfrak{v} .

It is planned to expand this part to include more detail, and to add further topics and results, such as truth-functional completeness.

Chapter 1

Syntax and Semantics

This is a very quick summary of definitions only. It should be expanded to provide a gentle intro to proofs by induction on formulas, with lots more examples.

1.1 Introduction

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Propositional logic deals with **formulas** that are built from **propositional variables** using the propositional connectives \neg , \wedge , \vee , \rightarrow , and \leftrightarrow . Intuitively, a **propositional variable** p stands for a sentence or proposition that is true or false. Whenever the “truth value” of the **propositional variable** in a **formula** are determined, so is the truth value of any **formulas** formed from them using propositional connectives. We say that propositional logic is *truth functional*, because its semantics is given by functions of truth values. In particular, in propositional logic we leave out of consideration any further determination of truth and falsity, e.g., whether something is necessarily true rather than just contingently true, or whether something is known to be true, or whether something is true now rather than was true or will be true. We only consider two truth values true (\mathbb{T}) and false (\mathbb{F}), and so exclude from discussion the possibility that a statement may be neither true nor false, or only half true. We also concentrate only on connectives where the truth value of a **formula** built from them is completely determined by the truth values of its parts (and not, say, on its meaning). In particular, whether the truth value of conditionals in English is truth functional in this sense is contentious. The material conditional \rightarrow is; other logics deal with conditionals that are not truth functional.

In order to develop the theory and metatheory of truth-functional propositional logic, we must first define the syntax and semantics of its expressions. We will describe one way of constructing **formulas** from **propositional variables** using the connectives. Alternative definitions are possible. Other systems will chose different symbols, will select different sets of connectives as primitive, will use parentheses differently (or even not at all, as in the case of so-called Polish

notation). What all approaches have in common, though, is that the formation rules define the set of **formulas** *inductively*. If done properly, every expression can result essentially in only one way according to the formation rules. The inductive definition resulting in expressions that are *uniquely readable* means we can give meanings to these expressions using the same method—inductive definition.

Giving the meaning of expressions is the domain of semantics. The central concept in semantics for propositional logic is that of satisfaction in a **valuation**. A **valuation** \mathbf{v} assigns truth values \mathbb{T} , \mathbb{F} to the **propositional variables**. Any **valuation** determines a truth value $\bar{\mathbf{v}}(\varphi)$ for any **formula** φ . A **formula** is satisfied in a **valuation** \mathbf{v} iff $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ —we write this as $\mathbf{v} \models \varphi$. This relation can also be defined by induction on the structure of φ , using the truth functions for the logical connectives to define, say, satisfaction of $\varphi \wedge \psi$ in terms of satisfaction (or not) of φ and ψ .

On the basis of the satisfaction relation $\mathbf{v} \models \varphi$ for sentences we can then define the basic semantic notions of tautology, entailment, and satisfiability. A **formula** is a tautology, $\models \varphi$, if every **valuation** satisfies it, i.e., $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ for any \mathbf{v} . It is entailed by a set of **formulas**, $\Gamma \models \varphi$, if every **valuation** that satisfies all the **formulas** in Γ also satisfies φ . And a set of **formulas** is satisfiable if some **valuation** satisfies all **formulas** in it at the same time. Because **formulas** are inductively defined, and satisfaction is in turn defined by induction on the structure of **formulas**, we can use induction to prove properties of our semantics and to relate the semantic notions defined.

1.2 Propositional Formulas

Formulas of propositional logic are built up from *propositional variables*, the propositional constant \perp and the propositional constant \top using *logical connectives*. pl:syn:fml:
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1. A denumerable set At_0 of **propositional variables** p_0, p_1, \dots
2. The propositional constant for **falsity** \perp .
3. The propositional constant for **truth** \top .
4. The logical connectives: \neg (negation), \wedge (conjunction), \vee (disjunction), \rightarrow (**conditional**), \leftrightarrow (**biconditional**)
5. Punctuation marks: $(,)$, and the comma.

intro You may be familiar with different terminology and symbols than the ones we use above. Logic texts (and teachers) commonly use either \sim , \neg , and $!$ for “negation”, \wedge , \cdot , and $\&$ for “conjunction”. Commonly used symbols for the “conditional” or “implication” are \rightarrow , \Rightarrow , and \supset . Symbols for “biconditional,” “bi-implication,” or “(material) equivalence” are \leftrightarrow , \Leftrightarrow , and \equiv . The \perp symbol is variously called “falsity,” “falsum,” “absurdity,” or “bottom.” The \top symbol is variously called “truth,” “verum,” or “top.”

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defn:formulas

Definition 1.1 (Formula). The set $\text{Frm}(\mathcal{L}_0)$ of *formulas* of propositional logic is defined inductively as follows:

1. \perp is an atomic formula.
2. \top is an atomic formula.
3. Every propositional variable p_i is an atomic formula.
4. If φ is a formula, then $\neg\varphi$ is formula.
5. If φ and ψ are formulas, then $(\varphi \wedge \psi)$ is a formula.
6. If φ and ψ are formulas, then $(\varphi \vee \psi)$ is a formula.
7. If φ and ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula.
8. If φ and ψ are formulas, then $(\varphi \leftrightarrow \psi)$ is a formula.
9. If φ is a formula and x is a variable, then $\forall x \varphi$ is a formula.
10. If φ is a formula and x is a variable, then $\exists x \varphi$ is a formula.
11. Nothing else is a formula.

The definitions of the set of terms and that of *formulas* are *inductive definitions*. Essentially, we construct the set of *formulas* in infinitely many stages. In the initial stage, we pronounce all atomic formulas to be formulas; this corresponds to the first few cases of the definition, i.e., the cases for \top , \perp , p_i . “Atomic formula” thus means any *formula* of this form. explanation

The other cases of the definition give rules for constructing new *formulas* out of *formulas* already constructed. At the second stage, we can use them to construct *formulas* out of atomic *formulas*. At the third stage, we construct new formulas from the atomic formulas and those obtained in the second stage, and so on. A *formula* is anything that is eventually constructed at such a stage, and nothing else.

Definition 1.2 (Syntactic identity). The symbol \equiv expresses syntactic identity between strings of symbols, i.e., $\varphi \equiv \psi$ iff φ and ψ are strings of symbols of the same length and which contain the same symbol in each place.

The \equiv symbol may be flanked by strings obtained by concatenation, e.g., $\varphi \equiv (\psi \vee \chi)$ means: the string of symbols φ is the same string as the one obtained by concatenating an opening parenthesis, the string ψ , the \vee symbol, the string χ , and a closing parenthesis, in this order. If this is the case, then we know that the first symbol of φ is an opening parenthesis, φ contains ψ as a substring (starting at the second symbol), that substring is followed by \vee , etc.

1.3 Preliminaries

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Theorem 1.3. Principle of induction on *formulas*: If some property P holds of all the atomic *formulas* and is such that

pl:syn:pre:
thm:induction

1. it holds for $\neg\varphi$ whenever it holds for φ ;
2. if holds for and $(\varphi \wedge \psi)$ whenever it holds for φ and ψ ;
3. if holds for and $(\varphi \vee \psi)$ whenever it holds for φ and ψ ;
4. if holds for and $(\varphi \rightarrow \psi)$ whenever it holds for φ and ψ ;
5. if holds for and $(\varphi \leftrightarrow \psi)$ whenever it holds for φ and ψ ;

then P holds of all *formulas*.

Proof. Let S be the collection of all *formulas* with property P . Clearly $S \subseteq \text{Frm}(\mathcal{L}_0)$. S satisfies all the conditions of [Definition 1.1](#): it contains all atomic *formulas* and is closed under the *logical operators*. $\text{Frm}(\mathcal{L}_0)$ is the smallest such class, so $\text{Frm} \subseteq S$. So $\text{Frm} = S$, and every formula has property P . \square

Proposition 1.4. Any *formula* in $\text{Frm}(\mathcal{L}_0)$ is balanced, in that it has as many left parentheses as right ones.

pl:syn:pre:
prop:balanced

Problem 1.1. Prove [Proposition 1.4](#)

Proposition 1.5. No proper initial segment of a *formula* is a *formula*.

pl:syn:pre:
prop:noinit

Problem 1.2. Prove [Proposition 1.5](#)

Proposition 1.6 (Unique Readability). Any *formula* φ in $\text{Frm}(\mathcal{L}_0)$ has exactly one parsing as one of the following

1. \perp .
2. \top .
3. p_n for some $p_n \in \text{At}_0$.
4. $\neg\psi$ for some ψ in $\text{Frm}(\mathcal{L}_0)$.
5. $(\psi \wedge \chi)$ for some *formulas* ψ and χ .
6. $(\psi \vee \chi)$ for some *formulas* ψ and χ .
7. $(\psi \rightarrow \chi)$ for some *formulas* ψ and χ .
8. $(\psi \leftrightarrow \chi)$ for some *formulas* ψ and χ .

Moreover, such parsing is unique.

Proof. By induction on φ . For instance, suppose that φ has two distinct readings as $(\psi \rightarrow \chi)$ and $(\psi' \rightarrow \chi')$. Then ψ and ψ' must be the same (or else one would be a proper initial segment of the other); so if the two readings of φ are distinct it must be because χ and χ' are distinct readings of the same sequence of symbols, which is impossible by the inductive hypothesis. \square

Definition 1.7 (Uniform Substitution). If φ and ψ are **formulas**, and p_i is a propositional **variable**, then $\varphi[\psi/p_i]$ denotes the result of replacing each occurrence of p_i by an occurrence of ψ in φ ; similarly, the simultaneous substitution of p_1, \dots, p_n by **formulas** ψ_1, \dots, ψ_n is denoted by $\varphi[\psi_1/p_1, \dots, \psi_n/p_n]$.

Problem 1.3. Give a mathematically rigorous definition of $\varphi[\psi/p]$ by induction.

1.4 Valuations and Satisfaction

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Definition 1.8 (**Valuations**). Let $\{\mathbb{T}, \mathbb{F}\}$ be the set of the two truth values, “true” and “false.” A **valuation** for \mathcal{L}_0 is a function \mathbf{v} assigning either \mathbb{T} or \mathbb{F} to the **propositional variables** of the language, i.e., $\mathbf{v}: \text{At}_0 \rightarrow \{\mathbb{T}, \mathbb{F}\}$.

Definition 1.9. Given a **valuation** v , define the evaluation function $\bar{\mathbf{v}}(\cdot) : \text{Frm}(\mathcal{L}_0) \rightarrow \{\mathbb{T}, \mathbb{F}\}$ inductively by:

$$\begin{aligned} \bar{\mathbf{v}}(\perp) &= \mathbb{F}; \\ \bar{\mathbf{v}}(\top) &= \mathbb{T}; \\ \bar{\mathbf{v}}(p_n) &= \mathbf{v}(p_n); \\ \bar{\mathbf{v}}(\neg\varphi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F}; \\ \mathbb{F} & \text{otherwise.} \end{cases} \\ \bar{\mathbf{v}}(\varphi \wedge \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{T} \text{ and } \bar{\mathbf{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F} \text{ or } \bar{\mathbf{v}}(\psi) = \mathbb{F}. \end{cases} \\ \bar{\mathbf{v}}(\varphi \vee \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{T} \text{ or } \bar{\mathbf{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F} \text{ and } \bar{\mathbf{v}}(\psi) = \mathbb{F}. \end{cases} \\ \bar{\mathbf{v}}(\varphi \rightarrow \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{F} \text{ or } \bar{\mathbf{v}}(\psi) = \mathbb{T}; \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) = \mathbb{T} \text{ and } \bar{\mathbf{v}}(\psi) = \mathbb{F}. \end{cases} \\ \bar{\mathbf{v}}(\varphi \leftrightarrow \psi) &= \begin{cases} \mathbb{T} & \text{if } \bar{\mathbf{v}}(\varphi) = \bar{\mathbf{v}}(\psi); \\ \mathbb{F} & \text{if } \bar{\mathbf{v}}(\varphi) \neq \bar{\mathbf{v}}(\psi). \end{cases} \end{aligned}$$

The **valuation** clauses correspond to the following truth tables:

[explanation](#)

| φ | ψ | $\varphi \wedge \psi$ | $\varphi \vee \psi$ | $\varphi \rightarrow \psi$ | $\varphi \leftrightarrow \psi$ |
|--------------|--------------|-----------------------|---------------------|----------------------------|--------------------------------|
| \mathbb{T} | \mathbb{T} | \mathbb{T} | \mathbb{T} | \mathbb{T} | \mathbb{T} |
| \mathbb{T} | \mathbb{F} | \mathbb{F} | \mathbb{T} | \mathbb{F} | \mathbb{F} |
| \mathbb{F} | \mathbb{T} | \mathbb{F} | \mathbb{T} | \mathbb{T} | \mathbb{F} |
| \mathbb{F} | \mathbb{F} | \mathbb{F} | \mathbb{F} | \mathbb{T} | \mathbb{T} |

Theorem 1.10 (Local Determination). Suppose that \mathbf{v}_1 and \mathbf{v}_2 are *valuations* that agree on the propositional letters occurring in φ , i.e., $\overline{\mathbf{v}_1}(p_n) = \overline{\mathbf{v}_2}(p_n)$ whenever p_n occurs in φ . Then they also agree on any φ , i.e., $\overline{\mathbf{v}_1}(\varphi) = \overline{\mathbf{v}_2}(\varphi)$.

pl:syn:val:
thm:LocalDetermination

Proof. By induction on φ . □

Definition 1.11 (Satisfaction). Using the evaluation function, we can define the notion of *satisfaction of a formula φ by a valuation \mathbf{v}* , $\mathbf{v} \models \varphi$, inductively as follows. (We write $\mathbf{v} \not\models \varphi$ to mean “not $\mathbf{v} \models \varphi$.”)

pl:syn:val:
defn:satisfaction

1. $\varphi \equiv \perp$: $\mathbf{v} \not\models \varphi$.
2. $\varphi \equiv \top$: $\mathbf{v} \models \varphi$.
3. $\varphi \equiv p_i$: $\mathfrak{M} \models \varphi$ iff $\overline{\mathbf{v}}(p_i) = \mathbb{T}$.
4. $\varphi \equiv \neg\psi$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \not\models \psi$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \models \psi$ and $\mathbf{v} \models \chi$.
6. $\varphi \equiv (\psi \vee \chi)$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \models \psi$ or $\mathbf{v} \models \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathbf{v} \models \varphi$ iff $\mathbf{v} \not\models \psi$ or $\mathbf{v} \models \chi$ (or both).
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathbf{v} \models \varphi$ iff either both $\mathbf{v} \models \psi$ and $\mathbf{v} \models \chi$, or neither $\mathbf{v} \models \psi$ nor $\mathbf{v} \models \chi$.

If Γ is a set of *formulas*, $\mathbf{v} \models \Gamma$ iff $\mathbf{v} \models \varphi$ for every $\varphi \in \Gamma$.

Proposition 1.12. $\mathbf{v} \models \varphi$ iff $\overline{\mathbf{v}}(\varphi) = \mathbb{T}$.

pl:syn:val:
prop:sat-value

Proof. By induction on φ . □

Problem 1.4. Prove Proposition 1.12

1.5 Semantic Notions

We define the following semantic notions:

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Definition 1.13. 1. A *formula* φ is *satisfiable* if for some \mathbf{v} , $\mathbf{v} \models \varphi$; it is *unsatisfiable* if for no \mathbf{v} , $\mathbf{v} \models \varphi$;

2. A *formula* φ is a *tautology* if $\mathbf{v} \models \varphi$ for all *valuations* \mathbf{v} ;

3. A *formula* φ is *contingent* if it is satisfiable but not a tautology;

4. If Γ is a set of formulas, $\Gamma \vDash \varphi$ (“ Γ entails φ ”) if and only if $\mathfrak{v} \vDash \varphi$ for every valuation \mathfrak{v} for which $\mathfrak{v} \vDash \Gamma$.
5. If Γ is a set of formulas, Γ is *satisfiable* if there is a valuation \mathfrak{v} for which $\mathfrak{v} \vDash \Gamma$, and Γ is *unsatisfiable* otherwise.

pl:syn:sem:
prop:semanticalfacts

Proposition 1.14.

1. φ is a tautology if and only if $\emptyset \vDash \varphi$;
2. If $\Gamma \vDash \varphi$ and $\Gamma \vDash \varphi \rightarrow \psi$ then $\Gamma \vDash \psi$;
3. If Γ is satisfiable then every finite subset of Γ is also satisfiable;
4. *Monotony:* if $\Gamma \subseteq \Delta$ and $\Gamma \vDash \varphi$ then also $\Delta \vDash \varphi$;
5. *Transitivity:* if $\Gamma \vDash \varphi$ and $\Delta \cup \{\varphi\} \vDash \psi$ then $\Gamma \cup \Delta \vDash \psi$;

pl:syn:sem:
def:Monotony

pl:syn:sem:
def:Cut

Proof. Exercise. □

Problem 1.5. Prove Proposition 1.14

pl:syn:sem:
prop:entails-unsat

Proposition 1.15. $\Gamma \vDash \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable;

Proof. Exercise. □

Problem 1.6. Prove Proposition 1.15

pl:syn:sem:
thm:sem-deduction

Theorem 1.16 (Semantic Deduction Theorem). $\Gamma \vDash \varphi \rightarrow \psi$ if and only if $\Gamma \cup \{\varphi\} \vDash \psi$.

Proof. Exercise. □

Problem 1.7. Prove Theorem 1.16

Chapter 2

Derivation Systems

This chapter collects general material on **derivation** systems. A text-book using a specific system can insert the introduction section plus the relevant survey section at the beginning of the chapter introducing that system.

2.1 Introduction

Logics commonly have both a semantics and a **derivation** system. The semantics concerns concepts such as truth, satisfiability, validity, and entailment. The purpose of **derivation** systems is to provide a purely syntactic method of establishing entailment and validity. They are purely syntactic in the sense that a **derivation** in such a system is a finite syntactic object, usually a sequence (or other finite arrangement) of **sentences** or **formulas**. Good **derivation** systems have the property that any given sequence or arrangement of **sentences** or **formulas** can be verified mechanically to be “correct.”

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The simplest (and historically first) **derivation** systems for first-order logic were *axiomatic*. A sequence of **formulas** counts as a **derivation** in such a system if each individual **formula** in it is either among a fixed set of “axioms” or follows from **formulas** coming before it in the sequence by one of a fixed number of “inference rules”—and it can be mechanically verified if a **formula** is an axiom and whether it follows correctly from other **formulas** by one of the inference rules. Axiomatic proof systems are easy to describe—and also easy to handle meta-theoretically—but **derivations** in them are hard to read and understand, and are also hard to produce.

Other **derivation** systems have been developed with the aim of making it easier to construct **derivations** or easier to understand **derivations** once they are complete. Examples are natural deduction, truth trees, also known as tableaux proofs, and the sequent calculus. Some **derivation** systems are designed especially with mechanization in mind, e.g., the resolution method is easy to implement in software (but its **derivations** are essentially impossible to

understand). Most of these other proof systems represent **derivations** as trees of **formulas** rather than sequences. This makes it easier to see which parts of a **derivation** depend on which other parts.

So for a given logic, such as first-order logic, the different **derivation** systems will give different explications of what it is for a **sentence** to be a *theorem* and what it means for a **sentence** to be **derivable** from some others. However that is done (via axiomatic **derivations**, natural deductions, sequent **derivations**, truth trees, resolution refutations), we want these relations to match the semantic notions of validity and entailment. Let's write $\vdash \varphi$ for “ φ is a theorem” and “ $\Gamma \vdash \varphi$ ” for “ φ is **derivable** from Γ .” However \vdash is defined, we want it to match up with \models , that is:

1. $\vdash \varphi$ if and only if $\models \varphi$
2. $\Gamma \vdash \varphi$ if and only if $\Gamma \models \varphi$

The “only if” direction of the above is called *soundness*. A **derivation** system is sound if **derivability** guarantees entailment (or validity). Every decent **derivation** system has to be sound; unsound **derivation** systems are not useful at all. After all, the entire purpose of a **derivation** is to provide a syntactic guarantee of validity or entailment. We'll prove soundness for the **derivation** systems we present.

The converse “if” direction is also important: it is called *completeness*. A complete **derivation** system is strong enough to show that φ is a theorem whenever φ is valid, and that there $\Gamma \vdash \varphi$ whenever $\Gamma \models \varphi$. Completeness is harder to establish, and some logics have no complete **derivation** systems. First-order logic does. Kurt Gödel was the first one to prove completeness for a **derivation** system of first-order logic in his 1929 dissertation.

Another concept that is connected to **derivation** systems is that of *consistency*. A set of **sentences** is called inconsistent if anything whatsoever can be **derived** from it, and consistent otherwise. Inconsistency is the syntactic counterpart to unsatisfiability: like unsatisfiable sets, inconsistent sets of **sentences** do not make good theories, they are defective in a fundamental way. Consistent sets of **sentences** may not be true or useful, but at least they pass that minimal threshold of logical usefulness. For different **derivation** systems the specific definition of consistency of sets of **sentences** might differ, but like \vdash , we want consistency to coincide with its semantic counterpart, satisfiability. We want it to always be the case that Γ is consistent if and only if it is satisfiable. Here, the “if” direction amounts to completeness (consistency guarantees satisfiability), and the “only if” direction amounts to soundness (satisfiability guarantees consistency). In fact, for classical first-order logic, the two versions of soundness and completeness are equivalent.

2.2 The Sequent Calculus

While many **derivation** systems operate with arrangements of **sentences**, the sequent calculus operates with *sequents*. A sequent is an expression of the form

$$\varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_n,$$

that is a pair of sequences of **sentences**, separated by the sequent symbol \Rightarrow . Either sequence may be empty. A **derivation** in the sequent calculus is a tree of sequents, where the topmost sequents are of a special form (they are called “initial sequents” or “axioms”) and every other sequent follows from the sequents immediately above it by one of the rules of inference. The rules of inference either manipulate the **sentences** in the sequents (adding, removing, or rearranging them on either the left or the right), or they introduce a complex **formula** in the conclusion of the rule. For instance, the $\wedge L$ rule allows the inference from $\varphi, \Gamma \Rightarrow \Delta$ to $A \wedge \psi, \Gamma \Rightarrow \Delta$, and the $\rightarrow R$ allows the inference from $\varphi, \Gamma \Rightarrow \Delta, \psi$ to $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$, for any Γ, Δ, φ , and ψ . (In particular, Γ and Δ may be empty.)

The \vdash relation based on the sequent calculus is defined as follows: $\Gamma \vdash \varphi$ iff there is some sequence Γ_0 such that every φ in Γ_0 is in Γ and there is a **derivation** with the sequent $\Gamma_0 \Rightarrow \varphi$ at its root. φ is a theorem in the sequent calculus if the sequent $\Rightarrow \varphi$ has a **derivation**. For instance, here is a **derivation** that shows that $\vdash (\varphi \wedge \psi) \rightarrow \varphi$:

$$\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi} \wedge L}{\Rightarrow (\varphi \wedge \psi) \rightarrow \varphi} \rightarrow R$$

A set Γ is inconsistent in the sequent calculus if there is a **derivation** of $\Gamma_0 \Rightarrow$ (where every $\varphi \in \Gamma_0$ is in Γ and the right side of the sequent is empty). Using the rule WR, any **sentence** can be **derived** from an inconsistent set.

The sequent calculus was invented in the 1930s by Gerhard Gentzen. Because of its systematic and symmetric design, it is a very useful formalism for developing a theory of **derivations**. It is relatively easy to find **derivations** in the sequent calculus, but these **derivations** are often hard to read and their connection to proofs are sometimes not easy to see. It has proved to be a very elegant approach to **derivation** systems, however, and many logics have sequent calculus systems.

2.3 Natural Deduction

Natural deduction is a **derivation** system intended to mirror actual reasoning (especially the kind of regimented reasoning employed by mathematicians). Actual reasoning proceeds by a number of “natural” patterns. For instance, proof by cases allows us to establish a conclusion on the basis of a disjunctive premise, by establishing that the conclusion follows from either of the disjuncts. Indirect proof allows us to establish a conclusion by showing that its negation leads to a contradiction. Conditional proof establishes a conditional claim “if ... then ...” by showing that the consequent follows from the antecedent.

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Natural deduction is a formalization of some of these natural inferences. Each of the logical connectives and quantifiers comes with two rules, an introduction and an elimination rule, and they each correspond to one such natural inference pattern. For instance, \rightarrow Intro corresponds to conditional proof, and \vee Elim to proof by cases. A particularly simple rule is \wedge Elim which allows the inference from $\varphi \wedge \psi$ to φ (or ψ).

One feature that distinguishes natural deduction from other **derivation** systems is its use of assumptions. A **derivation** in natural deduction is a tree of **formulas**. A single **formula** stands at the root of the tree of **formulas**, and the “leaves” of the tree are **formulas** from which the conclusion is derived. In natural deduction, some leaf **formulas** play a role inside the **derivation** but are “used up” by the time the **derivation** reaches the conclusion. This corresponds to the practice, in actual reasoning, of introducing hypotheses which only remain in effect for a short while. For instance, in a proof by cases, we assume the truth of each of the disjuncts; in conditional proof, we assume the truth of the antecedent; in indirect proof, we assume the truth of the negation of the conclusion. This way of introducing hypothetical assumptions and then doing away with them in the service of establishing an intermediate step is a hallmark of natural deduction. The formulas at the leaves of a natural deduction **derivation** are called assumptions, and some of the rules of inference may “**discharge**” them. For instance, if we have a **derivation** of ψ from some assumptions which include φ , then the \rightarrow Intro rule allows us to infer $\varphi \rightarrow \psi$ and discharge any assumption of the form φ . (To keep track of which assumptions are discharged at which inferences, we label the inference and the assumptions it discharges with a number.) The assumptions that remain **undischarged** at the end of the **derivation** are together sufficient for the truth of the conclusion, and so a **derivation** establishes that its **undischarged** assumptions entail its conclusion.

The relation $\Gamma \vdash \varphi$ based on natural deduction holds iff there is a **derivation** in which φ is the last **sentence** in the tree, and every leaf which is **undischarged** is in Γ . φ is a theorem in natural deduction iff there is a **derivation** in which φ is the last **sentence** and all assumptions are **discharged**. For instance, here is a **derivation** that shows that $\vdash (\varphi \wedge \psi) \rightarrow \varphi$:

$$1 \frac{\frac{[\varphi \wedge \psi]^1}{\varphi} \wedge\text{Elim}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow\text{Intro}$$

The label 1 indicates that the assumption $\varphi \wedge \psi$ is **discharged** at the \rightarrow Intro inference.

A set Γ is inconsistent iff $\Gamma \vdash \perp$ in natural deduction. The rule \perp_I makes it so that from an inconsistent set, any **sentence** can be **derived**.

Natural deduction systems were developed by Gerhard Gentzen and Stanisław Jaśkowski in the 1930s, and later developed by Dag Prawitz and Frederic Fitch. Because its inferences mirror natural methods of proof, it is favored by philosophers. The versions developed by Fitch are often used in introductory

logic textbooks. In the philosophy of logic, the rules of natural deduction have sometimes been taken to give the meanings of the logical operators (“proof-theoretic semantics”).

2.4 Tableaux

While many **derivation** systems operate with arrangements of **sentences**, **tableaux** operate with **signed formulas**. A **signed formula** is a pair consisting of a truth value sign (\mathbb{T} or \mathbb{F}) and a **sentence** pl:prftab:
sec

$$\mathbb{T}\varphi \text{ or } \mathbb{F}\varphi.$$

A **tableau** consists of **signed formulas** arranged in a downward-branching tree. It begins with a number of *assumptions* and continues with **signed formulas** which result from one of the **signed formulas** above it by applying one of the rules of inference. Each rule allows us to add one or more **signed formulas** to the end of a branch, or two **signed formulas** side by side—in this case a branch splits into two, with the two added **signed formulas** forming the ends of the two branches.

A rule applied to a complex **signed formula** results in the addition of **signed formulas** which are immediate sub-formulas. They come in pairs, one rule for each of the two signs. For instance, the $\wedge\mathbb{T}$ rule applies to $\mathbb{T}\varphi \wedge \psi$, and allows the addition of both the two **signed formulas** $\mathbb{T}\varphi$ and $\mathbb{T}\psi$ to the end of any branch containing $\mathbb{T}\varphi \wedge \psi$, and the rule $\varphi \wedge \psi\mathbb{F}$ allows a branch to be split by adding $\mathbb{F}\varphi$ and $\mathbb{F}\psi$ side-by-side. A **tableau** is closed if every one of its branches contains a matching pair of **signed formulas** $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$.

The \vdash relation based on **tableaux** is defined as follows: $\Gamma \vdash \varphi$ iff there is some finite set $\Gamma_0 = \{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ such that there is a closed **tableau** for the assumptions

$$\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$$

For instance, here is a closed **tableau** that shows that $\vdash (\varphi \wedge \psi) \rightarrow \varphi$:

| | | |
|----|---|--------------------------|
| 1. | $\mathbb{F}(\varphi \wedge \psi) \rightarrow \varphi$ | Assumption |
| 2. | $\mathbb{T}\varphi \wedge \psi$ | $\rightarrow\mathbb{F}1$ |
| 3. | $\mathbb{F}\varphi$ | $\rightarrow\mathbb{F}1$ |
| 4. | $\mathbb{T}\varphi$ | $\rightarrow\mathbb{T}2$ |
| 5. | $\mathbb{T}\psi$ | $\rightarrow\mathbb{T}2$ |
| | \otimes | |

A set Γ is inconsistent in the **tableau** calculus if there is a closed **tableau** for assumptions

$$\{\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$$

for some $\psi_i \in \Gamma$.

The sequent calculus was invented in the 1950s independently by Evert Beth and Jaakko Hintikka, and simplified and popularized by Raymond Smullyan. It

is very easy to use, since constructing a **tableau** is a very systematic procedure. Because of the systematic nature of **tableaux**, they also lend themselves to implementation by computer. However, **tableau** is often hard to read and their connection to proofs are sometimes not easy to see. The approach is also quite general, and many different logics have **tableau** systems. **Tableaux** also help us to find **structures** that satisfy given (sets of) **sentences**: if the set is satisfiable, it won't have a closed **tableau**, i.e., any **tableau** will have an open branch. The satisfying **structure** can be “read off” an open branch, provided all rules it is possible to apply have been applied on that branch. There is also a very close connection to the sequent calculus: essentially, a closed **tableau** is a condensed **derivation** in the sequent calculus, written upside-down.

2.5 Axiomatic Derivations

pl:prf:axd:
sec

Axiomatic **derivations** are the oldest and simplest logical **derivation** systems. Its **derivations** are simply sequences of **sentences**. A sequence of **sentences** counts as a correct **derivation** if every **sentence** φ in it satisfies one of the following conditions:

1. φ is an axiom, or
2. φ is an **element** of a given set Γ of **sentences**, or
3. φ is justified by a rule of inference.

To be an axiom, φ has to have the form of one of a number of fixed **sentence** schemas. There are many sets of axiom schemas that provide a satisfactory (sound and complete) **derivation** system for first-order logic. Some are organized according to the connectives they govern, e.g., the schemas

$$\varphi \rightarrow (\psi \rightarrow \varphi) \quad \psi \rightarrow (\psi \vee \chi) \quad (\psi \wedge \chi) \rightarrow \psi$$

are common axioms that govern \rightarrow , \vee and \wedge . Some axiom systems aim at a minimal number of axioms. Depending on the connectives that are taken as primitives, it is even possible to find axiom systems that consist of a single axiom.

A rule of inference is a conditional statement that gives a sufficient condition for a **sentence** in a **derivation** to be justified. Modus ponens is one very common such rule: it says that if φ and $\varphi \rightarrow \psi$ are already justified, then ψ is justified. This means that a line in a **derivation** containing the **sentence** ψ is justified, provided that both φ and $\varphi \rightarrow \psi$ (for some **sentence** φ) appear in the **derivation** before ψ .

The \vdash relation based on axiomatic **derivations** is defined as follows: $\Gamma \vdash \varphi$ iff there is a **derivation** with the **sentence** φ as its last formula (and Γ is taken as the set of **sentences** in that derivation which are justified by (2) above). φ is a theorem if φ has a **derivation** where Γ is empty, i.e., every **sentence** in the derivation is justified either by (1) or (3). For instance, here is a **derivation** that shows that $\vdash \varphi \rightarrow (\psi \rightarrow (\psi \vee \varphi))$:

1. $\psi \rightarrow (\psi \vee \varphi)$
2. $(\psi \rightarrow (\psi \vee \varphi)) \rightarrow (\varphi \rightarrow (\psi \rightarrow (\psi \vee \varphi)))$
3. $\varphi \rightarrow (\psi \rightarrow (\psi \vee \varphi))$

The **sentence** on line 1 is of the form of the axiom $\varphi \rightarrow (\varphi \vee \psi)$ (with the roles of φ and ψ reversed). The sentence on line 2 is of the form of the axiom $\varphi \rightarrow (\psi \rightarrow \varphi)$. Thus, both lines are justified. Line 3 is justified by modus ponens: if we abbreviate it as θ , then line 2 has the form $\chi \rightarrow \theta$, where χ is $\psi \rightarrow (\psi \vee \varphi)$, i.e., line 1.

A set Γ is inconsistent if $\Gamma \vdash \perp$. A complete axiom system will also prove that $\perp \rightarrow \varphi$ for any φ , and so if Γ is inconsistent, then $\Gamma \vdash \varphi$ for any φ .

Systems of axiomatic **derivations** for logic were first given by Gottlob Frege in his 1879 *Begriffsschrift*, which for this reason is often considered the first work of modern logic. They were perfected in Alfred North Whitehead and Bertrand Russell's *Principia Mathematica* and by David Hilbert and his students in the 1920s. They are thus often called “Frege systems” or “Hilbert systems.” They are very versatile in that it is often easy to find an axiomatic system for a logic. Because **derivations** have a very simple structure and only one or two inference rules, it is also relatively easy to prove things *about* them. However, they are very hard to use in practice, i.e., it is difficult to find and write proofs.

Chapter 3

The Sequent Calculus

This chapter presents Gentzen’s standard sequent calculus LK for classical first-order logic. It could use more examples and exercises. To include or exclude material relevant to the sequent calculus as a proof system, use the “prfLK” tag.

3.1 Rules and Derivations

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sec

For the following, let $\Gamma, \Delta, \Pi, \Lambda$ represent finite sequences of **sentences**.

Definition 3.1 (Sequent). A *sequent* is an expression of the form

$$\Gamma \Rightarrow \Delta$$

where Γ and Δ are finite (possibly empty) sequences of **sentences** of the language \mathcal{L} . Γ is called the *antecedent*, while Δ is the *succedent*.

The intuitive idea behind a sequent is: if all of the **sentences** in the antecedent hold, then at least one of the **sentences** in the succedent holds. That is, if $\Gamma = \langle \varphi_1, \dots, \varphi_m \rangle$ and $\Delta = \langle \psi_1, \dots, \psi_n \rangle$, then $\Gamma \Rightarrow \Delta$ holds iff

explanation

$$(\varphi_1 \wedge \dots \wedge \varphi_m) \rightarrow (\psi_1 \vee \dots \vee \psi_n)$$

holds. There are two special cases: where Γ is empty and when Δ is empty. When Γ is empty, i.e., $m = 0$, $\Rightarrow \Delta$ holds iff $\psi_1 \vee \dots \vee \psi_n$ holds. When Δ is empty, i.e., $n = 0$, $\Gamma \Rightarrow$ holds iff $\neg(\varphi_1 \wedge \dots \wedge \varphi_m)$ does. We say a sequent is valid iff the corresponding **sentence** is valid.

If Γ is a sequence of **sentences**, we write Γ, φ for the result of appending φ to the right end of Γ (and φ, Γ for the result of appending φ to the left end of Γ). If Δ is a sequence of **sentences** also, then Γ, Δ is the concatenation of the two sequences.

Definition 3.2 (Initial Sequent). An *initial sequent* is a sequent of one of the following forms:

1. $\varphi \Rightarrow \varphi$
2. $\Rightarrow \top$
3. $\perp \Rightarrow$

for any **sentence** φ in the language.

Derivations in the sequent calculus are certain trees of sequents, where the topmost sequents are initial sequents, and if a sequent stands below one or two other sequents, it must follow correctly by a rule of inference. The rules for **LK** are divided into two main types: *logical* rules and *structural* rules. The logical rules are named for the **main operator** of the **sentence** containing φ and/or ψ in the lower sequent. Each one comes in two versions, one for inferring a sequent with the **sentence** containing the **logical operator** on the left, and one with the **sentence** on the right.

3.2 Propositional Rules

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sec

Rules for \neg

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \neg\text{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi} \neg\text{R}$$

Rules for \wedge

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge\text{R}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\text{L}$$

Rules for \vee

$$\frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \vee\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee\text{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee\text{R}$$

Rules for \rightarrow

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Lambda}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \rightarrow\text{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \rightarrow\text{R}$$

3.3 Structural Rules

pl:seq:srl:
sec We also need a few rules that allow us to rearrange **sentences** in the left and right side of a sequent. Since the logical rules require that the **sentences** in the premise which the rule acts upon stand either to the far left or to the far right, we need an “exchange” rule that allows us to move **sentences** to the right position. It’s also important sometimes to be able to combine two identical **sentences** into one, and to add a **sentence** on either side.

Weakening

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{WL} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \text{WR}$$

Contraction

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{CL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \text{CR}$$

Exchange

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \text{XL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda} \text{XR}$$

A series of weakening, contraction, and exchange inferences will often be indicated by double inference lines.

The following rule, called “cut,” is not strictly speaking necessary, but makes it a lot easier to reuse and combine derivations.

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \text{Cut}$$

3.4 Derivations

explanation

We’ve said what an initial sequent looks like, and we’ve given the rules of inference. **Derivations** in the sequent calculus are inductively generated from these: each **derivation** either is an initial sequent on its own, or consists of one or two **derivations** followed by an inference.

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sec

Definition 3.3 (LK derivation). An **LK-derivation** of a sequent S is a tree of sequents satisfying the following conditions:

1. The topmost sequents of the tree are initial sequents.
2. The bottommost sequent of the tree is S .
3. Every sequent in the tree except S is a premise of a correct application of an inference rule whose conclusion stands directly below that sequent in the tree.

We then say that S is the *end-sequent* of the **derivation** and that S is *derivable in LK* (or **LK-derivable**).

Example 3.4. Every initial sequent, e.g., $\chi \Rightarrow \chi$ is a **derivation**. We can obtain a new **derivation** from this by applying, say, the WL rule,

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{WL}$$

The rule, however, is meant to be general: we can replace the φ in the rule with any **sentence**, e.g., also with θ . If the premise matches our initial sequent $\chi \Rightarrow \chi$, that means that both Γ and Δ are just χ , and the conclusion would then be $\theta, \chi \Rightarrow \chi$. So, the following is a **derivation**:

$$\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}$$

We can now apply another rule, say XL, which allows us to switch two **sentences** on the left. So, the following is also a correct **derivation**:

$$\frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \chi} \text{XL}$$

In this application of the rule, which was given as

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \text{XL}$$

both Γ and Π were empty, Δ is χ , and the roles of φ and ψ are played by θ and χ , respectively. In much the same way, we also see that

$$\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{WL}$$

is a **derivation**. Now we can take these two derivations, and combine them using $\wedge\text{R}$. That rule was

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge\text{R}$$

In our case, the premises must match the last sequents of the **derivations** ending in the premises. That means that Γ is χ, θ , Δ is empty, φ is χ and ψ is θ . So the conclusion, if the inference should be correct, is $\chi, \theta \Rightarrow \chi \wedge \theta$. Of course, we can also reverse the premises, then φ would be θ and ψ would be χ . So both of the following are correct **derivations**.

$$\frac{\frac{\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi} \text{WL}}{\chi, \theta \Rightarrow \chi} \text{XL}}{\chi, \theta \Rightarrow \chi \wedge \theta} \wedge\text{R} \quad \frac{\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta} \text{WL}}{\chi, \theta \Rightarrow \theta \wedge \chi} \wedge\text{R}$$

3.5 Examples of Derivations

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sec

Example 3.5. Give an **LK**-derivation for the sequent $\varphi \wedge \psi \Rightarrow \varphi$.

We begin by writing the desired end-sequent at the bottom of the derivation.

$$\overline{\varphi \wedge \psi \Rightarrow \varphi}$$

Next, we need to figure out what kind of inference could have a lower sequent of this form. This could be a structural rule, but it is a good idea to start by looking for a logical rule. The only logical connective occurring in the lower sequent is \wedge , so we're looking for an \wedge rule, and since the \wedge symbol occurs in the antecedent, we're looking at the $\wedge\text{L}$ rule.

$$\overline{\varphi \wedge \psi \Rightarrow \varphi} \wedge\text{L}$$

There are two options for what could have been the upper sequent of the $\wedge\text{L}$ inference: we could have an upper sequent of $\varphi \Rightarrow \varphi$, or of $\psi \Rightarrow \varphi$. Clearly, $\varphi \Rightarrow \varphi$ is an initial sequent (which is a good thing), while $\psi \Rightarrow \varphi$ is not derivable in general. We fill in the upper sequent:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi} \wedge L$$

We now have a correct **LK**-derivation of the sequent $\varphi \wedge \psi \Rightarrow \varphi$.

Example 3.6. Give an **LK**-derivation for the sequent $\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi$.

Begin by writing the desired end-sequent at the bottom of the derivation.

$$\overline{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi}$$

To find a logical rule that could give us this end-sequent, we look at the logical connectives in the end-sequent: \neg , \vee , and \rightarrow . We only care at the moment about \vee and \rightarrow because they are **main operators of sentences** in the end-sequent, while \neg is inside the scope of another connective, so we will take care of it later. Our options for logical rules for the final inference are therefore the $\vee L$ rule and the $\rightarrow R$ rule. We could pick either rule, really, but let's pick the $\rightarrow R$ rule (if for no reason other than it allows us to put off splitting into two branches). According to the form of $\rightarrow R$ inferences which can yield the lower sequent, this must look like:

$$\frac{\overline{\varphi, \neg\varphi \vee \psi \Rightarrow \psi}}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow R$$

If we move $\neg\varphi \vee \psi$ to the outside of the antecedent, we can apply the $\vee L$ rule. According to the schema, this must split into two upper sequents as follows:

$$\frac{\frac{\overline{\neg\varphi, \varphi \Rightarrow \psi} \quad \overline{\psi, \varphi \Rightarrow \psi}}{\neg\varphi \vee \psi, \varphi \Rightarrow \psi} \vee L}{\frac{\varphi, \neg\varphi \vee \psi \Rightarrow \psi}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow R} \text{XR}$$

Remember that we are trying to wind our way up to initial sequents; we seem to be pretty close! The right branch is just one weakening and one exchange away from an initial sequent and then it is done:

$$\frac{\frac{\overline{\neg\varphi, \varphi \Rightarrow \psi} \quad \frac{\frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} \text{WL}}{\psi, \varphi \Rightarrow \psi} \text{XL}}{\neg\varphi \vee \psi, \varphi \Rightarrow \psi} \vee L}{\frac{\varphi, \neg\varphi \vee \psi \Rightarrow \psi}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow R} \text{XR}$$

Now looking at the left branch, the only logical connective in any **sentence** is the \neg symbol in the antecedent **sentences**, so we're looking at an instance of the $\neg L$ rule.

$$\begin{array}{c}
\frac{\frac{\varphi \Rightarrow \psi, \varphi}{\neg\varphi, \varphi \Rightarrow \psi} \neg\text{L} \quad \frac{\frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} \text{WL} \quad \frac{\psi, \varphi \Rightarrow \psi}{\psi, \varphi \Rightarrow \psi} \text{XL}}{\frac{\neg\varphi \vee \psi, \varphi \Rightarrow \psi}{\varphi, \neg\varphi \vee \psi \Rightarrow \psi} \text{XR}} \vee\text{L} \\
\frac{\varphi, \neg\varphi \vee \psi \Rightarrow \psi}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow\text{R}
\end{array}$$

Similarly to how we finished off the right branch, we are just one weakening and one exchange away from finishing off this left branch as well.

$$\begin{array}{c}
\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi, \psi} \text{WR} \quad \frac{\varphi \Rightarrow \psi, \varphi}{\varphi, \psi \Rightarrow \psi} \text{XR} \quad \frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} \text{WL}}{\frac{\neg\varphi, \varphi \Rightarrow \psi}{\psi, \varphi \Rightarrow \psi} \neg\text{L}} \vee\text{L} \\
\frac{\frac{\neg\varphi \vee \psi, \varphi \Rightarrow \psi}{\varphi, \neg\varphi \vee \psi \Rightarrow \psi} \text{XR}}{\neg\varphi \vee \psi \Rightarrow \varphi \rightarrow \psi} \rightarrow\text{R}
\end{array}$$

Example 3.7. Give an **LK**-derivation of the sequent $\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$

Using the techniques from above, we start by writing the desired end-sequent at the bottom.

$$\overline{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)}$$

The available main connectives of **sentences** in the end-sequent are the \vee symbol and the \neg symbol. It would work to apply either the $\vee\text{L}$ or the $\neg\text{R}$ rule here, but we start with the $\neg\text{R}$ rule because it avoids splitting up into two branches for a moment:

$$\frac{\overline{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow}}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg\text{R}$$

Now we have a choice of whether to look at the $\wedge\text{L}$ or the $\vee\text{L}$ rule. Let's see what happens when we apply the $\wedge\text{L}$ rule: we have a choice to start with either the sequent $\varphi, \neg\varphi \vee \neg\psi \Rightarrow$ or the sequent $\psi, \neg\varphi \vee \neg\psi \Rightarrow$. Since the proof is symmetric with regards to φ and ψ , let's go with the former:

$$\frac{\frac{\overline{\varphi, \neg\varphi \vee \neg\psi \Rightarrow}}{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow} \wedge\text{L}}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg\text{R}$$

Continuing to fill in the derivation, we see that we run into a problem:

$$\begin{array}{c}
\frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg\text{L} \quad \frac{\overline{\varphi \Rightarrow \psi} \quad ?}{\neg\psi, \varphi \Rightarrow} \neg\text{L}}{\frac{\neg\varphi \vee \neg\psi, \varphi \Rightarrow}{\varphi, \neg\varphi \vee \neg\psi \Rightarrow} \text{XL}} \vee\text{L} \\
\frac{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \wedge\text{L} \\
\frac{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg\text{R}
\end{array}$$

The top of the right branch cannot be reduced any further, and it cannot be brought by way of structural inferences to an initial sequent, so this is not the right path to take. So clearly, it was a mistake to apply the $\wedge L$ rule above. Going back to what we had before and carrying out the $\vee L$ rule instead, we get

$$\frac{\frac{\frac{\overline{\neg\varphi, \varphi \wedge \psi \Rightarrow}}{\neg\varphi \vee \neg\psi, \varphi \wedge \psi \Rightarrow} \vee L}{\frac{\overline{\neg\psi, \varphi \wedge \psi \Rightarrow}}{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow} \text{XL}}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg R$$

Completing each branch as we've done before, we get

$$\frac{\frac{\frac{\frac{\overline{\varphi \Rightarrow \varphi}}{\varphi \wedge \psi \Rightarrow \varphi} \wedge L}{\neg\varphi, \varphi \wedge \psi \Rightarrow} \neg L}{\frac{\frac{\frac{\overline{\psi \Rightarrow \psi}}{\varphi \wedge \psi \Rightarrow \psi} \wedge L}{\neg\psi, \varphi \wedge \psi \Rightarrow} \neg L} \vee L}{\frac{\frac{\overline{\neg\varphi \vee \neg\psi, \varphi \wedge \psi \Rightarrow}}{\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow} \text{XL}}{\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)} \neg R$$

(We could have carried out the \wedge rules lower than the \neg rules in these steps and still obtained a correct derivation).

Example 3.8. So far we haven't used the contraction rule, but it is sometimes required. Here's an example where that happens. Suppose we want to prove $\Rightarrow A \vee \neg\varphi$. Applying $\vee R$ backwards would give us one of these two **derivations**:

$$\frac{\overline{\Rightarrow \varphi}}{\Rightarrow \varphi \vee \neg\varphi} \vee R \qquad \frac{\frac{\overline{\varphi \Rightarrow}}{\Rightarrow \neg\varphi} \neg R}{\Rightarrow \varphi \vee \neg\varphi} \vee R$$

Neither of these of course ends in an initial sequent. The trick is to realize that the contraction rule allows us to combine two copies of a **sentence** into one—and when we're searching for a proof, i.e., going from bottom to top, we can keep a copy of $\varphi \vee \neg\varphi$ in the premise, e.g.,

$$\frac{\frac{\overline{\Rightarrow \varphi \vee \neg\varphi, \varphi}}{\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi} \vee R}{\Rightarrow \varphi \vee \neg\varphi} \text{CR}$$

Now we can apply $\vee R$ a second time, and also get $\neg\varphi$, which leads to a complete **derivation**.

$$\frac{\frac{\frac{\frac{\overline{\varphi \Rightarrow \varphi}}{\Rightarrow \varphi, \neg\varphi} \neg R}{\Rightarrow \varphi, \varphi \vee \neg\varphi} \vee R}{\Rightarrow \varphi \vee \neg\varphi, \varphi} \text{XR}}{\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi} \vee R}{\Rightarrow \varphi \vee \neg\varphi} \text{CR}$$

Problem 3.1. Give **derivations** of the following sequents:

1. $\Rightarrow \neg(\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \neg\psi)$
2. $(\varphi \wedge \psi) \rightarrow \chi \Rightarrow (\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)$

This section collects the definitions of the provability relation and consistency for natural deduction.

3.6 Proof-Theoretic Notions

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain sequents. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorem*.

Definition 3.9 (Theorems). A **sentence** φ is a *theorem* if there is a **derivation** in **LK** of the sequent $\Rightarrow \varphi$. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition 3.10 (Derivability). A **sentence** φ is *derivable* from a set of **sentences** Γ , $\Gamma \vdash \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence Γ'_0 of the **sentences** in Γ_0 such that **LK** derives $\Gamma'_0 \Rightarrow \varphi$. If φ is not **derivable** from Γ we write $\Gamma \not\vdash \varphi$.

Because of the contraction, weakening, and exchange rules, the order and number of **sentences** in Γ'_0 does not matter: if a sequent $\Gamma'_0 \Rightarrow \varphi$ is **derivable**, then so is $\Gamma''_0 \Rightarrow \varphi$ for any Γ''_0 that contains the same **sentences** as Γ'_0 . For instance, if $\Gamma_0 = \{\psi, \chi\}$ then both $\Gamma'_0 = \langle \psi, \psi, \chi \rangle$ and $\Gamma''_0 = \langle \chi, \chi, \psi \rangle$ are sequences containing just the **sentences** in Γ_0 . If a sequent containing one is **derivable**, so is the other, e.g.:

$$\begin{array}{r}
 \vdots \\
 \vdots \\
 \vdots \\
 \frac{\psi, \psi, \chi \Rightarrow \varphi}{\psi, \chi \Rightarrow \varphi} \text{CL} \\
 \frac{\psi, \chi \Rightarrow \varphi}{\chi, \psi \Rightarrow \varphi} \text{XL} \\
 \frac{\chi, \psi \Rightarrow \varphi}{\chi, \chi, \psi \Rightarrow \varphi} \text{WL}
 \end{array}$$

From now on we'll say that if Γ_0 is a finite set of **sentences** then $\Gamma_0 \Rightarrow \varphi$ is any sequent where the antecedent is a sequence of **sentences** in Γ_0 and tacitly include contractions, exchanges, and weakenings if necessary.

Definition 3.11 (Consistency). A set of sentences Γ is *inconsistent* iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow \perp$. If Γ is not inconsistent, i.e., if for every finite $\Gamma_0 \subseteq \Gamma$, **LK** does not derive $\Gamma_0 \Rightarrow \perp$, we say it is *consistent*.

Proposition 3.12 (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

*pl:seq:ptn:
prop:reflexivity*

Proof. The initial sequent $\varphi \Rightarrow \varphi$ is **derivable**, and $\{\varphi\} \subseteq \Gamma$. □

Proposition 3.13 (Monotony). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

*pl:seq:ptn:
prop:monotony*

Proof. Suppose $\Gamma \vdash \varphi$, i.e., there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \Rightarrow \varphi$ is **derivable**. Since $\Gamma \subseteq \Delta$, then Γ_0 is also a finite subset of Δ . The **derivation** of $\Gamma_0 \Rightarrow \varphi$ thus also shows $\Delta \vdash \varphi$. □

Proposition 3.14 (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

*pl:seq:ptn:
prop:transitivity*

Proof. If $\Gamma \vdash \varphi$, there is a finite $\Gamma_0 \subseteq \Gamma$ and a **derivation** π_0 of $\Gamma_0 \Rightarrow \varphi$. If $\{\varphi\} \cup \Delta \vdash \psi$, then for some finite subset $\Delta_0 \subseteq \Delta$, there is a **derivation** π_1 of $\varphi, \Delta_0 \Rightarrow \psi$. Consider the following **derivation**:

$$\frac{\begin{array}{c} \vdots \\ \vdots \pi_0 \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \pi_1 \\ \vdots \\ \varphi, \Delta_0 \Rightarrow \psi \end{array}}{\Gamma_0, \Delta_0 \Rightarrow \psi} \text{Cut}$$

Since $\Gamma_0 \cup \Delta_0 \subseteq \Gamma \cup \Delta$, this shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

Proposition 3.15. *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence φ .*

*pl:seq:ptn:
prop:incons*

Proof. Exercise. □

Problem 3.2. Prove ??

Proposition 3.16 (Compactness).

*pl:seq:ptn:
prop:proves-compact*

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*
2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Rightarrow \varphi$ has a **derivation**. Consequently, $\Gamma_0 \vdash \varphi$.

2. If Γ is inconsistent, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow \perp$. But then Γ_0 is a finite subset of Γ that is inconsistent. □

3.7 Derivability and Consistency

pl:seq:prv:
sec

We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

pl:seq:prv:
prop:provability-contr

Proposition 3.17. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. There are finite Γ_0 and $\Gamma_1 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow \varphi$ and $\varphi, \Gamma_1 \Rightarrow$. Let the **LK-derivation** of $\Gamma_0 \Rightarrow \varphi$ be π_0 and the **LK-derivation** of $\Gamma_1, \varphi \Rightarrow$ be π_1 . We can then **derive**

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \pi_0 \\ \vdots \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ \pi_1 \\ \vdots \\ \vdots \\ \varphi, \Gamma_1 \Rightarrow \end{array}}{\Gamma_0, \Gamma_1 \Rightarrow} \text{Cut}$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence Γ is inconsistent. \square

pl:seq:prv:
prop:prov-incons

Proposition 3.18. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a **derivation** π_0 of $\Gamma \Rightarrow \varphi$. By adding a $\neg\text{L}$ rule, we obtain a **derivation** of $\neg\varphi, \Gamma \Rightarrow$, i.e., $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

If $\Gamma \cup \{\neg\varphi\}$ is inconsistent, there is a **derivation** π_1 of $\neg\varphi, \Gamma \Rightarrow$. The following is a **derivation** of $\Gamma \Rightarrow \varphi$:

$$\frac{\frac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi} \neg\text{R} \quad \begin{array}{c} \vdots \\ \vdots \\ \pi_1 \\ \vdots \\ \vdots \\ \neg\varphi, \Gamma \Rightarrow \end{array}}{\Gamma \Rightarrow \varphi} \text{Cut}$$

\square

Problem 3.3. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

pl:seq:prv:
prop:explicit-inc

Proposition 3.19. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a **derivation** π of a sequent $\Gamma_0 \Rightarrow \varphi$. The sequent $\neg\varphi, \Gamma_0 \Rightarrow$ is also **derivable**:

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \pi \\ \vdots \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg\text{L}}{\varphi, \neg\varphi \Rightarrow} \text{XL}}{\Gamma, \neg\varphi \Rightarrow} \text{Cut}$$

Since $\neg\varphi \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, this shows that Γ is inconsistent. \square

Proposition 3.20. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.* pl:seq:prv:
prop:provability-exhaustive

Proof. There are finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ and **LK-derivations** π_0 and π_1 of $\varphi, \Gamma_0 \Rightarrow$ and $\neg\varphi, \Gamma_1 \Rightarrow$, respectively. We can then derive

$$\frac{\frac{\frac{\vdots \pi_0}{\varphi, \Gamma_0 \Rightarrow}}{\Gamma_0 \Rightarrow \neg\varphi} \neg\text{R} \quad \frac{\vdots \pi_1}{\neg\varphi, \Gamma_1 \Rightarrow}}{\Gamma_0, \Gamma_1 \Rightarrow} \text{Cut}$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$. Hence Γ is inconsistent. □

3.8 Derivability and the Propositional Connectives

Proposition 3.21.

1. Both $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$.
2. $\varphi, \psi \vdash \varphi \wedge \psi$.

Proof. 1. Both sequents $\varphi \wedge \psi \Rightarrow \varphi$ and $\varphi \wedge \psi \Rightarrow \psi$ are derivable:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi} \wedge\text{L} \quad \frac{\psi \Rightarrow \psi}{\varphi \wedge \psi \Rightarrow \psi} \wedge\text{L}$$

2. Here is a derivation of the sequent $\varphi, \psi \Rightarrow \varphi \wedge \psi$:

$$\frac{\varphi \Rightarrow \varphi \quad \psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \varphi \wedge \psi} \wedge\text{R}$$

□

Proposition 3.22.

1. $\varphi \vee \psi, \neg\varphi, \neg\psi$ is inconsistent.
2. Both $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$.

Proof. 1. We give a derivation of the sequent $\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow$:

$$\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg\text{L}}{\varphi, \neg\varphi, \neg\psi \Rightarrow} \quad \frac{\frac{\frac{\psi \Rightarrow \psi}{\neg\psi, \psi \Rightarrow} \neg\text{L}}{\psi, \neg\varphi, \neg\psi \Rightarrow} \neg\text{L}}{\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow} \vee\text{L}$$

pl:seq:ppr:
sec
pl:seq:ppr:
prop:provability-land
pl:seq:ppr:
prop:provability-land-left
pl:seq:ppr:
prop:provability-land-right

(Recall that double inference lines indicate several weakening, contraction, and exchange inferences.)

2. Both sequents $\varphi \Rightarrow \varphi \vee \psi$ and $\psi \Rightarrow \varphi \vee \psi$ have **derivations**:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi} \vee R \qquad \frac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \vee \psi} \vee R$$

□

Proposition 3.23.

pl:seq:ppr: prop:provability-lif

1. $\varphi, \varphi \rightarrow \psi \vdash \psi$.
2. Both $\neg\varphi \vdash \varphi \rightarrow \psi$ and $\psi \vdash \varphi \rightarrow \psi$.

pl:seq:ppr: prop:provability-lif-left

pl:seq:ppr: prop:provability-lif-right

Proof. 1. The sequent $\varphi \rightarrow \psi, \varphi \Rightarrow \psi$ is **derivable**:

$$\frac{\varphi \Rightarrow \varphi \quad \psi \Rightarrow \psi}{\varphi \rightarrow \psi, \varphi \Rightarrow \psi} \rightarrow L$$

2. Both sequents $\neg\varphi \Rightarrow \varphi \rightarrow \psi$ and $\psi \Rightarrow \varphi \rightarrow \psi$ are **derivable**:

$$\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg L}{\varphi, \neg\varphi \Rightarrow} XL}{\varphi, \neg\varphi \Rightarrow \psi} WR \qquad \frac{\frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} WL}{\psi \Rightarrow \varphi \rightarrow \psi} \rightarrow R$$

□

3.9 Soundness

pl:seq:sou: sec

A **derivation** system, such as the sequent calculus, is *sound* if it cannot **derive** things that do not actually hold. Soundness is thus a kind of guaranteed safety property for **derivation** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

explanation

1. every **derivable** φ is a tautology;
2. if a **sentence** is **derivable** from some others, it is also a consequence of them;
3. if a set of **sentences** is inconsistent, it is unsatisfiable.

These are important properties of a **derivation** system. If any of them do not hold, the **derivation** system is deficient—it would **derive** too much. Consequently, establishing the soundness of a **derivation** system is of the utmost importance.

Because all these proof-theoretic properties are defined via **derivability** in the sequent calculus of certain sequents, proving (1)–(3) above requires proving something about the semantic properties of **derivable** sequents. We will first define what it means for a sequent to be *valid*, and then show that every **derivable** sequent is valid. (1)–(3) then follow as corollaries from this result.

Definition 3.24. A **valuation** \mathbf{v} *satisfies* a sequent $\Gamma \Rightarrow \Delta$ iff either $\mathbf{v} \not\models \varphi$ for some $\varphi \in \Gamma$ or $\mathbf{v} \models \varphi$ for some $\varphi \in \Delta$.

A sequent is *valid* iff every **valuation** \mathbf{v} satisfies it.

Theorem 3.25 (Soundness). *If **LK** derives $\Theta \Rightarrow \Xi$, then $\Theta \Rightarrow \Xi$ is valid.*

*pl:seq:sou:
thm:sequent-soundness*

Proof. Let π be a **derivation** of $\Theta \Rightarrow \Xi$. We proceed by induction on the number of inferences n in π .

If the number of inferences is 0, then π consists only of an initial sequent. Every initial sequent $\varphi \Rightarrow \varphi$ is obviously valid, since for every \mathbf{v} , either $\mathbf{v} \not\models \varphi$ or $\mathbf{v} \models \varphi$.

If the number of inferences is greater than 0, we distinguish cases according to the type of the lowermost inference. By induction hypothesis, we can assume that the premises of that inference are valid, since the number of inferences in the proof of any premise is smaller than n .

First, we consider the possible inferences with only one premise.

1. The last inference is a weakening. Then $\Theta \Rightarrow \Xi$ is either $A, \Gamma \Rightarrow \Delta$ (if the last inference is WL) or $\Gamma \Rightarrow \Delta, \varphi$ (if it's WR), and the **derivation** ends in one of

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array}}{\varphi, \Gamma \Rightarrow \Delta} \text{WL} \qquad \frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array}}{\Gamma \Rightarrow \Delta, \varphi} \text{WR}$$

By induction hypothesis, $\Gamma \Rightarrow \Delta$ is valid, i.e., for every **valuation** \mathbf{v} , either there is some $\chi \in \Gamma$ such that $\mathbf{v} \not\models \chi$ or there is some $\chi \in \Delta$ such that $\mathbf{v} \models \chi$.

If $\mathbf{v} \not\models \chi$ for some $\chi \in \Gamma$, then $\chi \in \Theta$ as well since $\Theta = \varphi, \Gamma$, and so $\mathbf{v} \not\models \chi$ for some $\chi \in \Theta$. Similarly, if $\mathbf{v} \models \chi$ for some $\chi \in \Delta$, as $\chi \in \Xi$, $\mathbf{v} \models \chi$ for some $\chi \in \Xi$. Consequently, $\Theta \Rightarrow \Xi$ is valid.

2. The last inference is \neg L: Then the premise of the last inference is $\Gamma \Rightarrow \Delta, \varphi$ and the conclusion is $\neg\varphi, \Gamma \Rightarrow \Delta$, i.e., the **derivation** ends in

$$\frac{\begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array}}{\neg\varphi, \Gamma \Rightarrow \Delta} \neg\text{L}$$

and $\Theta = \neg\varphi, \Gamma$ while $\Xi = \Delta$.

The induction hypothesis tells us that $\Gamma \Rightarrow \Delta, \varphi$ is valid, i.e., for every \mathbf{v} , either (a) for some $\chi \in \Gamma$, $\mathbf{v} \not\models \chi$, or (b) for some $\chi \in \Delta$, $\mathbf{v} \models \chi$, or (c) $\mathbf{v} \models \varphi$. We want to show that $\Theta \Rightarrow \Xi$ is also valid. Let \mathbf{v} be a valuation. If (a) holds, then there is $\chi \in \Gamma$ so that $\mathbf{v} \not\models \varphi$, but $\varphi \in \Theta$ as well. If (b) holds, there is $\chi \in \Delta$ such that $\mathbf{v} \models \chi$, but $\chi \in \Xi$ as well. Finally, if $\mathbf{v} \models \varphi$, then $\mathbf{v} \not\models \neg\varphi$. Since $\neg\varphi \in \Theta$, there is $\chi \in \Theta$ such that $\mathbf{v} \not\models \chi$. Consequently, $\Theta \Rightarrow \Xi$ is valid.

3. The last inference is $\neg\text{R}$: Exercise.
4. The last inference is $\wedge\text{L}$: There are two variants: $\varphi \wedge \psi$ may be inferred on the left from φ or from ψ on the left side of the premise. In the first case, the π ends in

$$\frac{\begin{array}{c} \vdots \\ \varphi, \Gamma \Rightarrow \Delta \end{array}}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \wedge\text{L}$$

and $\Theta = \varphi \wedge \psi, \Gamma$ while $\Xi = \Delta$. Consider a valuation \mathbf{v} . Since by induction hypothesis, $\varphi, \Gamma \Rightarrow \Delta$ is valid, (a) $\mathbf{v} \not\models \varphi$, (b) $\mathbf{v} \not\models \chi$ for some $\chi \in \Gamma$, or (c) $\mathbf{v} \models \chi$ for some $\chi \in \Delta$. In case (a), $\mathbf{v} \not\models \varphi \wedge \psi$, so there is $\chi \in \Theta$ (namely, $\varphi \wedge \psi$) such that $\mathbf{v} \not\models \chi$. In case (b), there is $\chi \in \Gamma$ such that $\mathbf{v} \not\models \chi$, and $\chi \in \Theta$ as well. In case (c), there is $\chi \in \Delta$ such that $\mathbf{v} \models \chi$, and $\chi \in \Xi$ as well since $\Xi = \Delta$. So in each case, \mathbf{v} satisfies $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$. Since \mathbf{v} was arbitrary, $\Gamma \Rightarrow \Delta$ is valid. The case where $\varphi \wedge \psi$ is inferred from ψ is handled the same, changing φ to ψ .

5. The last inference is $\vee\text{R}$: There are two variants: $\varphi \vee \psi$ may be inferred on the right from φ or from ψ on the right side of the premise. In the first case, π ends in

$$\frac{\begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee\text{R}$$

Now $\Theta = \Gamma$ and $\Xi = \Delta, \varphi \vee \psi$. Consider a valuation \mathbf{v} . Since $\Gamma \Rightarrow \Delta, \varphi$ is valid, (a) $\mathbf{v} \models \varphi$, (b) $\mathbf{v} \not\models \chi$ for some $\chi \in \Gamma$, or (c) $\mathbf{v} \models \chi$ for some $\chi \in \Delta$. In case (a), $\mathbf{v} \models \varphi \vee \psi$. In case (b), there is $\chi \in \Gamma$ such that $\mathbf{v} \not\models \chi$. In case (c), there is $\chi \in \Delta$ such that $\mathbf{v} \models \chi$. So in each case, \mathbf{v} satisfies $\Gamma \Rightarrow \Delta, \varphi \vee \psi$, i.e., $\Theta \Rightarrow \Xi$. Since \mathbf{v} was arbitrary, $\Theta \Rightarrow \Xi$ is valid. The case where $\varphi \vee \psi$ is inferred from ψ is handled the same, changing φ to ψ .

6. The last inference is $\rightarrow R$: Then π ends in

$$\frac{\begin{array}{c} \vdots \\ \varphi, \Gamma \Rightarrow \Delta, \varphi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \rightarrow R$$

Again, the induction hypothesis says that the premise is valid; we want to show that the conclusion is valid as well. Let \mathbf{v} be arbitrary. Since $\varphi, \Gamma \Rightarrow \Delta, \psi$ is valid, at least one of the following cases obtains: (a) $\mathbf{v} \not\models \varphi$, (b) $\mathbf{v} \models \psi$, (c) $\mathbf{v} \not\models \chi$ for some $\chi \in \Gamma$, or (d) $\mathbf{v} \models \chi$ for some $\chi \in \Delta$. In cases (a) and (b), $\mathbf{v} \models \varphi \rightarrow \psi$ and so there is a $\chi \in \Delta, \varphi \rightarrow \psi$ such that $\mathbf{v} \models \chi$. In case (c), for some $\chi \in \Gamma$, $\mathbf{v} \not\models \chi$. In case (d), for some $\chi \in \Delta$, $\mathbf{v} \models \chi$. In each case, \mathbf{v} satisfies $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$. Since \mathbf{v} was arbitrary, $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$ is valid.

Now let's consider the possible inferences with two premises.

1. The last inference is a cut: then π ends in

$$\frac{\begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \varphi, \Pi \Rightarrow \Lambda \end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \text{Cut}$$

Let \mathbf{v} be a valuation. By induction hypothesis, the premises are valid, so \mathbf{v} satisfies both premises. We distinguish two cases: (a) $\mathbf{v} \not\models \varphi$ and (b) $\mathbf{v} \models \varphi$. In case (a), in order for \mathbf{v} to satisfy the left premise, it must satisfy $\Gamma \Rightarrow \Delta$. But then it also satisfies the conclusion. In case (b), in order for \mathbf{v} to satisfy the right premise, it must satisfy $\Pi \Rightarrow \Lambda$. Again, \mathbf{v} satisfies the conclusion.

2. The last inference is $\wedge R$. Then π ends in

$$\frac{\begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta, \psi \end{array}}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge R$$

Consider a valuation \mathbf{v} . If \mathbf{v} satisfies $\Gamma \Rightarrow \Delta$, we are done. So suppose it doesn't. Since $\Gamma \Rightarrow \Delta, \varphi$ is valid by induction hypothesis, $\mathbf{v} \models \varphi$. Similarly, since $\Gamma \Rightarrow \Delta, \psi$ is valid, $\mathbf{v} \models \psi$. But then $\mathbf{v} \models \varphi \wedge \psi$.

3. The last inference is $\forall\text{L}$: Exercise.
4. The last inference is $\rightarrow\text{L}$. Then π ends in

$$\frac{\begin{array}{c} \vdots \\ \Gamma \Rightarrow \Delta, \varphi \end{array} \quad \begin{array}{c} \vdots \\ \psi, \Pi \Rightarrow \Lambda \end{array}}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \rightarrow\text{L}$$

Again, consider a valuation \mathbf{v} and suppose \mathbf{v} doesn't satisfy $\Gamma, \Pi \Rightarrow \Lambda, \Pi$. We have to show that $\mathbf{v} \not\models \varphi \rightarrow \psi$. If \mathbf{v} doesn't satisfy $\Gamma, \Pi \Rightarrow \Lambda, \Pi$, it satisfies neither $\Gamma \Rightarrow \Delta$ nor $\Pi \Rightarrow \Lambda$. Since $\Gamma \Rightarrow \Delta, \varphi$ is valid, we have $\mathbf{v} \models \varphi$. Since $\psi, \Pi \Rightarrow \Lambda$ is valid, we have $\mathbf{v} \not\models \psi$. But then $\mathbf{v} \not\models \varphi \rightarrow \psi$, which is what we wanted to show.

□

Problem 3.4. Complete the proof of [Theorem 3.25](#).

*pl:seq:sou:
cor:weak-soundness*

Corollary 3.26. *If $\vdash \varphi$ then φ is a tautology.*

*pl:seq:sou:
cor:entailment-soundness*

Corollary 3.27. *If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*

Proof. If $\Gamma \vdash \varphi$ then for some finite subset $\Gamma_0 \subseteq \Gamma$, there is a derivation of $\Gamma_0 \Rightarrow \varphi$. By [Theorem 3.25](#), every valuation \mathbf{v} either makes some $\psi \in \Gamma_0$ false or makes φ true. Hence, if $\mathbf{v} \models \Gamma$ then also $\mathbf{v} \models \varphi$. □

*pl:seq:sou:
cor:consistency-soundness*

Corollary 3.28. *If Γ is satisfiable, then it is consistent.*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then there is a finite $\Gamma_0 \subseteq \Gamma$ and a derivation of $\Gamma_0 \Rightarrow \perp$. By [Theorem 3.25](#), $\Gamma_0 \Rightarrow \perp$ is valid. In other words, for every valuation \mathbf{v} , there is $\chi \in \Gamma_0$ so that $\mathbf{v} \not\models \chi$, and since $\Gamma_0 \subseteq \Gamma$, that χ is also in Γ . Thus, no \mathbf{v} satisfies Γ , and Γ is not satisfiable. □

Chapter 4

Natural Deduction

This chapter presents a natural deduction system in the style of Gentzen/Prawitz.

To include or exclude material relevant to natural deduction as a proof system, use the “prfND” tag.

4.1 Rules and Derivations

explanation

Natural deduction systems are meant to closely parallel the informal reasoning used in mathematical proof (hence it is somewhat “natural”). Natural deduction proofs begin with assumptions. Inference rules are then applied. Assumptions are “discharged” by the \neg Intro, \rightarrow Intro, and \forall Elim inference rules, and the label of the discharged assumption is placed beside the inference for clarity.

pl:ntd:rul:
sec

Definition 4.1 (Initial Formula). An *initial formula* or *assumption* is any formula in the topmost position of any branch.

Derivations in natural deduction are certain trees of sentences, where the topmost sentences are assumptions, and if a sentence stands below one, two, or three other sequents, it must follow correctly by a rule of inference. The sentences at the top of the inference are called the *premises* and the sentence below the *conclusion* of the inference. The rules come in pairs, an introduction and an elimination rule for each logical operator. They introduce a logical operator in the conclusion or remove a logical operator from a premise of the rule. Some of the rules allow an assumption of a certain type to be discharged. To indicate which assumption is discharged by which inference, we also assign labels to both the assumption and the inference. This is indicated by writing the assumption as “[φ]ⁿ”.

It is customary to consider rules for all logical operators, even for those (if any) that we consider as defined.

4.2 Propositional Rules

pl:ntd:prl:
sec

Rules for \wedge

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge\text{Intro} \qquad \frac{\varphi \wedge \psi}{\varphi} \wedge\text{Elim} \qquad \frac{\varphi \wedge \psi}{\psi} \wedge\text{Elim}$$

Rules for \vee

$$\frac{\varphi}{\varphi \vee \psi} \vee\text{Intro} \qquad \frac{\psi}{\varphi \vee \psi} \vee\text{Intro} \qquad \begin{array}{c} [\varphi]^n \quad [\psi]^n \\ \vdots \quad \vdots \\ \chi \quad \chi \\ \hline n \frac{\varphi \vee \psi}{\chi} \vee\text{Elim} \end{array}$$

Rules for \rightarrow

$$\begin{array}{c} [\varphi]^n \\ \vdots \\ \psi \\ \hline n \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \end{array} \qquad \frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \rightarrow\text{Elim}$$

Rules for \neg

$$\begin{array}{c} [\varphi]^n \\ \vdots \\ \perp \\ \hline n \frac{\perp}{\neg\varphi} \neg\text{Intro} \end{array} \qquad \frac{\neg\varphi \quad \varphi}{\perp} \neg\text{Elim}$$

Rules for \perp

| | |
|---------------------------------|---|
| $\frac{\perp}{\varphi} \perp_I$ | $\begin{array}{c} [\neg\varphi]^n \\ \vdots \\ n \frac{\perp}{\varphi} \perp_C \end{array}$ |
|---------------------------------|---|

Note that \neg -Intro and \perp_C are very similar: The difference is that \neg -Intro derives a negated **sentence** $\neg\varphi$ but \perp_C a positive **sentence** φ .

4.3 Derivations

explanation We've said what an assumption is, and we've given the rules of inference. pl:ntd:der:sec **Derivations** in natural deduction are inductively generated from these: each **derivation** either is an assumption on its own, or consists of one, two, or three **derivations** followed by a correct inference.

Definition 4.2 (Derivation). A *derivation* of a **sentence** φ from assumptions Γ is a tree of **sentences** satisfying the following conditions:

1. The topmost **sentences** of the tree are either in Γ or are **discharged** by an inference in the tree.
2. The bottommost **sentence** of the tree is φ .
3. Every **sentence** in the tree except φ is a premise of a correct application of an inference rule whose conclusion stands directly below that **sentence** in the tree.

We then say that φ is the *conclusion* of the **derivation** and that φ is *derivable* from Γ .

Example 4.3. Every assumption on its own is a **derivation**. So, e.g., χ by itself is a **derivation**, and so is θ by itself. We can obtain a new **derivation** from these by applying, say, the \wedge Intro rule,

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge\text{Intro}$$

These rules are meant to be general: we can replace the φ and ψ in it with any **sentences**, e.g., by χ and θ . Then the conclusion would be $\chi \wedge \theta$, and so

$$\frac{\chi \quad \theta}{\chi \wedge \theta} \wedge\text{Intro}$$

is a correct **derivation**. Of course, we can also switch the assumptions, so that θ plays the role of φ and χ that of ψ . Thus,

$$\frac{\theta \quad \chi}{\theta \wedge \chi} \wedge\text{Intro}$$

is also a correct derivation.

We can now apply another rule, say, \rightarrow Intro, which allows us to conclude a conditional and allows us to **discharge** any assumption that is identical to the conclusion of that conditional. So both of the following would be correct **derivations**:

$$1 \frac{\frac{[\chi]^1 \quad \theta}{\chi \wedge \theta} \wedge\text{Intro}}{\chi \rightarrow (\chi \wedge \theta)} \rightarrow\text{Intro} \quad 1 \frac{\frac{\chi \quad [\theta]^1}{\chi \wedge \theta} \wedge\text{Intro}}{\theta \rightarrow (\chi \wedge \theta)} \rightarrow\text{Intro}$$

4.4 Examples of Derivations

pl:ntd:pro:
sec

Example 4.4. Let's give a **derivation** of the **sentence** $(\varphi \wedge \psi) \rightarrow \varphi$.

We begin by writing the desired conclusion at the bottom of the **derivation**.

$$\overline{(\varphi \wedge \psi) \rightarrow \varphi}$$

Next, we need to figure out what kind of inference could result in a **sentence** of this form. The **main operator** of the conclusion is \rightarrow , so we'll try to arrive at the conclusion using the \rightarrow Intro rule. It is best to write down the assumptions involved and label the inference rules as you progress, so it is easy to see whether all assumptions have been **discharged** at the end of the proof.

$$1 \frac{\begin{array}{c} [\varphi \wedge \psi]^1 \\ \vdots \\ \vdots \\ \varphi \end{array}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow\text{Intro}$$

We now need to fill in the steps from the assumption $\varphi \wedge \psi$ to φ . Since we only have one connective to deal with, \wedge , we must use the \wedge elim rule. This gives us the following proof:

$$1 \frac{\frac{[\varphi \wedge \psi]^1}{\varphi} \wedge\text{Elim}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow\text{Intro}$$

We now have a correct **derivation** of $(\varphi \wedge \psi) \rightarrow \varphi$.

Example 4.5. Now let's give a **derivation** of $(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$.

We begin by writing the desired conclusion at the bottom of the derivation.

$$\overline{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)}$$

To find a logical rule that could give us this conclusion, we look at the logical connectives in the conclusion: \neg , \vee , and \rightarrow . We only care at the moment about the first occurrence of \rightarrow because it is the **main operator** of the **sentence** in the end-sequent, while \neg , \vee and the second occurrence of \rightarrow are inside the scope of another connective, so we will take care of those later. We therefore start with the \rightarrow Intro rule. A correct application must look as follows:

$$\begin{array}{c} [\neg\varphi \vee \psi]^1 \\ \vdots \\ \varphi \rightarrow \psi \\ \hline 1 \quad (\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \quad \rightarrow\text{Intro} \end{array}$$

This leaves us with two possibilities to continue. Either we can keep working from the bottom up and look for another application of the \rightarrow Intro rule, or we can work from the top down and apply a \vee Elim rule. Let us apply the latter. We will use the assumption $\neg\varphi \vee \psi$ as the leftmost premise of \vee Elim. For a valid application of \vee Elim, the other two premises must be identical to the conclusion $\varphi \rightarrow \psi$, but each may be derived in turn from another assumption, namely the two disjuncts of $\neg\varphi \vee \psi$. So our **derivation** will look like this:

$$\begin{array}{c} \begin{array}{c} [\neg\varphi]^2 \\ \vdots \\ \varphi \rightarrow \psi \end{array} \quad \begin{array}{c} [\psi]^2 \\ \vdots \\ \varphi \rightarrow \psi \end{array} \\ \hline 2 \quad \frac{[\neg\varphi \vee \psi]^1 \quad \varphi \rightarrow \psi \quad \varphi \rightarrow \psi}{\varphi \rightarrow \psi} \quad \vee\text{Elim} \\ \hline 1 \quad \frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \quad \rightarrow\text{Intro} \end{array}$$

In each of the two branches on the right, we want to **derive** $\varphi \rightarrow \psi$, which is best done using \rightarrow Intro.

$$\begin{array}{c} \begin{array}{c} [\neg\varphi]^2, [\varphi]^3 \\ \vdots \\ \psi \end{array} \quad \begin{array}{c} [\psi]^2, [\varphi]^4 \\ \vdots \\ \psi \end{array} \\ \hline 3 \quad \frac{\psi}{\varphi \rightarrow \psi} \quad \rightarrow\text{Intro} \quad 4 \quad \frac{\psi}{\varphi \rightarrow \psi} \quad \rightarrow\text{Intro} \\ \hline 2 \quad \frac{[\neg\varphi \vee \psi]^1 \quad \varphi \rightarrow \psi \quad \varphi \rightarrow \psi}{\varphi \rightarrow \psi} \quad \vee\text{Elim} \\ \hline 1 \quad \frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \quad \rightarrow\text{Intro} \end{array}$$

For the two missing parts of the **derivation**, we need **derivations** of ψ from $\neg\varphi$ and φ in the middle, and from φ and ψ on the left. Let's take the former

first. $\neg\varphi$ and φ are the two premises of \neg Elim:

$$\frac{[\neg\varphi]^2 \quad [\varphi]^3}{\perp} \neg\text{Elim}$$

$$\vdots$$

$$\psi$$

By using \perp_I , we can obtain ψ as a conclusion and complete the branch.

$$\frac{\frac{\frac{[\neg\varphi]^2 \quad [\varphi]^3}{\perp} \perp\text{Intro} \quad \frac{[\psi]^2, [\varphi]^4}{\vdots} \psi}{\frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro}} \rightarrow\text{Intro} \quad \frac{[\psi]^2, [\varphi]^4}{\psi} \rightarrow\text{Intro}}{\frac{[\neg\varphi \vee \psi]^1}{\varphi \rightarrow \psi} \vee\text{Elim}} \rightarrow\text{Intro} \quad \frac{[\psi]^2, [\varphi]^4}{\psi} \rightarrow\text{Intro}}{\frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}} \rightarrow\text{Intro}$$

Let's now look at the rightmost branch. Here it's important to realize that the definition of **derivation** allows assumptions to be discharged but does not require them to be. In other words, if we can derive ψ from one of the assumptions φ and ψ without using the other, that's ok. And to derive ψ from ψ is trivial: ψ by itself is such a **derivation**, and no inferences are needed. So we can simply delete the assumption φ .

$$\frac{\frac{\frac{[\neg\varphi]^2 \quad [\varphi]^3}{\perp} \neg\text{Elim} \quad \frac{[\psi]^2}{\psi} \rightarrow\text{Intro}}{\frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro}} \rightarrow\text{Intro} \quad \frac{[\psi]^2}{\varphi \rightarrow \psi} \rightarrow\text{Intro}}{\frac{[\neg\varphi \vee \psi]^1}{\varphi \rightarrow \psi} \vee\text{Elim}} \rightarrow\text{Intro} \quad \frac{[\psi]^2}{\psi} \rightarrow\text{Intro}}{\frac{\varphi \rightarrow \psi}{(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)} \rightarrow\text{Intro}} \rightarrow\text{Intro}$$

Note that in the finished **derivation**, the rightmost \rightarrow Intro inference does not actually discharge any assumptions.

Example 4.6. So far we have not needed the \perp_C rule. It is special in that it allows us to discharge an assumption that isn't a sub-formula of the conclusion of the rule. It is closely related to the \perp_I rule. In fact, the \perp_I rule is a special case of the \perp_C rule—there is a logic called “intuitionistic logic” in which only \perp_I is allowed. The \perp_C rule is a last resort when nothing else works. For instance, suppose we want to derive $\varphi \vee \neg\varphi$. Our usual strategy would be to attempt to derive $\varphi \vee \neg\varphi$ using \vee Intro. But this would require us to derive either φ or $\neg\varphi$ from no assumptions, and this can't be done. \perp_C to the rescue!

$$\frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots \quad \perp}{1 \quad \frac{\perp}{\varphi \vee \neg\varphi} \perp_C} \perp_C$$

Now we're looking for a **derivation** of \perp from $\neg(\varphi \vee \neg\varphi)$. Since \perp is the conclusion of \neg -Elim we might try that:

$$\frac{\frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots \quad \neg\varphi}{\neg\varphi} \quad \frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots \quad \varphi}{\varphi}}{1 \quad \frac{\perp}{\varphi \vee \neg\varphi} \perp_C} \neg\text{-Elim}$$

Our strategy for finding a **derivation** of $\neg\varphi$ calls for an application of \neg -Intro:

$$\frac{\frac{[\neg(\varphi \vee \neg\varphi)]^1, [\varphi]^2 \quad \vdots \quad \perp}{2 \quad \frac{\perp}{\neg\varphi} \neg\text{-Intro}} \quad \frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots \quad \varphi}{\varphi} \neg\text{-Elim}}{1 \quad \frac{\perp}{\varphi \vee \neg\varphi} \perp_C} \neg\text{-Elim}$$

Here, we can get \perp easily by applying \neg -Elim to the assumption $\neg(\varphi \vee \neg\varphi)$ and $\varphi \vee \neg\varphi$ which follows from our new assumption φ by \vee -Intro:

$$\frac{\frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \frac{[\varphi]^2}{\varphi \vee \neg\varphi} \vee\text{-Intro}}{\neg\text{-Elim}} \quad \frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots \quad \varphi}{\varphi} \neg\text{-Elim}}{2 \quad \frac{\perp}{\neg\varphi} \neg\text{-Intro}} \neg\text{-Elim} \quad \frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots \quad \perp}{1 \quad \frac{\perp}{\varphi \vee \neg\varphi} \perp_C} \perp_C$$

On the right side we use the same strategy, except we get φ by \perp_C :

$$\frac{\frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \frac{[\varphi]^2}{\varphi \vee \neg\varphi} \vee\text{-Intro}}{\neg\text{-Elim}} \quad \frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \frac{[\neg\varphi]^3}{\varphi \vee \neg\varphi} \vee\text{-Intro}}{\neg\text{-Elim}}}{2 \quad \frac{\perp}{\neg\varphi} \neg\text{-Intro} \quad 3 \quad \frac{\perp}{\varphi} \perp_C} \neg\text{-Elim} \quad \frac{[\neg(\varphi \vee \neg\varphi)]^1 \quad \vdots \quad \perp}{1 \quad \frac{\perp}{\varphi \vee \neg\varphi} \perp_C} \perp_C$$

Problem 4.1. Give **derivations** of the following:

1. $\neg(\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \neg\psi)$
2. $(\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)$ from the assumption $(\varphi \wedge \psi) \rightarrow \chi$

4.5 Proof-Theoretic Notions

pl:ntd:ptn:
sec

This section collects the definitions the provability relation and consistency for natural deduction.

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sentences from others. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

Definition 4.7 (Theorems). A sentence φ is a *theorem* if there is a derivation of φ in natural deduction in which all assumptions are discharged. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition 4.8 (Derivability). A sentence φ is *derivable* from a set of sentences Γ , $\Gamma \vdash \varphi$, if there is a derivation with conclusion φ and in which every assumption is either discharged or is in Γ . If φ is not derivable from Γ we write $\Gamma \not\vdash \varphi$.

Definition 4.9 (Consistency). A set of sentences Γ is *inconsistent* iff $\Gamma \vdash \perp$. If Γ is not inconsistent, i.e., if $\Gamma \not\vdash \perp$, we say it is *consistent*.

pl:ntd:ptn:
prop:reflexivity

Proposition 4.10 (Reflexivity). If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.

Proof. The assumption φ by itself is a derivation of φ where every undischarged assumption (i.e., φ) is in Γ . \square

pl:ntd:ptn:
prop:monotony

Proposition 4.11 (Monotony). If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.

Proof. Any derivation of φ from Γ is also a derivation of φ from Δ . \square

pl:ntd:ptn:
prop:transitivity

Proposition 4.12 (Transitivity). If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.

Proof. If $\Gamma \vdash \varphi$, there is a derivation δ_0 of φ with all undischarged assumptions in Γ . If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a derivation δ_1 of ψ with all undischarged assumptions in $\{\varphi\} \cup \Delta$. Now consider:

$$\begin{array}{c}
 \Delta, [\varphi]^1 \\
 \vdots \\
 \delta_1 \\
 \vdots \\
 \psi \\
 \hline
 \varphi \rightarrow \psi \xrightarrow{1} \text{Intro} \\
 \hline
 \psi
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma \\
 \vdots \\
 \delta_0 \\
 \vdots \\
 \varphi \\
 \hline
 \varphi \xrightarrow{\text{Elim}}
 \end{array}$$

The **undischarged** assumptions are now all among $\Gamma \cup \Delta$, so this shows $\Gamma \cup \Delta \vdash \psi$. \square

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

Proposition 4.13. Γ is inconsistent iff $\Gamma \vdash \varphi$ for every *sentence* φ .

*pl:ntd:ptn:
prop:incons*

Proof. Exercise. \square

Problem 4.2. Prove [Proposition 4.13](#)

Proposition 4.14 (Compactness).

*pl:ntd:ptn:
prop:proves-compact*

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a **derivation** δ of φ from Γ . Let Γ_0 be the set of **undischarged** assumptions of δ . Since any **derivation** is finite, Γ_0 can only contain finitely many **sentences**. So, δ is a **derivation** of φ from a finite $\Gamma_0 \subseteq \Gamma$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \perp$.

\square

4.6 Derivability and Consistency

We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

*pl:ntd:prv:
sec*

Proposition 4.15. If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.

*pl:ntd:prv:
prop:provability-contr*

Proof. Let the **derivation** of φ from Γ be δ_1 and the **derivation** of \perp from $\Gamma \cup \{\varphi\}$ be δ_2 . We can then **derive**:

$$\frac{\begin{array}{c} \Gamma, [\varphi]^1 \\ \vdots \\ \vdots \delta_2 \\ \vdots \\ \perp \\ \hline \neg\varphi \end{array} \text{ } \neg\text{-Intro} \quad \begin{array}{c} \Gamma \\ \vdots \\ \vdots \delta_1 \\ \vdots \\ \varphi \end{array} \text{ } \neg\text{-Elim}}{\perp}$$

In the new **derivation**, the assumption φ is **discharged**, so it is a **derivation** from Γ . \square

Proposition 4.16. $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

*pl:ntd:prv:
prop:prov-incons*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a **derivation** δ_0 of φ from **undischarged** assumptions Γ . We obtain a **derivation** of \perp from $\Gamma \cup \{\neg\varphi\}$ as follows:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \delta_0 \\ \vdots \\ \varphi \end{array}}{\frac{\neg\varphi \quad \varphi}{\perp} \neg\text{Elim}}$$

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, and let δ_1 be the corresponding derivation of \perp from **undischarged** assumptions in $\Gamma \cup \{\neg\varphi\}$. We obtain a **derivation** of φ from Γ alone by using \perp_C :

$$\frac{\begin{array}{c} \Gamma, [\neg\varphi]^1 \\ \vdots \\ \vdots \delta_1 \\ \vdots \\ \perp \end{array}}{\varphi} \perp_C$$

□

Problem 4.3. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

*pl:ntd:prv:
prop:explicit-inc*

Proposition 4.17. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a **derivation** δ of φ from Γ . Consider this simple application of the \neg -Elim rule:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \vdots \delta \\ \vdots \\ \varphi \end{array}}{\frac{\neg\varphi \quad \varphi}{\perp} \neg\text{Elim}}$$

Since $\neg\varphi \in \Gamma$, all **undischarged** assumptions are in Γ , this shows that $\Gamma \vdash \perp$. □

*pl:ntd:prv:
prop:provability-exhaustive*

Proposition 4.18. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.*

Proof. There are **derivations** δ_1 and δ_2 of \perp from $\Gamma \cup \{\varphi\}$ and \perp from $\Gamma \cup \{\neg\varphi\}$, respectively. We can then **derive**

$$\frac{\begin{array}{c} \Gamma, [\neg\varphi]^2 \\ \vdots \\ \vdots \delta_2 \\ \vdots \\ \perp \end{array} \quad \begin{array}{c} \Gamma, [\varphi]^1 \\ \vdots \\ \vdots \delta_1 \\ \vdots \\ \perp \end{array}}{\frac{2 \frac{\perp}{\neg\neg\varphi} \neg\text{Intro} \quad 1 \frac{\perp}{\neg\varphi} \neg\text{Intro}}{\perp} \neg\text{Elim}}$$

Since the assumptions φ and $\neg\varphi$ are **discharged**, this is a **derivation** of \perp from Γ alone. Hence Γ is inconsistent. \square

4.7 Derivability and the Propositional Connectives

Proposition 4.19.

1. Both $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$
2. $\varphi, \psi \vdash \varphi \wedge \psi$.

Proof. 1. We can **derive** both

$$\frac{\varphi \wedge \psi}{\varphi} \wedge\text{Elim} \qquad \frac{\varphi \wedge \psi}{\psi} \wedge\text{Elim}$$

2. We can **derive**:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge\text{Intro}$$

\square

Proposition 4.20.

1. $\varphi \vee \psi, \neg\varphi, \neg\psi$ is inconsistent.
2. Both $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$.

Proof. 1. Consider the following **derivation**:

$$\frac{\varphi \vee \psi \quad \frac{\neg\varphi \quad [\varphi]^1}{\perp} \neg\text{Elim} \quad \frac{\neg\psi \quad [\psi]^1}{\perp} \neg\text{Elim}}{\perp} \vee\text{Elim}$$

This is a **derivation** of \perp from **undischarged** assumptions $\varphi \vee \psi$, $\neg\varphi$, and $\neg\psi$.

2. We can **derive** both

$$\frac{\varphi}{\varphi \vee \psi} \vee\text{Intro} \qquad \frac{\psi}{\varphi \vee \psi} \vee\text{Intro}$$

\square

Proposition 4.21.

1. $\varphi, \varphi \rightarrow \psi \vdash \psi$.

*pl:ntd:ppr:
sec*

*pl:ntd:ppr:
prop:provability-land*

*pl:ntd:ppr:
prop:provability-land-left*

*pl:ntd:ppr:
prop:provability-land-right*

*pl:ntd:ppr:
prop:provability-lor*

*pl:ntd:ppr:
prop:provability-lif*

*pl:ntd:ppr:
prop:provability-lif-left*

pl:ntd:ppr:
prop:provability-lif-right

2. Both $\neg\varphi \vdash \varphi \rightarrow \psi$ and $\psi \vdash \varphi \rightarrow \psi$.

Proof. 1. We can derive:

$$\frac{\varphi \rightarrow \psi \quad \psi}{\psi} \rightarrow\text{Elim}$$

2. This is shown by the following two derivations:

$$\frac{\frac{\frac{\neg\varphi}{\perp} \quad \perp_I}{\psi} \rightarrow\text{Intro} \quad \frac{[\varphi]^1}{\perp} \neg\text{Elim}}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \quad \frac{\psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro}$$

Note that $\rightarrow\text{Intro}$ may, but does not have to, discharge the assumption φ . \square

4.8 Soundness

pl:ntd:sou:
sec

A derivation system, such as natural deduction, is *sound* if it cannot derive things that do not actually follow. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

1. every derivable sentence is a tautology;
2. if a sentence is derivable from some others, it is also a consequence of them;
3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

pl:ntd:sou:
thm:soundness

Theorem 4.22 (Soundness). *If φ is derivable from the undischarged assumptions Γ , then $\Gamma \models \varphi$.*

Proof. Let δ be a derivation of φ . We proceed by induction on the number of inferences in δ .

For the induction basis we show the claim if the number of inferences is 0. In this case, δ consists only of an initial formula. Every initial formula φ is an undischarged assumption, and as such, any valuation \mathfrak{v} that satisfies all of the undischarged assumptions of the proof also satisfies φ .

Now for the inductive step. Suppose that δ contains n inferences. The premise(s) of the lowermost inference are derived using sub-derivations, each of which contains fewer than n inferences. We assume the induction hypothesis:

The premises of the last inference follow from the **undischarged** assumptions of the sub-derivations ending in those premises. We have to show that φ follows from the **undischarged** assumptions of the entire proof.

We distinguish cases according to the type of the lowermost inference. First, we consider the possible inferences with only one premise.

1. Suppose that the last inference is \neg Intro: The **derivation** has the form

$$\begin{array}{c} \Gamma, [\varphi]^n \\ \vdots \\ \delta_1 \\ \vdots \\ \perp \\ \hline \neg\varphi \quad \neg\text{Intro} \end{array}$$

By inductive hypothesis, \perp follows from the **undischarged** assumptions $\Gamma \cup \{\varphi\}$ of δ_1 . Consider a **valuation** \mathbf{v} . We need to show that, if $\mathbf{v} \models \Gamma$, then $\mathbf{v} \models \neg\varphi$. Suppose for reductio that $\mathbf{v} \models \Gamma$, but $\mathbf{v} \not\models \neg\varphi$, i.e., $\mathbf{v} \models \varphi$. This would mean that $\mathbf{v} \models \Gamma \cup \{\varphi\}$. This is contrary to our inductive hypothesis. So, $\mathbf{v} \models \neg\varphi$.

2. The last inference is \wedge Elim: There are two variants: φ or ψ may be inferred from the premise $\varphi \wedge \psi$. Consider the first case. The derivation δ looks like this:

$$\begin{array}{c} \Gamma \\ \vdots \\ \delta_1 \\ \vdots \\ \varphi \wedge \psi \\ \hline \varphi \quad \wedge\text{Elim} \end{array}$$

By inductive hypothesis, $\varphi \wedge \psi$ follows from the **undischarged** assumptions Γ of δ_1 . Consider a **structure** \mathbf{v} . We need to show that, if $\mathbf{v} \models \Gamma$, then $\mathbf{v} \models \varphi$. Suppose $\mathbf{v} \models \Gamma$. By our inductive hypothesis ($\Gamma \models \varphi \wedge \psi$), we know that $\mathbf{v} \models \varphi \wedge \psi$. By definition, $\mathbf{v} \models \varphi \wedge \psi$ iff $\mathbf{v} \models \varphi$ and $\mathbf{v} \models \psi$. (The case where ψ is inferred from $\varphi \wedge \psi$ is handled similarly.)

3. The last inference is \vee Intro: There are two variants: $\varphi \vee \psi$ may be inferred from the premise φ or the premise ψ . Consider the first case. The derivation has the form

$$\begin{array}{c} \Gamma \\ \vdots \\ \delta_1 \\ \vdots \\ \varphi \\ \hline \varphi \vee \psi \quad \vee\text{Intro} \end{array}$$

By inductive hypothesis, φ follows from the **undischarged** assumptions Γ of δ_1 . Consider a **valuation** \mathbf{v} . We need to show that, if $\mathbf{v} \models \Gamma$, then $\mathbf{v} \models \varphi \vee \psi$. Suppose that $\mathbf{v} \models \Gamma$; then $\mathbf{v} \models \varphi$ since $\Gamma \models \varphi$ (the inductive hypothesis). So it must also be the case that $\mathbf{v} \models \varphi \vee \psi$. (The case where $\varphi \vee \psi$ is inferred from ψ is handled similarly.)

4. The last inference is \rightarrow Intro: $\varphi \rightarrow \psi$ is inferred from a subproof with assumption φ and conclusion ψ , i.e.,

$$\frac{\begin{array}{c} \Gamma, [\varphi]^n \\ \vdots \\ \delta_1 \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \rightarrow\text{Intro}$$

By inductive hypothesis, ψ follows from the **undischarged** assumptions of δ_1 , i.e., $\Gamma \cup \{\varphi\} \models \psi$. Consider a **valuation** \mathbf{v} . The **undischarged** assumptions of δ are just Γ , since φ is discharged at the last inference. So we need to show that $\Gamma \models \varphi \rightarrow \psi$. For reductio, suppose that for some **valuation** \mathbf{v} , $\mathbf{v} \models \Gamma$ but $\mathbf{v} \not\models \varphi \rightarrow \psi$. So, $\mathbf{v} \models \varphi$ and $\mathbf{v} \not\models \psi$. But by hypothesis, ψ is a consequence of $\Gamma \cup \{\varphi\}$, i.e., $\mathbf{v} \models \psi$, which is a contradiction. So, $\Gamma \models \varphi \rightarrow \psi$.

5. The last inference is \perp_I : Here, δ ends in

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ \delta_1 \\ \vdots \\ \perp \end{array}}{\varphi} \perp_I$$

By induction hypothesis, $\Gamma \models \perp$. We have to show that $\Gamma \models \varphi$. Suppose not; then for some \mathbf{v} we have $\mathbf{v} \models \Gamma$ and $\mathbf{v} \not\models \varphi$. But we always have $\mathbf{v} \models \perp$, so this would mean that $\Gamma \not\models \perp$, contrary to the induction hypothesis.

6. The last inference is \perp_C : Exercise.

Now let's consider the possible inferences with several premises: \vee Elim, \wedge Intro, and \rightarrow Elim.

1. The last inference is \wedge Intro. $\varphi \wedge \psi$ is inferred from the premises φ and ψ and δ has the form

$$\frac{\begin{array}{cc} \Gamma_1 & \Gamma_2 \\ \vdots & \vdots \\ \delta_1 & \delta_2 \\ \vdots & \vdots \\ \varphi & \psi \end{array}}{\varphi \wedge \psi} \wedge\text{Intro}$$

By induction hypothesis, φ follows from the **undischarged** assumptions Γ_1 of δ_1 and ψ follows from the **undischarged** assumptions Γ_2 of δ_2 . The **undischarged** assumptions of δ are $\Gamma_1 \cup \Gamma_2$, so we have to show that $\Gamma_1 \cup \Gamma_2 \models \varphi \wedge \psi$. Consider a **valuation** \mathbf{v} with $\mathbf{v} \models \Gamma_1 \cup \Gamma_2$. Since $\mathbf{v} \models \Gamma_1$, it must be the case that $\mathbf{v} \models \varphi$ as $\Gamma_1 \models \varphi$, and since $\mathbf{v} \models \Gamma_2$, $\mathbf{v} \models \psi$ since $\Gamma_2 \models \psi$. Together, $\mathbf{v} \models \varphi \wedge \psi$.

2. The last inference is \vee Elim: Exercise.
3. The last inference is \rightarrow Elim. ψ is inferred from the premises $\varphi \rightarrow \psi$ and φ . The derivation δ looks like this:

$$\frac{\begin{array}{c} \Gamma_1 \\ \vdots \\ \delta_1 \\ \vdots \\ \varphi \rightarrow \psi \end{array} \quad \begin{array}{c} \Gamma_2 \\ \vdots \\ \delta_2 \\ \vdots \\ \varphi \end{array}}{\psi} \rightarrow\text{Elim}$$

By induction hypothesis, $\varphi \rightarrow \psi$ follows from the **undischarged** assumptions Γ_1 of δ_1 and φ follows from the **undischarged** assumptions Γ_2 of δ_2 . Consider a **valuation** \mathbf{v} . We need to show that, if $\mathbf{v} \models \Gamma_1 \cup \Gamma_2$, then $\mathbf{v} \models \psi$. Suppose $\mathbf{v} \models \Gamma_1 \cup \Gamma_2$. Since $\Gamma_1 \models \varphi \rightarrow \psi$, $\mathbf{v} \models \varphi \rightarrow \psi$. Since $\Gamma_2 \models \varphi$, we have $\mathbf{v} \models \varphi$. This means that $\mathbf{v} \models \psi$ (For if $\mathbf{v} \not\models \psi$, since $\mathbf{v} \models \varphi$, we'd have $\mathbf{v} \not\models \varphi \rightarrow \psi$, contradicting $\mathbf{v} \models \varphi \rightarrow \psi$).

4. The last inference is \neg Elim: Exercise.

□

Problem 4.4. Complete the proof of [Theorem 4.22](#).

Corollary 4.23. *If $\vdash \varphi$, then φ is a tautology.*

[pl:ntd:sou:](#)
[cor:weak-soundness](#)

Corollary 4.24. *If Γ is satisfiable, then it is consistent.*

[pl:ntd:sou:](#)
[cor:consistency-soundness](#)

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \perp$, i.e., there is a **derivation** of \perp from **undischarged** assumptions in Γ . By [Theorem 4.22](#), any **valuation** \mathbf{v} that satisfies Γ must satisfy \perp . Since $\mathbf{v} \not\models \perp$ for every **valuation** \mathbf{v} , no \mathbf{v} can satisfy Γ , i.e., Γ is not satisfiable. □

Chapter 5

Tableaux

This chapter presents a signed analytic tableaux system.
To include or exclude material relevant to natural deduction as a proof system, use the “prfTab” tag.

5.1 Rules and Tableaux

pl:tab:rul:
sec A **tableau** is a systematic survey of the possible ways a **sentence** can be true or false in a **structure**. The building blocks of a tableau are **signed formulas: sentences** plus a truth value “sign,” either \mathbb{T} or \mathbb{F} . These signed **formulas** are arranged in a (downward growing) tree.

Definition 5.1. A *signed formula* is a pair consisting of a truth value and a **sentence**, i.e., either:

$$\mathbb{T}\varphi \text{ or } \mathbb{F}\varphi.$$

Intuitively, we might read $\mathbb{T}\varphi$ as “ φ might be true” and $\mathbb{F}\varphi$ as “ φ might be false” (in some **structure**).

Each **signed formula** in the tree is either an *assumption* (which are listed at the very top of the tree), or it is obtained from a **signed formula** above it by one of a number of rules of inference. There are two rules for each possible **main operator** of the preceding **formula**, one for the case when the sign is \mathbb{T} , and one for the case where the sign is \mathbb{F} . Some rules allow the tree to branch, and some only add **signed formulas** to the branch. A rule may be (and often must be) applied not to the immediately preceding **signed formula**, but to any **signed formula** in the branch from the root to the place the rule is applied.

A branch is *closed* when it contains both $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$. A closed **tableau** is one where every branch is closed. Under the intuitive interpretation, any branch describes a joint possibility, but $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$ are not jointly possible. In other words, if a branch is closed, the possibility it describes has been ruled out. In particular, that means that a closed **tableau** rules out all possibilities

of simultaneously making every assumption of the form $\mathbb{T}\varphi$ true and every assumption of the form $\mathbb{F}\varphi$ false.

A closed **tableau for φ** is a closed **tableau** with root $\mathbb{F}\varphi$. If such a closed **tableau** exists, all possibilities for φ being false have been ruled out; i.e., φ must be true in every **structure**.

5.2 Propositional Rules

pl:tab:prl:
sec

Rules for \neg

$$\frac{\mathbb{T}\neg\varphi}{\mathbb{F}\varphi} \neg\mathbb{T} \qquad \frac{\mathbb{F}\neg\varphi}{\mathbb{T}\varphi} \neg\mathbb{F}$$

Rules for \wedge

$$\frac{\mathbb{T}\varphi \wedge \psi}{\mathbb{T}\varphi \quad \mathbb{T}\psi} \wedge\mathbb{T} \qquad \frac{\mathbb{F}\varphi \wedge \psi}{\mathbb{F}\varphi \quad | \quad \mathbb{F}\psi} \wedge\mathbb{F}$$

Rules for \vee

$$\frac{\mathbb{T}\varphi \vee \psi}{\mathbb{T}\varphi \quad | \quad \mathbb{T}\psi} \vee\mathbb{T} \qquad \frac{\mathbb{F}\varphi \vee \psi}{\mathbb{F}\varphi \quad \mathbb{F}\psi} \vee\mathbb{F}$$

Rules for \rightarrow

$$\frac{\mathbb{T}\varphi \rightarrow \psi}{\mathbb{F}\varphi \quad | \quad \mathbb{T}\psi} \rightarrow\mathbb{T} \qquad \frac{\mathbb{F}\varphi \rightarrow \psi}{\mathbb{T}\varphi \quad \mathbb{F}\psi} \rightarrow\mathbb{F}$$

The Cut Rule

$$\frac{\mathbb{T}\varphi \quad | \quad \mathbb{F}\varphi}{\text{Cut}}$$

The Cut rule is not applied “to” a previous **signed formula**; rather, it allows every branch in a **tableau** to be split in two, one branch containing $\mathbb{T}\varphi$, the other $\mathbb{F}\varphi$. It is not necessary—any set of **signed formulas** with a closed **tableau** has one not using Cut—but it allows us to combine **tableaux** in a convenient way.

5.3 Tableaux

We’ve said what an assumption is, and we’ve given the rules of inference. **Tableaux** are inductively generated from these: each **tableau** either is a single branch consisting of one or more assumptions, or it results from a **tableau** by applying one of the rules of inference on a branch.

Definition 5.2 (Tableau). A **tableau** for assumptions $S_{\mathbb{T}\varphi_1}, \dots, S_{\mathbb{F}\varphi_n}$ (where each S_i is either \mathbb{T} or \mathbb{F}) is a tree of **signed formulas** satisfying the following conditions:

1. The n topmost **signed formulas** of the tree are $S_{\mathbb{T}\varphi_i}$, one below the other.
2. Every **signed formula** in the tree that is not one of the assumptions results from a correct application of an inference rule to a **signed formula** in the branch above it.

A branch of a **tableau** is *closed* iff it contains both $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$, and *open* otherwise. A **tableau** in which every branch is closed is a *closed tableau* (for its set of assumptions). If a **tableau** is not closed, i.e., if it contains at least one open branch, it is *open*.

Example 5.3. Every set of assumptions on its own is a **tableau**, but it will generally not be closed. (Obviously, it is closed only if the assumptions already contain a pair of **signed formulas** $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$.)

From a **tableau** (open or closed) we can obtain a new, larger one by applying one of the rules of inference to a **signed formula** φ in it. The rule will append one or more **signed formulas** to the end of any branch containing the occurrence of φ to which we apply the rule.

For instance, consider the assumption $\mathbb{T}\varphi \wedge \neg\varphi$. Here is the (open) **tableau** consisting of just that assumption:

1. $\mathbb{T}\varphi \wedge \neg\varphi$ Assumption

We obtain a new **tableau** from it by applying the $\wedge\mathbb{T}$ rule to the assumption. That rule allows us to add two new lines to the **tableau**, $\mathbb{T}\varphi$ and $\mathbb{T}\neg\varphi$:

- | | | |
|----|--|-----------------------|
| 1. | $\mathbb{T} \varphi \wedge \neg \varphi$ | Assumption |
| 2. | $\mathbb{T} \varphi$ | $\wedge \mathbb{T} 1$ |
| 3. | $\mathbb{T} \neg \varphi$ | $\wedge \mathbb{T} 1$ |

When we write down **tableaux**, we record the rules we've applied on the right (e.g., $\wedge \mathbb{T} 1$ means that the **signed formula** on that line is the result of applying the $\wedge \mathbb{T}$ rule to the **signed formula** on line 1). This new **tableau** now contains additional **signed formulas**, but to only one ($\mathbb{T} \neg \varphi$) can we apply a rule (in this case, the $\neg \mathbb{T}$ rule). This results in the closed **tableau**

- | | | |
|----|--|-----------------------|
| 1. | $\mathbb{T} \varphi \wedge \neg \varphi$ | Assumption |
| 2. | $\mathbb{T} \varphi$ | $\wedge \mathbb{T} 1$ |
| 3. | $\mathbb{T} \neg \varphi$ | $\wedge \mathbb{T} 1$ |
| 4. | $\mathbb{F} \varphi$ | $\neg \mathbb{T} 3$ |
| | \otimes | |

5.4 Examples of Tableaux

pl:tab:pro:
sec

Example 5.4. Let's find a closed **tableau** for the **sentence** $(\varphi \wedge \psi) \rightarrow \varphi$.

We begin by writing the corresponding assumption at the top of the **tableau**.

- | | | |
|----|--|------------|
| 1. | $\mathbb{F} (\varphi \wedge \psi) \rightarrow \varphi$ | Assumption |
|----|--|------------|

There is only one assumption, so only one **signed formula** to which we can apply a rule. (For every **signed formula**, there is always at most one rule that can be applied: it's the rule for the corresponding sign and **main operator** of the **sentence**.) In this case, this means, we must apply $\rightarrow \mathbb{F}$.

- | | | |
|----|---|----------------------------|
| 1. | $\mathbb{F} (\varphi \wedge \psi) \rightarrow \varphi \checkmark$ | Assumption |
| 2. | $\mathbb{T} \varphi \wedge \psi$ | $\rightarrow \mathbb{F} 1$ |
| 3. | $\mathbb{F} \varphi$ | $\rightarrow \mathbb{F} 1$ |

To keep track of which **signed formulas** we have applied their corresponding rules to, we write a checkmark next to the sentence. However, *only* write a checkmark if the rule has been applied to all open branches. Once a **signed formula** has had the corresponding rule applied in every open branch, we will not have to return to it and apply the rule again. In this case, there is only one branch, so the rule only has to be applied once. (Note that checkmarks are only a convenience for constructing tableaux and are not officially part of the syntax of tableaux.)

There is one new **signed formula** to which we can apply a rule: the $\mathbb{T} \varphi \wedge \psi$ on line 3. Applying the $\wedge \mathbb{T}$ rule results in:

| | | |
|----|--|---------------------------|
| 1. | $\mathbb{F}(\varphi \wedge \psi) \rightarrow \varphi \checkmark$ | Assumption |
| 2. | $\mathbb{T}\varphi \wedge \psi \checkmark$ | $\rightarrow\mathbb{F} 1$ |
| 3. | $\mathbb{F}\varphi$ | $\rightarrow\mathbb{F} 1$ |
| 4. | $\mathbb{T}\varphi$ | $\wedge\mathbb{T} 2$ |
| 5. | $\mathbb{T}\psi$ | $\wedge\mathbb{T} 2$ |
| | \otimes | |

Since the branch now contains both $\mathbb{T}\varphi$ (on line 4) and $\mathbb{F}\varphi$ (on line 3), the branch is closed. Since it is the only branch, the **tableau** is closed. We have found a closed **tableau** for $(\varphi \wedge \psi) \rightarrow \varphi$.

Example 5.5. Now let's find a closed **tableau** for $(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$.

We begin with the corresponding assumption:

| | | |
|----|--|------------|
| 1. | $\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$ | Assumption |
|----|--|------------|

The one **signed formula** in this **tableau** has **main operator** \rightarrow and sign \mathbb{F} , so we apply the $\rightarrow\mathbb{F}$ rule to it to obtain:

| | | |
|----|---|---------------------------|
| 1. | $\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$ | Assumption |
| 2. | $\mathbb{T}\neg\varphi \vee \psi$ | $\rightarrow\mathbb{F} 1$ |
| 3. | $\mathbb{F}(\varphi \rightarrow \psi)$ | $\rightarrow\mathbb{F} 1$ |

We now have a choice as to whether to apply $\vee\mathbb{T}$ to line 2 or $\rightarrow\mathbb{F}$ to line 3. It actually doesn't matter which order we pick, as long as each **signed formula** has its corresponding rule applied in every branch. So let's pick the first one. The $\vee\mathbb{T}$ rule allows the **tableau** to branch, and the two conclusions of the rule will be the new **signed formulas** added to the two new branches. This results in:

| | | |
|----|--|---------------------------|
| 1. | $\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$ | Assumption |
| 2. | $\mathbb{T}\neg\varphi \vee \psi \checkmark$ | $\rightarrow\mathbb{F} 1$ |
| 3. | $\mathbb{F}(\varphi \rightarrow \psi)$ | $\rightarrow\mathbb{F} 1$ |
| 4. | $\begin{array}{c} \diagup \quad \diagdown \\ \mathbb{T}\neg\varphi \quad \mathbb{T}\psi \end{array}$ | $\vee\mathbb{T} 2$ |

We have not applied the $\rightarrow\mathbb{F}$ rule to line 3 yet: let's do that now. To save time, we apply it to both branches. Recall that we write a checkmark next to a **signed formula** only if we have applied the corresponding rule in every open branch. So it's a good idea to apply a rule at the end of every branch that contains the **signed formula** the rule applies to. That way we won't have to return to that **signed formula** lower down in the various branches.

| | | | |
|----|---|--------------------------|--------------------------|
| 1. | $\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$ | Assumption | |
| 2. | $\mathbb{T}\neg\varphi \vee \psi \checkmark$ | $\rightarrow\mathbb{F}1$ | |
| 3. | $\mathbb{F}(\varphi \rightarrow \psi) \checkmark$ | $\rightarrow\mathbb{F}1$ | |
| | $\swarrow \quad \searrow$ | | |
| 4. | $\mathbb{T}\neg\varphi$ | $\mathbb{T}\psi$ | $\vee\mathbb{T}2$ |
| 5. | $\mathbb{T}\varphi$ | $\mathbb{T}\varphi$ | $\rightarrow\mathbb{F}3$ |
| 6. | $\mathbb{F}\psi$ | $\mathbb{F}\psi$ | $\rightarrow\mathbb{F}3$ |
| | \otimes | | |

The right branch is now closed. On the left branch, we can still apply the $\neg\mathbb{T}$ rule to line 4. This results in $\mathbb{F}\varphi$ and closes the left branch:

| | | | |
|----|---|--------------------------|--------------------------|
| 1. | $\mathbb{F}(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi) \checkmark$ | Assumption | |
| 2. | $\mathbb{T}\neg\varphi \vee \psi \checkmark$ | $\rightarrow\mathbb{F}1$ | |
| 3. | $\mathbb{F}(\varphi \rightarrow \psi) \checkmark$ | $\rightarrow\mathbb{F}1$ | |
| | $\swarrow \quad \searrow$ | | |
| 4. | $\mathbb{T}\neg\varphi$ | $\mathbb{T}\psi$ | $\vee\mathbb{T}2$ |
| 5. | $\mathbb{T}\varphi$ | $\mathbb{T}\varphi$ | $\rightarrow\mathbb{F}3$ |
| 6. | $\mathbb{F}\psi$ | $\mathbb{F}\psi$ | $\rightarrow\mathbb{F}3$ |
| 7. | $\mathbb{F}\varphi$ | \otimes | $\neg\mathbb{T}4$ |
| | \otimes | | |

Example 5.6. We can give **tableaux** for any number of **signed formulas** as assumptions. Often it is also necessary to apply more than one rule that allows branching; and in general a **tableau** can have any number of branches. For instance, consider a **tableau** for $\{\mathbb{T}\varphi \vee (\psi \wedge \chi), \mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi)\}$. We start by applying the $\vee\mathbb{T}$ to the first assumption:

| | | | |
|----|---|------------------------------|-------------------|
| 1. | $\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$ | Assumption | |
| 2. | $\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$ | Assumption | |
| | $\swarrow \quad \searrow$ | | |
| 3. | $\mathbb{T}\varphi$ | $\mathbb{T}\psi \wedge \chi$ | $\vee\mathbb{T}1$ |

Now we can apply the $\wedge\mathbb{F}$ rule to line 2. We do this on both branches simultaneously, and can therefore check off line 2:

| | | | | | |
|----|---|-------------------------------|-------------------------------|-------------------------------|---------------------|
| 1. | $\mathbb{T}\varphi \vee (\psi \wedge \chi) \checkmark$ | Assumption | | | |
| 2. | $\mathbb{F}(\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$ | Assumption | | | |
| | $\swarrow \quad \searrow$ | | | | |
| 3. | $\mathbb{T}\varphi$ | $\mathbb{T}\psi \wedge \chi$ | $\vee\mathbb{T}1$ | | |
| | $\swarrow \quad \searrow \quad \swarrow \quad \searrow$ | | | | |
| 4. | $\mathbb{F}\varphi \vee \psi$ | $\mathbb{F}\varphi \vee \chi$ | $\mathbb{F}\varphi \vee \psi$ | $\mathbb{F}\varphi \vee \chi$ | $\wedge\mathbb{F}2$ |

Now we can apply $\vee\mathbb{F}$ to all the branches containing $\varphi \vee \psi$:

| | | |
|--|---|----------------------|
| 1. | $\mathbb{T} \varphi \vee (\psi \wedge \chi) \checkmark$ | Assumption |
| 2. | $\mathbb{F} (\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$ | Assumption |
| $\swarrow \qquad \qquad \searrow$ | | |
| 3. | $\mathbb{T} \varphi$ $\mathbb{T} \psi \wedge \chi$ | $\vee\mathbb{T} 1$ |
| $\swarrow \qquad \searrow \qquad \swarrow \qquad \searrow$ | | |
| 4. | $\mathbb{F} \varphi \vee \psi \checkmark$ $\mathbb{F} \varphi \vee \chi$ $\mathbb{F} \varphi \vee \psi \checkmark$ $\mathbb{F} \varphi \vee \chi$ | $\wedge\mathbb{F} 2$ |
| 5. | $\mathbb{F} \varphi$ $\mathbb{F} \varphi$ | $\vee\mathbb{F} 4$ |
| 6. | $\mathbb{F} \psi$ $\mathbb{F} \psi$ | $\vee\mathbb{F} 4$ |
| $\otimes \qquad \qquad \qquad \otimes$ | | |

The leftmost branch is now closed. Let's now apply $\vee\mathbb{F}$ to $\varphi \vee \chi$:

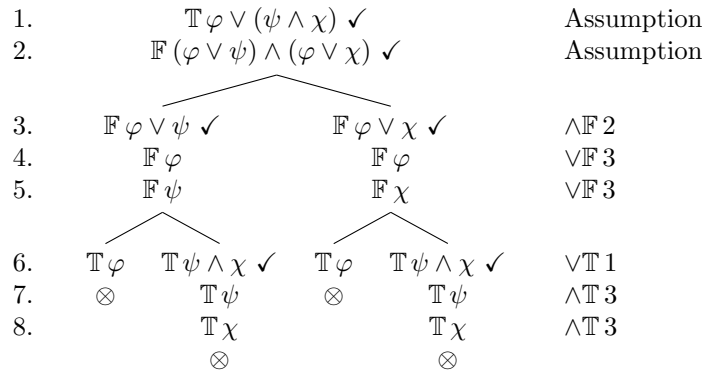
| | | |
|--|---|----------------------|
| 1. | $\mathbb{T} \varphi \vee (\psi \wedge \chi) \checkmark$ | Assumption |
| 2. | $\mathbb{F} (\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$ | Assumption |
| $\swarrow \qquad \qquad \searrow$ | | |
| 3. | $\mathbb{T} \varphi$ $\mathbb{T} \psi \wedge \chi$ | $\vee\mathbb{T} 1$ |
| $\swarrow \qquad \searrow \qquad \swarrow \qquad \searrow$ | | |
| 4. | $\mathbb{F} \varphi \vee \psi \checkmark$ $\mathbb{F} \varphi \vee \chi \checkmark$ $\mathbb{F} \varphi \vee \psi \checkmark$ $\mathbb{F} \varphi \vee \chi \checkmark$ | $\wedge\mathbb{F} 2$ |
| 5. | $\mathbb{F} \varphi$ $\mathbb{F} \varphi$ | $\vee\mathbb{F} 4$ |
| 6. | $\mathbb{F} \psi$ $\mathbb{F} \psi$ | $\vee\mathbb{F} 4$ |
| 7. | \otimes $\mathbb{F} \varphi$ $\mathbb{F} \varphi$ | $\vee\mathbb{F} 4$ |
| 8. | \otimes $\mathbb{F} \chi$ $\mathbb{F} \chi$ | $\vee\mathbb{F} 4$ |
| $\otimes \qquad \qquad \qquad \otimes$ | | |

Note that we moved the result of applying $\vee\mathbb{F}$ a second time below for clarity. In this instance it would not have been needed, since the justifications would have been the same.

Two branches remain open, and $\mathbb{T} \psi \wedge \chi$ on line 3 remains unchecked. We apply $\wedge\mathbb{T}$ to it to obtain a closed **tableau**:

| | | |
|--|---|----------------------|
| 1. | $\mathbb{T} \varphi \vee (\psi \wedge \chi) \checkmark$ | Assumption |
| 2. | $\mathbb{F} (\varphi \vee \psi) \wedge (\varphi \vee \chi) \checkmark$ | Assumption |
| $\swarrow \qquad \qquad \searrow$ | | |
| 3. | $\mathbb{T} \varphi$ $\mathbb{T} \psi \wedge \chi \checkmark$ | $\vee\mathbb{T} 1$ |
| $\swarrow \qquad \searrow \qquad \swarrow \qquad \searrow$ | | |
| 4. | $\mathbb{F} \varphi \vee \psi \checkmark$ $\mathbb{F} \varphi \vee \chi \checkmark$ $\mathbb{F} \varphi \vee \psi \checkmark$ $\mathbb{F} \varphi \vee \chi \checkmark$ | $\wedge\mathbb{F} 2$ |
| 5. | $\mathbb{F} \varphi$ $\mathbb{F} \varphi$ $\mathbb{F} \varphi$ $\mathbb{F} \varphi$ | $\vee\mathbb{F} 4$ |
| 6. | $\mathbb{F} \psi$ $\mathbb{F} \chi$ $\mathbb{F} \psi$ $\mathbb{F} \chi$ | $\vee\mathbb{F} 4$ |
| 7. | \otimes \otimes $\mathbb{T} \psi$ $\mathbb{T} \psi$ | $\wedge\mathbb{T} 3$ |
| 8. | \otimes \otimes $\mathbb{T} \chi$ $\mathbb{T} \chi$ | $\wedge\mathbb{T} 3$ |
| $\otimes \qquad \qquad \qquad \otimes$ | | |

For comparison, here's a closed **tableau** for the same set of assumptions in which the rules are applied in a different order:



Problem 5.1. Give closed **tableaux** of the following:

1. $\mathbb{F} \neg(\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \neg\psi)$
2. $\mathbb{F} (\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi), \mathbb{T} (\varphi \wedge \psi) \rightarrow \chi$

5.5 Proof-Theoretic Notions

pl:tab:ptn:
sec

This section collects the definitions of the provability relation and consistency for tableaux.

explanation

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the existence of certain closed **tableaux**. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

Definition 5.7 (Theorems). A **sentence** φ is a *theorem* if there is a closed **tableau** for $\mathbb{F} \varphi$. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition 5.8 (Derivability). A **sentence** φ is *derivable* from a set of **sentences** Γ , $\Gamma \vdash \varphi$, iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed **tableau** for the set

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n, \}$$

If φ is not **derivable** from Γ we write $\Gamma \not\vdash \varphi$.

Definition 5.9 (Consistency). A set of **sentences** Γ is *inconsistent* iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed **tableau** for the set

$$\{\mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n, \}.$$

If Γ is not inconsistent, we say it is *consistent*.

pl:tab:ptn:
prop:reflexivity

Proposition 5.10 (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. If $\varphi \in \Gamma$, $\{\varphi\}$ is a finite subset of Γ and the **tableau**

1. $\mathbb{F} \varphi$ Assumption
 2. $\mathbb{T} \varphi$ Assumption
- \otimes

is closed. \square

pl:tab:ptn:
prop:monotony

Proposition 5.11 (Monotony). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any finite subset of Γ is also a finite subset of Δ . \square

pl:tab:ptn:
prop:transitivity

Proposition 5.12 (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Proof. If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a finite subset $\Delta_0 = \{\chi_1, \dots, \chi_n\} \subseteq \Delta$ such that

$$\{\mathbb{F} \psi, \mathbb{T} \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n\}$$

has a closed **tableau**. If $\Gamma \vdash \varphi$ then there are $\theta_1, \dots, \theta_m$ such that

$$\{\mathbb{F} \varphi, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m\}$$

has a closed **tableau**.

Now consider the **tableau** with assumptions

$$\mathbb{F} \psi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m.$$

Apply the Cut rule on φ . This generates two branches, one has $\mathbb{T} \varphi$ in it, the other $\mathbb{F} \varphi$. Thus, on the one branch, all of

$$\{\mathbb{F} \psi, \mathbb{T} \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_n\}$$

are available. Since there is a closed **tableau** for these assumptions, we can attach it to that branch; every branch through $\mathbb{T} \varphi$ closes. On the other branch, all of

$$\{\mathbb{F} \varphi, \mathbb{T} \theta_1, \dots, \mathbb{T} \theta_m\}$$

are available, so we can also complete the other side to obtain a closed **tableau**. This shows $\Gamma \cup \Delta \vdash \psi$. \square

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

pl:tab:ptn:
prop:incons

Proposition 5.13. *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every **sentence** φ .*

Proof. Exercise. \square

Problem 5.2. Prove Proposition 5.13

Proposition 5.14 (Compactness).

*pl:tab:ptn:
prop:proves-compact*

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and a closed **tableau** for

$$\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n$$

This **tableau** also shows $\Gamma_0 \vdash \varphi$.

2. If Γ is inconsistent, then for some finite subset $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ there is a closed **tableau** for

$$\mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n$$

This closed **tableau** shows that Γ_0 is inconsistent. □

5.6 Derivability and Consistency

We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

*pl:tab:prv:
sec*

Proposition 5.15. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

*pl:tab:prv:
prop:provability-contr*

Proof. There are finite $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and $\Gamma_1 = \{\chi_1, \dots, \chi_m\} \subseteq \Gamma$ such that

$$\begin{aligned} &\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\} \\ &\{\mathbb{T} \neg \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_m\} \end{aligned}$$

have closed **tableaux**. Using the Cut rule on φ we can combine these into a single closed **tableau** that shows $\Gamma_0 \cup \Gamma_1$ is inconsistent. Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence Γ is inconsistent. □

Proposition 5.16. *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg \varphi\}$ is inconsistent.*

*pl:tab:prv:
prop:prov-incons*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a closed **tableau** for

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}$$

Using the $\neg\mathbb{T}$ rule, this can be turned into a closed **tableau** for

$$\{\mathbb{T} \neg \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}.$$

On the other hand, if there is a closed **tableau** for the latter, we can turn it into a closed **tableau** of the former by removing every formula that results from

$\neg\mathbb{T}$ applied to the first assumption $\mathbb{T}\neg\varphi$ as well as that assumption, and adding the assumption $\mathbb{F}\varphi$. For if a branch was closed before because it contained the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{T}\neg\varphi$, i.e., $\mathbb{F}\varphi$, the corresponding branch in the new **tableau** is also closed. If a branch in the old tableau was closed because it contained the assumption $\mathbb{T}\neg\varphi$ as well as $\mathbb{F}\neg\varphi$ we can turn it into a closed branch by applying $\neg\mathbb{F}$ to $\mathbb{F}\neg\varphi$ to obtain $\mathbb{T}\varphi$. This closes the branch since we added $\mathbb{F}\varphi$ as an assumption. \square

Problem 5.3. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

*pl:tab:prv:
prop:explicit-inc*

Proposition 5.17. *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ such that

$$\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$$

has a closed tableau. Replace the assumption $\mathbb{F}\varphi$ by $\mathbb{T}\neg\varphi$, and insert the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{F}\varphi$ after the assumptions. Any **sentence** in the **tableau** justified by appeal to line 1 in the old **tableau** is now justified by appeal to line $n+1$. So if the old **tableau** was closed, the new one is. It shows that Γ is inconsistent, since all assumptions are in Γ . \square

*pl:tab:prv:
prop:provability-exhaustive*

Proposition 5.18. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.*

Proof. If there are $\psi_1, \dots, \psi_n \in \Gamma$ and $\chi_1, \dots, \chi_m \in \Gamma$ such that

$$\begin{aligned} &\{\mathbb{T}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\} \\ &\{\mathbb{T}\neg\varphi, \mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m\} \end{aligned}$$

both have closed **tableaux**, we can construct a **tableau** that shows that Γ is inconsistent by using as assumptions $\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n$ together with $\mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m$, followed by an application of the Cut rule, yielding two branches, one starting with $\mathbb{T}\varphi$, the other with $\mathbb{F}\varphi$. Add on the part below the assumptions of the first **tableau** on the left side. Here, every rule application is still correct, and every branch closes. On the right side, add the part below the assumptions of the second **tableau**, with the results of any applications of $\neg\mathbb{T}$ to $\mathbb{T}\neg\varphi$ removed.

For if a branch was closed before because it contained the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{T}\neg\varphi$, i.e., $\mathbb{F}\varphi$, as well as $\mathbb{F}\varphi$, the corresponding branch in the new **tableau** is also closed. If a branch in the old tableau was closed because it contained the assumption $\mathbb{T}\neg\varphi$ as well as $\mathbb{F}\neg\varphi$ we can turn it into a closed branch by applying $\neg\mathbb{F}$ to $\mathbb{F}\neg\varphi$ to obtain $\mathbb{T}\varphi$. \square

5.7 Derivability and the Propositional Connectives

*pl:tab:ppr:
sec*

*pl:tab:ppr:
prop:provability-land*

Proposition 5.19.

1. Both $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$.

2. $\varphi, \psi \vdash \varphi \wedge \psi$.

*pl:tab:ppr:
prop:provability-land-left
pl:tab:ppr:
prop:provability-land-right*

Proof. 1. Both $\{\mathbb{F} \varphi, \mathbb{T} \varphi \wedge \psi\}$ and $\{\mathbb{F} \psi, \mathbb{T} \varphi \wedge \psi\}$ have closed **tableaux**

| | | |
|----|----------------------------------|-----------------------|
| 1. | $\mathbb{F} \varphi$ | Assumption |
| 2. | $\mathbb{T} \varphi \wedge \psi$ | Assumption |
| 3. | $\mathbb{T} \varphi$ | $\wedge \mathbb{T} 2$ |
| 4. | $\mathbb{T} \psi$ | $\wedge \mathbb{T} 2$ |
| | \otimes | |

| | | |
|----|----------------------------------|-----------------------|
| 1. | $\mathbb{F} \psi$ | Assumption |
| 2. | $\mathbb{T} \varphi \wedge \psi$ | Assumption |
| 3. | $\mathbb{T} \varphi$ | $\wedge \mathbb{T} 2$ |
| 4. | $\mathbb{T} \psi$ | $\wedge \mathbb{T} 2$ |
| | \otimes | |

2. Here is a closed **tableau** for $\{\mathbb{T} \varphi, \mathbb{T} \psi, \mathbb{F} \varphi \wedge \psi\}$:

| | | |
|----|--|-----------------------|
| 1. | $\mathbb{F} \varphi \wedge \psi$ | Assumption |
| 2. | $\mathbb{T} \varphi$ | Assumption |
| 3. | $\mathbb{T} \psi$ | Assumption |
| | $\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \end{array}$ | |
| 4. | $\mathbb{F} \varphi$ $\mathbb{F} \psi$ | $\wedge \mathbb{F} 1$ |
| | \otimes \otimes | |

□

Proposition 5.20.

*pl:tab:ppr:
prop:provability-lor*

1. $\varphi \vee \psi, \neg \varphi, \neg \psi$ is inconsistent.

2. Both $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$.

Proof. 1. We give a closed **tableau** of $\{\mathbb{T} \varphi \vee \psi, \mathbb{T} \neg \varphi, \mathbb{T} \neg \psi\}$:

| | | |
|----|--|---------------------|
| 1. | $\mathbb{T} \varphi \vee \psi$ | Assumption |
| 2. | $\mathbb{T} \neg \varphi$ | Assumption |
| 3. | $\mathbb{T} \neg \psi$ | Assumption |
| 4. | $\mathbb{F} \varphi$ | $\neg \mathbb{T} 2$ |
| 5. | $\mathbb{F} \psi$ | $\neg \mathbb{T} 3$ |
| | $\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \end{array}$ | |
| 6. | $\mathbb{T} \varphi$ $\mathbb{T} \psi$ | $\vee \mathbb{T} 1$ |
| | \otimes \otimes | |

2. Both $\{\mathbb{F} \varphi \vee \psi, \mathbb{T} \varphi\}$ and $\{\mathbb{F} \varphi \vee \psi, \mathbb{T} \psi\}$ have closed **tableaux**:

| | | |
|----|--------------------------------|---------------------|
| 1. | $\mathbb{F} \varphi \vee \psi$ | Assumption |
| 2. | $\mathbb{T} \varphi$ | Assumption |
| 3. | $\mathbb{F} \varphi$ | $\vee \mathbb{F} 1$ |
| 4. | $\mathbb{F} \psi$ | $\vee \mathbb{F} 1$ |
| | \otimes | |

| | | |
|----|--------------------------------|---------------------|
| 1. | $\mathbb{F} \varphi \vee \psi$ | Assumption |
| 2. | $\mathbb{T} \psi$ | Assumption |
| 3. | $\mathbb{F} \varphi$ | $\vee \mathbb{F} 1$ |
| 4. | $\mathbb{F} \psi$ | $\vee \mathbb{F} 1$ |
| | \otimes | |

□

Proposition 5.21.

pl:tab:ppr: prop:provability-lif

1. $\varphi, \varphi \rightarrow \psi \vdash \psi$.

pl:tab:ppr: prop:provability-lif-left

2. Both $\neg \varphi \vdash \varphi \rightarrow \psi$ and $\psi \vdash \varphi \rightarrow \psi$.

pl:tab:ppr: prop:provability-lif-right

Proof. 1. $\{\mathbb{F} \psi, \mathbb{T} \varphi \rightarrow \psi, \mathbb{T} \varphi\}$ has a closed **tableau**:

| | | |
|----|--|----------------------------|
| 1. | $\mathbb{F} \psi$ | Assumption |
| 2. | $\mathbb{T} \varphi \rightarrow \psi$ | Assumption |
| 3. | $\mathbb{T} \varphi$ | Assumption |
| | $\swarrow \quad \searrow$ $\mathbb{F} \varphi \quad \mathbb{T} \psi$ | |
| 4. | <div style="text-align: center;">$\mathbb{F} \varphi$ \otimes</div> <div style="text-align: center;">$\mathbb{T} \psi$ \otimes</div> | $\rightarrow \mathbb{T} 2$ |

2. Both $s\{\mathbb{F} \varphi \rightarrow \psi, \mathbb{T} \neg \varphi\}$ and $\{\mathbb{F} \varphi \rightarrow \psi, \mathbb{T} \neg \psi\}$ have closed **tableaux**:

| | | |
|----|---------------------------------------|----------------------------|
| 1. | $\mathbb{F} \varphi \rightarrow \psi$ | Assumption |
| 2. | $\mathbb{T} \neg \varphi$ | Assumption |
| 3. | $\mathbb{T} \varphi$ | $\rightarrow \mathbb{F} 1$ |
| 4. | $\mathbb{F} \psi$ | $\rightarrow \mathbb{F} 1$ |
| 5. | $\mathbb{F} \varphi$ | $\neg \mathbb{T} 2$ |
| | \otimes | |

- | | | |
|----|---------------------------------------|---------------------------|
| 1. | $\mathbb{F} \varphi \rightarrow \psi$ | Assumption |
| 2. | $\mathbb{T} \neg\psi$ | Assumption |
| 3. | $\mathbb{T} \varphi$ | $\rightarrow\mathbb{F} 1$ |
| 4. | $\mathbb{F} \psi$ | $\rightarrow\mathbb{F} 1$ |
| 5. | $\mathbb{F} \psi$ | $\neg\mathbb{T} 2$ |
| | \otimes | |

□

5.8 Soundness

explanation A **derivation** system, such as tableaux, is *sound* if it cannot **derive** things that do not actually hold. Soundness is thus a kind of guaranteed safety property for **derivation** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that pl:tab:sou: sec

1. every **derivable** φ is a tautology;
2. if a **sentence** is **derivable** from some others, it is also a consequence of them;
3. if a set of **sentences** is inconsistent, it is unsatisfiable.

These are important properties of a **derivation** system. If any of them do not hold, the **derivation** system is deficient—it would **derive** too much. Consequently, establishing the soundness of a **derivation** system is of the utmost importance.

Because all these proof-theoretic properties are defined via closed **tableaux** of some kind or other, proving (1)–(3) above requires proving something about the semantic properties of closed **tableaux**. We will first define what it means for a **signed formula** to be satisfied in a structure, and then show that if a **tableau** is closed, no structure satisfies all its assumptions. (1)–(3) then follow as corollaries from this result.

Definition 5.22. A valuation \mathbf{v} *satisfies* a signed formula $\mathbb{T}\varphi$ iff $\mathbf{v} \models \varphi$, and it satisfies $\mathbb{F}\varphi$ iff $\mathbf{v} \not\models \varphi$. \mathbf{v} satisfies a set of signed formulas Γ iff it satisfies every $S\varphi \in \Gamma$. Γ is *satisfiable* if there is a valuation that satisfies it, and *unsatisfiable* otherwise.

Theorem 5.23 (Soundness). *If Γ has a closed tableau, Γ is unsatisfiable.*

pl:tab:sou: thm:tableau-soundness

Proof. Let’s call a branch of a **tableau** *satisfiable* iff the set of **signed formulas** on it is satisfiable, and let’s call a **tableau** *satisfiable* if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable **tableau** by one of the rules of inference always results in a satisfiable **tableau**. This will prove the theorem: any closed **tableau** results by applying rules of inference to the **tableau**

consisting only of assumptions from Γ . So if Γ were satisfiable, any **tableau** for it would be satisfiable. A closed **tableau**, however, is clearly not satisfiable: every branch contains both $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$, and no structure can both satisfy and not satisfy φ .

Suppose we have a satisfiable **tableau**, i.e., a **tableau** with at least one satisfiable branch. Applying a rule of inference either adds **signed formulas** to a branch, or splits a branch in two. If the **tableau** has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended **tableau**, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let Γ be the set of **signed formulas** on that branch, and let $S\varphi \in \Gamma$ be the **signed formula** to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, i.e., Γ together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable.

First, we consider the possible inferences with only one premise.

1. The branch is expanded by applying $\neg\mathbb{T}$ to $\mathbb{T}\neg\psi \in \Gamma$. Then the extended branch contains the **signed formulas** $\Gamma \cup \{\mathbb{F}\psi\}$. Suppose $\mathfrak{v} \models \Gamma$. In particular, $\mathfrak{v} \models \neg\psi$. Thus, $\mathfrak{v} \not\models \psi$, i.e., \mathfrak{v} satisfies $\mathbb{F}\psi$.
2. The branch is expanded by applying $\neg\mathbb{F}$ to $\mathbb{F}\neg\psi \in \Gamma$: Exercise.
3. The branch is expanded by applying $\wedge\mathbb{T}$ to $\mathbb{T}\psi \wedge \chi \in \Gamma$, which results in two new **signed formulas** on the branch: $\mathbb{T}\psi$ and $\mathbb{T}\chi$. Suppose $\mathfrak{v} \models \Gamma$, in particular $\mathfrak{v} \models \psi \wedge \chi$. Then $\mathfrak{v} \models \psi$ and $\mathfrak{v} \models \chi$. This means that \mathfrak{v} satisfies both $\mathbb{T}\psi$ and $\mathbb{T}\chi$.
4. The branch is expanded by applying $\vee\mathbb{F}$ to $\mathbb{T}\psi \vee \chi \in \Gamma$: Exercise.
5. The branch is expanded by applying $\rightarrow\mathbb{F}$ to $\mathbb{T}\psi \rightarrow \chi \in \Gamma$: This results in two new **signed formulas** on the branch: $\mathbb{T}\psi$ and $\mathbb{F}\chi$. Suppose $\mathfrak{v} \models \Gamma$, in particular $\mathfrak{v} \models \psi \rightarrow \chi$. Then $\mathfrak{v} \models \psi$ and $\mathfrak{v} \not\models \chi$. This means that \mathfrak{v} satisfies both $\mathbb{T}\psi$ and $\mathbb{F}\chi$.

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying $\wedge\mathbb{F}$ to $\mathbb{F}\psi \wedge \chi \in \Gamma$, which results in two branches, a left one continuing through $\mathbb{F}\psi$ and a right one through $\mathbb{F}\chi$. Suppose $\mathfrak{v} \models \Gamma$, in particular $\mathfrak{v} \not\models \psi \wedge \chi$. Then $\mathfrak{v} \not\models \psi$ or $\mathfrak{v} \not\models \chi$. In the former case, \mathfrak{v} satisfies $\mathbb{F}\psi$, i.e., \mathfrak{v} satisfies the formulas on the left branch. In the latter, \mathfrak{v} satisfies $\mathbb{F}\chi$, i.e., \mathfrak{v} satisfies the formulas on the right branch.
2. The branch is expanded by applying $\vee\mathbb{T}$ to $\mathbb{T}\psi \vee \chi \in \Gamma$: Exercise.
3. The branch is expanded by applying $\rightarrow\mathbb{T}$ to $\mathbb{T}\psi \rightarrow \chi \in \Gamma$: Exercise.

4. The branch is expanded by Cut: This results in two branches, one containing $\mathbb{T}\psi$, the other containing $\mathbb{F}\psi$. Since $\mathfrak{v} \models \Gamma$ and either $\mathfrak{v} \models \psi$ or $\mathfrak{v} \not\models \psi$, \mathfrak{v} satisfies either the left or the right branch.

□

Problem 5.4. Complete the proof of [Theorem 5.23](#).

Corollary 5.24. *If $\vdash \varphi$ then φ is a tautology.*

*pl:tab:sou:
cor:weak-soundness*

Corollary 5.25. *If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*

*pl:tab:sou:
cor:entailment-soundness*

Proof. If $\Gamma \vdash \varphi$ then for some $\psi_1, \dots, \psi_n \in \Gamma$, $\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$ has a closed **tableau**. By [Theorem 5.23](#), every **valuation** \mathfrak{v} either makes some ψ_i false or makes φ true. Hence, if $\mathfrak{v} \models \Gamma$ then also $\mathfrak{v} \models \varphi$. □

Corollary 5.26. *If Γ is satisfiable, then it is consistent.*

*pl:tab:sou:
cor:consistency-soundness*

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ and a closed **tableau** for $\{\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$. By [Theorem 5.23](#), there is no \mathfrak{v} such that $\mathfrak{v} \models \psi_i$ for all $i = 1, \dots, n$. But then Γ is not satisfiable. □

Chapter 6

Axiomatic Derivations

No effort has been made yet to ensure that the material in this chapter respects various tags indicating which connectives and quantifiers are primitive or defined: all are assumed to be primitive. If the FOL tag is true, we produce a version with quantifiers, otherwise without.

6.1 Rules and Derivations

pl:axd:rul:
sec Axiomatic **derivations** are perhaps the simplest proof system for logic. A explanation **derivation** is just a sequence of **formulas**. To count as a **derivation**, every **formula** in the sequence must either be an instance of an axiom, or must follow from one or more **formulas** that precede it in the sequence by a rule of inference. A **derivation derives** its last **formula**.

Definition 6.1 (Derivability). If Γ is a set of **formulas** of \mathcal{L} then a **derivation** from Γ is a finite sequence $\varphi_1, \dots, \varphi_n$ of **formulas** where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. φ_i follows from some φ_j (and φ_k) with $j < i$ (and $k < i$) by a rule of inference.

What counts as a correct **derivation** depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step A_i in is a correct inference step.

Definition 6.2 (Rule of inference). A *rule of inference* gives a sufficient condition for what counts as a correct inference step in a **derivation** from Γ .

For instance, since any one-element sequence φ with $\varphi \in \Gamma$ trivially counts as a **derivation**, the following might be a very simple rule of inference:

If $\varphi \in \Gamma$, then φ is always a correct inference step in any **derivation** from Γ .

Similarly, if φ is one of the axioms, then φ by itself is a **derivation**, and so this is also a rule of inference:

If φ is an axiom, then φ is a correct inference step.

It gets more interesting if the rule of inference appeals to **formulas** that appear before the step considered. The following rule is called *modus ponens*:

If $\psi \rightarrow \varphi$ and ψ occur higher up in the **derivation**, then φ is a correct inference step.

If this is the only rule of inference, then our definition of **derivation** above amounts to this: $\varphi_1, \dots, \varphi_n$ is a **derivation** iff for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. for some $j < i$, φ_j is $\psi \rightarrow \varphi_i$, and for some $k < i$, φ_k is ψ .

The last clause says that φ_i follows from φ_j (ψ) and φ_k ($\psi \rightarrow \varphi_i$) by modus ponens. If we can go from 1 to n , and each time we find a **formula** φ_i that is either in Γ , an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct **derivation**.

Definition 6.3 (Derivability). A **formula** φ is *derivable* from Γ , written $\Gamma \vdash \varphi$, if there is a **derivation** from Γ ending in φ .

Definition 6.4 (Theorems). A **formula** φ is a *theorem* if there is a **derivation** of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

6.2 Axiom and Rules for the Propositional Connectives

pl:axd:prp:
sec

Definition 6.5 (Axioms). The set of Ax_0 of *axioms* for the propositional connectives comprises all **formulas** of the following forms:

$$\text{pl:axd:prp:} \quad (\varphi \wedge \psi) \rightarrow \varphi \quad (6.1)$$

$$\text{pl:ax:land1:axd:prp:} \quad (\varphi \wedge \psi) \rightarrow \psi \quad (6.2)$$

$$\text{pl:ax:land2:axd:prp:} \quad \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi)) \quad (6.3)$$

$$\text{pl:ax:land3:axd:prp:} \quad \varphi \rightarrow (\varphi \vee \psi) \quad (6.4)$$

$$\text{pl:ax:lor1:axd:prp:} \quad \varphi \rightarrow (\psi \vee \varphi) \quad (6.5)$$

$$\text{pl:ax:lor2:axd:prp:} \quad (\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)) \quad (6.6)$$

$$\text{pl:ax:lor3:axd:prp:} \quad \varphi \rightarrow (\psi \rightarrow \varphi) \quad (6.7)$$

$$\text{pl:ax:lif1:axd:prp:} \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \quad (6.8)$$

$$\text{pl:ax:lif2:axd:prp:} \quad (\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg\psi) \rightarrow \neg\varphi) \quad (6.9)$$

$$\text{pl:ax:not1:axd:prp:} \quad \neg\varphi \rightarrow (\varphi \rightarrow \psi) \quad (6.10)$$

$$\text{pl:ax:not2:axd:prp:} \quad \top \quad (6.11)$$

$$\text{pl:ax:true:axd:prp:} \quad \perp \rightarrow \varphi \quad (6.12)$$

$$\text{pl:ax:false1:axd:prp:} \quad (\varphi \rightarrow \perp) \rightarrow \neg\varphi \quad (6.13)$$

$$\text{pl:ax:false2:axd:prp:} \quad \neg\neg\varphi \rightarrow \varphi \quad (6.14)$$

ax:dne

Definition 6.6 (Modus ponens). If ψ and $\psi \rightarrow \varphi$ already occur in a derivation, then φ is a correct inference step.

We'll abbreviate the rule modus ponens as “MP.”

6.3 Examples of Derivations

pl:axd:pro:
sec

Example 6.7. Suppose we want to prove $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$. Clearly, this is not an instance of any of our axioms, so we have to use the MP rule to **derive** it. Our only rule is MP, which given φ and $\varphi \rightarrow \psi$ allows us to justify ψ . One strategy would be to use [eq. \(6.6\)](#) with φ being $\neg\theta$, ψ being α , and χ being $\theta \rightarrow \alpha$, i.e., the instance

$$(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha))).$$

Why? Two applications of MP yield the last part, which is what we want. And we easily see that $\neg\theta \rightarrow (\theta \rightarrow \alpha)$ is an instance of [eq. \(6.10\)](#), and $\alpha \rightarrow (\theta \rightarrow \alpha)$ is an instance of [eq. \(6.7\)](#). So our derivation is:

1. $\neg\theta \rightarrow (\theta \rightarrow \alpha)$ eq. (6.7)
2. $(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow$
 $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$ eq. (6.6)
3. $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$ 1, 2, MP
4. $\alpha \rightarrow (\theta \rightarrow \alpha)$ eq. (6.7)
5. $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$ 3, 4, MP

Example 6.8. Let's try to find a **derivation** of $\theta \rightarrow \theta$. It is not an instance of an axiom, so we have to use MP to **derive** it. eq. (6.7) is an axiom of the form $\varphi \rightarrow \psi$ to which we could apply MP. To be useful, of course, the ψ which MP would justify as a correct step in this case would have to be $\theta \rightarrow \theta$, since this is what we want to **derive**. That means φ would also have to be θ , i.e., we might look at this instance of eq. (6.7):

$$\theta \rightarrow (\theta \rightarrow \theta)$$

In order to apply MP, we would also need to justify the corresponding second premise, namely φ . But in our case, that would be θ , and we won't be able to **derive** θ by itself. So we need a different strategy.

The other axiom involving just \rightarrow is eq. (6.8), i.e.,

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

We could get to the last nested conditional by applying MP twice. Again, that would mean that we want an instance of eq. (6.8) where $\varphi \rightarrow \chi$ is $\theta \rightarrow \theta$, the **formula** we are aiming for. Then of course, φ and χ are both θ . How should we pick ψ so that both $\varphi \rightarrow (\psi \rightarrow \chi)$ and $\varphi \rightarrow \psi$, i.e., in our case $\theta \rightarrow (\psi \rightarrow \theta)$ and $\theta \rightarrow \psi$, are also **derivable**? Well, the first of these is already an instance of eq. (6.7), whatever we decide ψ to be. And $\theta \rightarrow \psi$ would be another instance of eq. (6.7) if ψ were $(\theta \rightarrow \theta)$. So, our derivation is:

1. $\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)$ eq. (6.7)
2. $(\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)) \rightarrow$
 $((\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta))$ eq. (6.8)
3. $(\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta)$ 1, 2, MP
4. $\theta \rightarrow (\theta \rightarrow \theta)$ eq. (6.7)
5. $\theta \rightarrow \theta$ 3, 4, MP

Example 6.9. Sometimes we want to show that there is a derivation of some **formula** from some other **formulas** Γ . For instance, let's show that we can **derive** $\varphi \rightarrow \chi$ from $\Gamma = \{\varphi \rightarrow \psi, \psi \rightarrow \chi\}$.

1. $\varphi \rightarrow \psi$ HYP
2. $\psi \rightarrow \chi$ HYP
3. $(\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$ eq. (6.7)
4. $\varphi \rightarrow (\psi \rightarrow \chi)$ 2, 3, MP
5. $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow$
 $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ eq. (6.8)
6. $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ 4, 5, MP
7. $\varphi \rightarrow \chi$ 1, 6, MP

The lines labelled "HYP" (for "hypothesis") indicate that the **formula** on that line is an **element** of Γ .

Proposition 6.10. *If $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \psi \rightarrow \chi$, then $\Gamma \vdash \varphi \rightarrow \chi$*

Proof. Suppose $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \psi \rightarrow \chi$. Then there is a **derivation** of $\varphi \rightarrow \psi$ from Γ ; and a **derivation** of $\psi \rightarrow \chi$ from Γ as well. Combine these into a single **derivation** by concatenating them. Now add lines 3–7 of the **derivation** in the preceding example. This is a **derivation** of $\varphi \rightarrow \chi$ —which is the last line of the new **derivation**—from Γ . Note that the justifications of lines 4 and 7 remain valid if the reference to line number 2 is replaced by reference to the last line of the **derivation** of $\varphi \rightarrow \psi$, and reference to line number 1 by reference to the last line of the **derivation** of $B \rightarrow \chi$. \square

Problem 6.1. Show that the following hold by exhibiting **derivations** from the axioms:

1. $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
2. $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
3. $\neg(\varphi \vee \psi) \rightarrow \neg\varphi$

6.4 Proof-Theoretic Notions

pl:axd:ptn:
sec Just as we’ve defined a number of important semantic notions (tautology, explanation entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain formulas. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

Definition 6.11 (Derivability). A formula φ is *derivable* from Γ , written $\Gamma \vdash \varphi$, if there is a **derivation** from Γ ending in φ .

Definition 6.12 (Theorems). A formula φ is a *theorem* if there is a **derivation** of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition 6.13 (Consistency). A set Γ of **formulas** is *consistent* if and only if $\Gamma \not\vdash \perp$; it is *inconsistent* otherwise.

pl:axd:ptn:
prop:reflexivity **Proposition 6.14 (Reflexivity).** *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. The **formula** φ by itself is a **derivation** of φ from Γ . \square

pl:axd:ptn:
prop:monotony **Proposition 6.15 (Monotony).** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any **derivation** of φ from Γ is also a **derivation** of φ from Δ . \square

pl:axd:ptn:
prop:transitivity **Proposition 6.16 (Transitivity).** *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Proof. Suppose $\{\varphi\} \cup \Delta \vdash \psi$. Then there is a **derivation** $\psi_1, \dots, \psi_l = \psi$ from $\{\varphi\} \cup \Delta$. Some of the steps in that derivation will be correct because of a rule which refers to a prior line $\psi_i = \varphi$. By hypothesis, there is a **derivation** of φ from Γ , i.e., a **derivation** $\varphi_1, \dots, \varphi_k = \varphi$ where every φ_i is an axiom, an **element** of Γ , or correct by a rule of inference. Now consider the sequence

$$\varphi_1, \dots, \varphi_k = \varphi, \psi_1, \dots, \psi_l = \psi.$$

This is a correct **derivation** of ψ from $\Gamma \cup \Delta$ since every $B_i = \varphi$ is now justified by the same rule which justifies $\varphi_k = \varphi$. \square

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

Proposition 6.17. Γ is inconsistent iff $\Gamma \vdash \varphi$ for every φ .

*pl:axd:ptn:
prop:incons*

Proof. Exercise. \square

Problem 6.2. Prove [Proposition 6.17](#).

Proposition 6.18 (Compactness).

*pl:axd:ptn:
prop:proves-compact*

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite sequence of **formulas** $\varphi_1, \dots, \varphi_n$ so that $\varphi \equiv \varphi_n$ and each φ_i is either a logical axiom, an **element** of Γ or follows from previous **formulas** by modus ponens. Take Γ_0 to be those φ_i which are in Γ . Then the **derivation** is likewise a **derivation** from Γ_0 , and so $\Gamma_0 \vdash \varphi$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \perp$. \square

6.5 The Deduction Theorem

As we've seen, giving **derivations** in an axiomatic system is cumbersome, and **derivations** may be hard to find. Rather than actually write out long lists of **formulas**, it is generally easier to argue that such **derivations** exist, by making use of a few simple results. We've already established three such results: [Proposition 6.14](#) says we can always assert that $\Gamma \vdash \varphi$ when we know that $\varphi \in \Gamma$. [Proposition 6.15](#) says that if $\Gamma \vdash \varphi$ then also $\Gamma \cup \{\psi\} \vdash \varphi$. And [Proposition 6.16](#) implies that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. Here's another simple result, a "meta"-version of modus ponens:

*pl:axd:ded:
sec*

Proposition 6.19. If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$, then $\Gamma \vdash \psi$.

*pl:axd:ded:
prop:mp*

Proof. We have that $\{\varphi, \varphi \rightarrow \psi\} \vdash \psi$:

1. φ Hyp.
2. $\varphi \rightarrow \psi$ Hyp.
3. ψ 1, 2, MP

By Proposition 6.16, $\Gamma \vdash \psi$. □

The most important result we'll use in this context is the deduction theorem:

*pl:axd:ded:
thm:deduction-thm*

Theorem 6.20 (Deduction Theorem). $\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \rightarrow \psi$.

Proof. The “if” direction is immediate. If $\Gamma \vdash \varphi \rightarrow \psi$ then also $\Gamma \cup \{\varphi\} \vdash \varphi \rightarrow \psi$ by Proposition 6.15. Also, $\Gamma \cup \{\varphi\} \vdash \varphi$ by Proposition 6.14. So, by Proposition 6.19, $\Gamma \cup \{\varphi\} \vdash \psi$.

For the “only if” direction, we proceed by induction on the length of the derivation of ψ from $\Gamma \cup \{\varphi\}$.

For the induction basis, we prove the claim for every derivation of length 1. A derivation of ψ from $\Gamma \cup \{\varphi\}$ of length 1 consists of ψ by itself; and if it is correct ψ is either $\in \Gamma \cup \{\varphi\}$ or is an axiom. If $\psi \in \Gamma$ or is an axiom, then $\Gamma \vdash \psi$. We also have that $\Gamma \vdash \psi \rightarrow (\varphi \rightarrow \psi)$ by eq. (6.7), and Proposition 6.19 gives $\Gamma \vdash \varphi \rightarrow \psi$. If $\psi \in \{\varphi\}$ then $\Gamma \vdash \varphi \rightarrow \psi$ because then last sentence $\varphi \rightarrow \psi$ is the same as $\varphi \rightarrow \varphi$, and we have derived that in Example 6.8.

For the inductive step, suppose a derivation of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by modus ponens. (If it is not justified by modus ponens, $\psi \in \Gamma$, $\psi \equiv \varphi$, or ψ is an axiom, and the same reasoning as in the induction basis applies.) Then some previous steps in the derivation are $\chi \rightarrow \psi$ and χ , for some formula χ , i.e., $\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \psi$ and $\Gamma \cup \{\varphi\} \vdash \chi$, and the respective derivations are shorter, so the inductive hypothesis applies to them. We thus have both:

$$\begin{aligned} \Gamma \vdash \varphi \rightarrow (\chi \rightarrow \psi); \\ \Gamma \vdash \varphi \rightarrow \chi. \end{aligned}$$

But also

$$\Gamma \vdash (\varphi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi)),$$

by eq. (6.8), and two applications of Proposition 6.19 give $\Gamma \vdash \varphi \rightarrow \psi$, as required. □

Notice how eq. (6.7) and eq. (6.8) were chosen precisely so that the Deduction Theorem would hold.

The following are some useful facts about derivability, which we leave as exercises.

*pl:axd:ded:
prop:derivfacts*

Proposition 6.21.

*pl:axd:ded:
derivfacts:a
pl:axd:ded:
derivfacts:b
pl:axd:ded:
derivfacts:c*

1. $\vdash (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi));$
2. If $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$ then $\Gamma \cup \{\psi\} \vdash \varphi$ (Contraposition);
3. $\{\varphi, \neg\varphi\} \vdash \psi$ (Ex Falso Quodlibet, Explosion);

4. $\{\neg\neg\varphi\} \vdash \varphi$ (Double Negation Elimination);

*pl:axd:ded:
derivfacts:d*

5. If $\Gamma \vdash \neg\neg\varphi$ then $\Gamma \vdash \varphi$;

*pl:axd:ded:
derivfacts:e*

Problem 6.3. Prove Proposition 6.21

6.6 Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

*pl:axd:prv:
sec*

Proposition 6.22. If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.

*pl:axd:prv:
prop:provability-contr*

Proof. If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \perp$. By Proposition 6.14, $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{\varphi\}$. By Proposition 6.16, $\Gamma \vdash \perp$, i.e., Γ is inconsistent. \square

Proposition 6.23. $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

*pl:axd:prv:
prop:prov-incons*

Proof. First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by Proposition 6.15. $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by Proposition 6.14. We also have $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$ by eq. (6.10). So by two applications of Proposition 6.19, we have $\Gamma \cup \{\neg\varphi\} \vdash \perp$.

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, i.e., $\Gamma \cup \{\neg\varphi\} \vdash \perp$. By the deduction theorem, $\Gamma \vdash \neg\varphi \rightarrow \perp$. $\Gamma \vdash (\neg\varphi \rightarrow \perp) \rightarrow \neg\neg\varphi$ by eq. (6.13), so $\Gamma \vdash \neg\neg\varphi$ by Proposition 6.19. Since $\Gamma \vdash \neg\neg\varphi \rightarrow \varphi$ (eq. (6.14)), we have $\Gamma \vdash \varphi$ by Proposition 6.19 again. \square

Problem 6.4. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

Proposition 6.24. If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.

*pl:axd:prv:
prop:explicit-inc*

Proof. $\Gamma \vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$ by eq. (6.10). $\Gamma \vdash \perp$ by two applications of Proposition 6.19. \square

Proposition 6.25. If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.

*pl:axd:prv:
prop:provability-exhaustive*

Proof. Exercise. \square

Problem 6.5. Prove Proposition 6.25

6.7 Derivability and the Propositional Connectives

pl:axd:ppr:
sec

Proposition 6.26.

pl:axd:ppr:
prop:provability-land

pl:axd:ppr:
prop:provability-land-left

pl:axd:ppr:
prop:provability-land-right

1. Both $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$
2. $\varphi, \psi \vdash \varphi \wedge \psi$.

Proof. 1. From eq. (6.1) and eq. (6.1) by modus ponens.
2. From eq. (6.3) by two applications of modus ponens. □

Proposition 6.27.

pl:axd:ppr:
prop:provability-lor

1. $\varphi \vee \psi, \neg\varphi, \neg\psi$ is inconsistent.
2. Both $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$.

Proof. 1. From eq. (6.9) we get $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$ and $\vdash \neg\psi \rightarrow (\psi \rightarrow \perp)$. So by the deduction theorem, we have $\{\neg\varphi\} \vdash \varphi \rightarrow \perp$ and $\{\neg\psi\} \vdash \psi \rightarrow \perp$. From eq. (6.6) we get $\{\neg\varphi, \neg\psi\} \vdash (\varphi \vee \psi) \rightarrow \perp$. By the deduction theorem, $\{\varphi \vee \psi, \neg\varphi, \neg\psi\} \vdash \perp$.
2. From eq. (6.4) and eq. (6.5) by modus ponens. □

Proposition 6.28.

pl:axd:ppr:
prop:provability-lif

pl:axd:ppr:
prop:provability-lif-left

pl:axd:ppr:
prop:provability-lif-right

1. $\varphi, \varphi \rightarrow \psi \vdash \psi$.
2. Both $\neg\varphi \vdash \varphi \rightarrow \psi$ and $\psi \vdash \varphi \rightarrow \psi$.

Proof. 1. We can derive:

- | | | |
|----|----------------------------|----------|
| 1. | φ | HYP |
| 2. | $\varphi \rightarrow \psi$ | HYP |
| 3. | ψ | 1, 2, MP |

2. By eq. (6.10) and eq. (6.7) and the deduction theorem, respectively. □

6.8 Soundness

explanation A **derivation** system, such as axiomatic deduction, is *sound* if it cannot **derive** things that do not actually hold. Soundness is thus a kind of guaranteed safety property for **derivation** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that pl:axd:sou:sec

1. every **derivable** φ is valid;
2. if φ is **derivable** from some others Γ , it is also a consequence of them;
3. if a set of **formulas** Γ is inconsistent, it is unsatisfiable.

These are important properties of a **derivation** system. If any of them do not hold, the **derivation** system is deficient—it would **derive** too much. Consequently, establishing the soundness of a **derivation** system is of the utmost importance.

Proposition 6.29. *If φ is an axiom, then $\mathfrak{v} \models \varphi$ for each **valuation** \mathfrak{v} .*

Proof. Do truth tables for each axiom to verify that they are tautologies. \square

Theorem 6.30 (Soundness). *If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.* pl:axd:sou:thm:soundness

Proof. By induction on the length of the **derivation** of φ from Γ . If there are no steps justified by inferences, then all **formulas** in the derivation are either instances of axioms or are in Γ . By the previous proposition, all the axioms are tautologies, and hence if φ is an axiom then $\Gamma \models \varphi$. If $\varphi \in \Gamma$, then trivially $\Gamma \models \varphi$.

If the last step of the derivation of φ is justified by modus ponens, then there are **formulas** ψ and $\psi \rightarrow \varphi$ in the **derivation**, and the induction hypothesis applies to the part of the **derivation** ending in those **formulas** (since they contain at least one fewer steps justified by an inference). So, by induction hypothesis, $\Gamma \models \psi$ and $\Gamma \models \psi \rightarrow \varphi$. Then $\Gamma \models \varphi$ by [Theorem 1.16](#). \square

Corollary 6.31. *If $\vdash \varphi$, then φ is a tautology.* pl:axd:sou:cor:weak-soundness

Corollary 6.32. *If Γ is satisfiable, then it is consistent.* pl:axd:sou:cor:consistency-soundness

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then $\Gamma \vdash \perp$, i.e., there is a **derivation** of \perp from Γ . By [Theorem 6.30](#), any **valuation** \mathfrak{v} that satisfies Γ must satisfy \perp . Since $\mathfrak{v} \not\models \perp$ for every **valuation** \mathfrak{v} , no \mathfrak{v} can satisfy Γ , i.e., Γ is not satisfiable. \square

Chapter 7

The Completeness Theorem

7.1 Introduction

pl:com:int:
sec

The completeness theorem is one of the most fundamental results about logic. It comes in two formulations, the equivalence of which we'll prove. In its first formulation it says something fundamental about the relationship between semantic consequence and our proof system: if a **sentence** φ follows from some **sentences** Γ , then there is also a **derivation** that establishes $\Gamma \vdash \varphi$. Thus, the proof system is as strong as it can possibly be without proving things that don't actually follow.

In its second formulation, it can be stated as a model existence result: every consistent set of **sentences** is satisfiable. Consistency is a proof-theoretic notion: it says that our proof system is unable to produce certain **derivations**. But who's to say that just because there are no **derivations** of a certain sort from Γ , it's guaranteed that there is **valuation** \mathfrak{v} with $\mathfrak{v} \models \Gamma$? Before the completeness theorem was first proved—in fact before we had the proof systems we now do—the great German mathematician David Hilbert held the view that consistency of mathematical theories guarantees the existence of the objects they are about. He put it as follows in a letter to Gottlob Frege:

If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.

Frege vehemently disagreed. The second formulation of the completeness theorem shows that Hilbert was right in at least the sense that if the axioms are consistent, then *some* **valuation** exists that makes them all true.

These aren't the only reasons the completeness theorem—or rather, its proof—is important. It has a number of important consequences, some of which we'll discuss separately. For instance, since any **derivation** that shows $\Gamma \vdash \varphi$ is finite and so can only use finitely many of the **sentences** in Γ , it follows by the completeness theorem that if φ is a consequence of Γ , it is already a

consequence of a finite subset of Γ . This is called *compactness*. Equivalently, if every finite subset of Γ is consistent, then Γ itself must be consistent.

Although the compactness theorem follows from the completeness theorem via the detour through *derivations*, it is also possible to use the *the proof of* the completeness theorem to establish it directly. For what the proof does is take a set of *sentences* with a certain property—consistency—and constructs a *structure* out of this set that has certain properties (in this case, that it satisfies the set). Almost the very same construction can be used to directly establish compactness, by starting from “finitely satisfiable” sets of *sentences* instead of consistent ones.

7.2 Outline of the Proof

The proof of the completeness theorem is a bit complex, and upon first reading it, it is easy to get lost. So let us outline the proof. The first step is a shift of perspective, that allows us to see a route to a proof. When completeness is thought of as “whenever $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$,” it may be hard to even come up with an idea: for to show that $\Gamma \vdash \varphi$ we have to find a *derivation*, and it does not look like the hypothesis that $\Gamma \models \varphi$ helps us for this in any way. For some proof systems it is possible to directly construct a *derivation*, but we will take a slightly different tack. The shift in perspective required is this: completeness can also be formulated as: “if Γ is consistent, it has a model.” Perhaps we can use the information in Γ together with the hypothesis that it is consistent to construct a model. After all, we know what kind of model we are looking for: one that is as Γ describes it!

If Γ contains only *propositional variables*, it is easy to construct a model for it. All we have to do is come up with a *valuation* \mathfrak{v} such that $\mathfrak{v} \models pa$ for all $p \in \Gamma$. Well, let $\mathfrak{v}(p) = \mathbb{T}$ iff $p \in \Gamma$.

Now suppose Γ contains some *formula* $\neg\psi$, with ψ atomic. We might worry that the construction of \mathfrak{v} interferes with the possibility of making $\neg\psi$ true. But here’s where the consistency of Γ comes in: if $\neg\psi \in \Gamma$, then $\psi \notin \Gamma$, or else Γ would be inconsistent. And if $\psi \notin \Gamma$, then according to our construction of \mathfrak{v} , $\mathfrak{v} \not\models \psi$, so $\mathfrak{v} \models \neg\psi$. So far so good.

What if Γ contains complex, non-atomic formulas? Say it contains $\varphi \wedge \psi$. To make that true, we should proceed as if both φ and ψ were in Γ . And if $\varphi \vee \psi \in \Gamma$, then we will have to make at least one of them true, i.e., proceed as if one of them was in Γ .

This suggests the following idea: we add additional *formulas* to Γ so as to (a) keep the resulting set consistent and (b) make sure that for every possible atomic *sentence* φ , either φ is in the resulting set, or $\neg\varphi$ is, and (c) such that, whenever $\varphi \wedge \psi$ is in the set, so are both φ and ψ , if $\varphi \vee \psi$ is in the set, at least one of φ or ψ is also, etc. We keep doing this (potentially forever). Call the set of all *formulas* so added Γ^* . Then our construction above would provide us with a *structure* \mathfrak{v} for which we could prove, by induction, that all sentences in Γ^* are true in it, and hence also all sentence in Γ since $\Gamma \subseteq \Gamma^*$. It turns

out that guaranteeing (a) and (b) is enough. A set of sentences for which (b) holds is called *complete*. So our task will be to extend the consistent set Γ to a consistent and complete set Γ^* .

So here's what we'll do. First we investigate the properties of **complete** consistent sets, in particular we prove that a **complete** consistent set contains $\varphi \wedge \psi$ iff it contains both φ and ψ , $\varphi \vee \psi$ iff it contains at least one of them, etc. (Proposition 7.2). We'll then take the consistent set Γ and show that it can be extended to a consistent and **complete** set Γ^* (Lemma 7.3). This set Γ^* is what we'll use to define our **valuation** $\mathbf{v}(\Gamma^*)$. The valuation is determined by the **propositional variables** in Γ^* (Definition 7.4). We'll use the properties of complete consistent sets to show that indeed $\mathbf{v}(\Gamma^*) \models \varphi$ iff $\varphi \in \Gamma^*$ (Lemma 7.5), and thus in particular, $\mathbf{v}(\Gamma^*) \models \Gamma$.

7.3 Complete Consistent Sets of Sentences

pl:com:ccs:
sec

pl:com:ccs:
def:complete-set

Definition 7.1 (Complete set). A set Γ of **sentences** is *complete* iff for any **sentence** φ , either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Complete sets of sentences leave no questions unanswered. For any **sentence** A , Γ “says” if φ is true or false. The importance of **complete** sets extends beyond the proof of the completeness theorem. A theory which is **complete** and axiomatizable, for instance, is always decidable.

explanation

Complete consistent sets are important in the completeness proof since we can guarantee that every consistent set of **sentences** Γ is contained in a **complete** consistent set Γ^* . A **complete** consistent set contains, for each **sentence** φ , either φ or its negation $\neg\varphi$, but not both. This is true in particular for atomic **sentences**, so from a **complete** consistent set in a language suitably expanded by **constant symbols**, we can construct a **structure** where the interpretation of **predicate symbols** is defined according to which atomic **sentences** are in Γ^* . This **structure** can then be shown to make all **sentences** in Γ^* (and hence also all those in Γ) true. The proof of this latter fact requires that $\neg\varphi \in \Gamma^*$ iff $\varphi \notin \Gamma^*$, $(\varphi \vee \psi) \in \Gamma^*$ iff $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$, etc.

explanation

In what follows, we will often tacitly use the properties of reflexivity, monotonicity, and transitivity of \vdash (see sections 3.6, 4.5, 5.5 and 6.4).

pl:com:ccs:
prop:ccs

Proposition 7.2. Suppose Γ is **complete** and consistent. Then:

pl:com:ccs:
prop:ccs-prov-in

1. If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.

pl:com:ccs:
prop:ccs-and

2. $\varphi \wedge \psi \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$.

pl:com:ccs:
prop:ccs-or

3. $\varphi \vee \psi \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$.

pl:com:ccs:
prop:ccs-if

4. $\varphi \rightarrow \psi \in \Gamma$ iff either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.

Proof. Let us suppose for all of the following that Γ is **complete** and consistent.

1. If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.

Suppose that $\Gamma \vdash \varphi$. Suppose to the contrary that $\varphi \notin \Gamma$. Since Γ is **complete**, $\neg\varphi \in \Gamma$. By [Propositions 4.17, 5.17, 3.19](#) and [6.24](#), Γ is inconsistent. This contradicts the assumption that Γ is consistent. Hence, it cannot be the case that $\varphi \notin \Gamma$, so $\varphi \in \Gamma$.

2. $\varphi \wedge \psi \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$:

For the forward direction, suppose $\varphi \wedge \psi \in \Gamma$. Then by [Propositions 4.19, 5.19, 3.21](#) and [6.26](#), item (1), $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$. By (1), $\varphi \in \Gamma$ and $\psi \in \Gamma$, as required.

For the reverse direction, let $\varphi \in \Gamma$ and $\psi \in \Gamma$. By [Propositions 4.19, 5.19, 3.21](#) and [6.26](#), item (2), $\Gamma \vdash \varphi \wedge \psi$. By (1), $\varphi \wedge \psi \in \Gamma$.

3. First we show that if $\varphi \vee \psi \in \Gamma$, then either $\varphi \in \Gamma$ or $\psi \in \Gamma$. Suppose $\varphi \vee \psi \in \Gamma$ but $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since Γ is **complete**, $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$. By [Propositions 4.20, 5.20, 3.22](#) and [6.27](#), item (1), Γ is inconsistent, a contradiction. Hence, either $\varphi \in \Gamma$ or $\psi \in \Gamma$.

For the reverse direction, suppose that $\varphi \in \Gamma$ or $\psi \in \Gamma$. By [Propositions 4.20, 5.20, 3.22](#) and [6.27](#), item (2), $\Gamma \vdash \varphi \vee \psi$. By (1), $\varphi \vee \psi \in \Gamma$, as required.

4. For the forward direction, suppose $\varphi \rightarrow \psi \in \Gamma$, and suppose to the contrary that $\varphi \in \Gamma$ and $\psi \notin \Gamma$. On these assumptions, $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \in \Gamma$. By [Propositions 4.21, 5.21, 3.23](#) and [6.28](#), item (1), $\Gamma \vdash \psi$. But then by (1), $\psi \in \Gamma$, contradicting the assumption that $\psi \notin \Gamma$.

For the reverse direction, first consider the case where $\varphi \notin \Gamma$. Since Γ is **complete**, $\neg\varphi \in \Gamma$. By [Propositions 4.21, 5.21, 3.23](#) and [6.28](#), item (2), $\Gamma \vdash \varphi \rightarrow \psi$. Again by (1), we get that $\varphi \rightarrow \psi \in \Gamma$, as required.

Now consider the case where $\psi \in \Gamma$. By [Propositions 4.21, 5.21, 3.23](#) and [6.28](#), item (2) again, $\Gamma \vdash \varphi \rightarrow \psi$. By (1), $\varphi \rightarrow \psi \in \Gamma$.

□

Problem 7.1. Complete the proof of [Proposition 7.2](#).

7.4 Lindenbaum's Lemma

explanation We now prove a lemma that shows that any consistent set of **sentences** pl:com:lin:sec is contained in some set of sentences which is not just consistent, but also **complete**. The proof works by adding one **sentence** at a time, guaranteeing at each step that the set remains consistent. We do this so that for every φ , either φ or $\neg\varphi$ gets added at some stage. The union of all stages in that construction then contains either φ or its negation $\neg\varphi$ and is thus complete. It is also consistent, since we made sure at each stage not to introduce an inconsistency.

pl:com:lin:
lem:lindenbaum

Lemma 7.3 (Lindenbaum's Lemma). *Every consistent set Γ in a language \mathcal{L} can be extended to a complete and consistent set Γ^* .*

Proof. Let Γ be consistent. Let $\varphi_0, \varphi_1, \dots$ be an enumeration of all the sentences of \mathcal{L} . Define $\Gamma_0 = \Gamma$, and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_n\} & \text{otherwise.} \end{cases}$$

Let $\Gamma^* = \bigcup_{n \geq 0} \Gamma_n$.

Each Γ_n is consistent: Γ_0 is consistent by definition. If $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$, this is because the latter is consistent. If it isn't, $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}$. We have to verify that $\Gamma_n \cup \{\neg\varphi_n\}$ is consistent. Suppose it's not. Then both $\Gamma_n \cup \{\varphi_n\}$ and $\Gamma_n \cup \{\neg\varphi_n\}$ are inconsistent. This means that Γ_n would be inconsistent by Propositions 4.17, 5.17, 3.19 and 6.24, contrary to the induction hypothesis.

For every n and every $i < n$, $\Gamma_i \subseteq \Gamma_n$. This follows by a simple induction on n . For $n = 0$, there are no $i < 0$, so the claim holds automatically. For the inductive step, suppose it is true for n . We have $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ or $\Gamma_n \cup \{\neg\varphi_n\}$ by construction. So $\Gamma_n \subseteq \Gamma_{n+1}$. If $i < n$, then $\Gamma_i \subseteq \Gamma_n$ by inductive hypothesis, and so $\subseteq \Gamma_{n+1}$ by transitivity of \subseteq .

From this it follows that every finite subset of Γ^* is a subset of Γ_n for some n , since each $\psi \in \Gamma^*$ not already in Γ_0 is added at some stage i . If n is the last one of these, then all ψ in the finite subset are in Γ_n . So, every finite subset of Γ^* is consistent. By Propositions 4.14, 5.14, 3.16 and 6.18, Γ^* is consistent.

Every sentence of $\text{Frm}(\mathcal{L})$ appears on the list used to define Γ^* . If $\varphi_n \notin \Gamma^*$, then that is because $\Gamma_n \cup \{\varphi_n\}$ was inconsistent. But then $\neg\varphi_n \in \Gamma^*$, so Γ^* is complete. \square

7.5 Construction of a Model

pl:com:mod:
sec

We are now ready to define a valuation that makes all $\varphi \in \Gamma$ true. To do this, we first apply Lindenbaum's Lemma: we get a complete consistent $\Gamma^* \supseteq \Gamma$. We let the propositional variables in Γ^* determine $\mathbf{v}(\Gamma^*)$. explanation

pl:com:mod:
defn:termmodel

Definition 7.4. Suppose Γ^* is a complete consistent set of formulas. Then we let

$$\mathbf{v}(\Gamma^*)(p) = \begin{cases} \mathbb{T} & \text{if } p \in \Gamma^* \\ \mathbb{F} & \text{if } p \notin \Gamma^* \end{cases}$$

pl:com:mod:
lem:truth

Lemma 7.5 (Truth Lemma). $\mathbf{v}(\Gamma^*) \models \varphi$ iff $\varphi \in \Gamma^*$.

Proof. We prove both directions simultaneously, and by induction on φ .

1. $\varphi \equiv \perp$: $\mathbf{v}(\Gamma^*) \not\models \perp$ by definition of satisfaction. On the other hand, $\perp \notin \Gamma^*$ since Γ^* is consistent.

2. $\varphi \equiv \top$: $\mathfrak{v}(\Gamma^*) \models \top$ by definition of satisfaction. On the other hand, $\top \in \Gamma^*$ since Γ^* is consistent and **complete**, and $\Gamma^* \vdash \top$.
3. $\varphi \equiv p$: $\mathfrak{v}(\Gamma^*) \models p$ iff $\mathfrak{v}(\Gamma^*)(p) = \mathbb{T}$ (by the definition of satisfaction) iff $p \in \Gamma^*$ (by the construction of $\mathfrak{v}(\Gamma^*)$).
4. $\varphi \equiv \neg\psi$: $\mathfrak{v}(\Gamma^*) \models \varphi$ iff $\mathfrak{M}(\Gamma^*) \not\models \psi$ (by definition of satisfaction). By induction hypothesis, $\mathfrak{M}(\Gamma^*) \not\models \psi$ iff $\psi \notin \Gamma^*$. Since Γ^* is consistent and **complete**, $\psi \notin \Gamma^*$ iff $\neg\psi \in \Gamma^*$.
5. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{v}(\Gamma^*) \models \varphi$ iff we have both $\mathfrak{v}(\Gamma^*) \models \psi$ and $\mathfrak{v}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff both $\psi \in \Gamma^*$ and $\chi \in \Gamma^*$ (by the induction hypothesis). By [Proposition 7.2\(2\)](#), this is the case iff $(\psi \wedge \chi) \in \Gamma^*$.
6. $\varphi \equiv \psi \vee \chi$: $\mathfrak{v}(\Gamma^*) \models \varphi$ iff at $\mathfrak{v}(\Gamma^*) \models \psi$ or $\mathfrak{v}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \vee \chi) \in \Gamma^*$ (by [Proposition 7.2\(3\)](#)).
7. $\varphi \equiv \psi \rightarrow \chi$: $\mathfrak{v}(\Gamma^*) \models \varphi$ iff $\mathfrak{M}(\Gamma^*) \not\models \psi$ or $\mathfrak{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff $\psi \notin \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \rightarrow \chi) \in \Gamma^*$ (by [Proposition 7.2\(4\)](#)).

□

7.6 The Completeness Theorem

explanation

Let's combine our results: we arrive at the completeness theorem.

pl:com:cth:
sec

Theorem 7.6 (Completeness Theorem). *Let Γ be a set of **sentences**. If Γ is consistent, it is satisfiable.*

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thm:completeness

Proof. Suppose Γ is consistent. By [Lemma 7.3](#), there is a $\Gamma^* \supseteq \Gamma$ which is consistent and **complete**. By [Lemma 7.5](#), $\mathfrak{v}(\Gamma^*) \models \varphi$ iff $\varphi \in \Gamma^*$. From this it follows in particular that for all $\varphi \in \Gamma$, $\mathfrak{v}(\Gamma^*) \models \varphi$, so Γ is satisfiable. □

Corollary 7.7 (Completeness Theorem, Second Version). *For all Γ and φ **sentences**: if $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.*

pl:com:cth:
cor:completeness

Proof. Note that the Γ 's in [Corollary 7.7](#) and [Theorem 7.6](#) are universally quantified. To make sure we do not confuse ourselves, let us restate [Theorem 7.6](#) using a different variable: for any set of **sentences** Δ , if Δ is consistent, it is satisfiable. By contraposition, if Δ is not satisfiable, then Δ is inconsistent. We will use this to prove the corollary.

Suppose that $\Gamma \models \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable by [Proposition 1.15](#). Taking $\Gamma \cup \{\neg\varphi\}$ as our Δ , the previous version of [Theorem 7.6](#) gives us that $\Gamma \cup \{\neg\varphi\}$ is inconsistent. By [Propositions 4.16](#), [5.16](#), [3.18](#) and [6.23](#), $\Gamma \vdash \varphi$. □

Problem 7.2. Use [Corollary 7.7](#) to prove [Theorem 7.6](#), thus showing that the two formulations of the completeness theorem are equivalent.

Problem 7.3. In order for a **derivation** system to be complete, its rules must be strong enough to prove every unsatisfiable set inconsistent. Which of the rules of **derivation** were necessary to prove completeness? Are any of these rules not used anywhere in the proof? In order to answer these questions, make a list or diagram that shows which of the rules of **derivation** were used in which results that lead up to the proof of **Theorem 7.6**. Be sure to note any tacit uses of rules in these proofs.

7.7 The Compactness Theorem

pl:com:com:
sec One important consequence of the completeness theorem is the compactness theorem. The compactness theorem states that if each *finite* subset of a set of **sentences** is satisfiable, the entire set is satisfiable—even if the set itself is infinite. This is far from obvious. There is nothing that seems to rule out, at first glance at least, the possibility of there being infinite sets of **sentences** which are contradictory, but the contradiction only arises, so to speak, from the infinite number. The compactness theorem says that such a scenario can be ruled out: there are no unsatisfiable infinite sets of **sentences** each finite subset of which is satisfiable. Like the completeness theorem, it has a version related to entailment: if an infinite set of **sentences** entails something, already a finite subset does.

Definition 7.8. A set Γ of **formulas** is *finitely satisfiable* if and only if every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

pl:com:com:
thm:compactness **Theorem 7.9** (Compactness Theorem). *The following hold for any sentences Γ and φ :*

1. $\Gamma \models \varphi$ iff there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$.
2. Γ is satisfiable if and only if it is finitely satisfiable.

Proof. We prove (2). If Γ is satisfiable, then there is a **valuation** \mathbf{v} such that $\mathbf{v} \models \varphi$ for all $\varphi \in \Gamma$. Of course, this \mathbf{v} also satisfies every finite subset of Γ , so Γ is finitely satisfiable.

Now suppose that Γ is finitely satisfiable. Then every finite subset $\Gamma_0 \subseteq \Gamma$ is satisfiable. By soundness (**Corollaries 4.24, 5.26, 3.28 and 6.32**), every finite subset is consistent. Then Γ itself must be consistent by **Propositions 4.14, 5.14, 3.16 and 6.18**. By completeness (**Theorem 7.6**), since Γ is consistent, it is satisfiable. \square

Problem 7.4. Prove (1) of **Theorem 7.9**.

7.8 A Direct Proof of the Compactness Theorem

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sec

We can prove the Compactness Theorem directly, without appealing to the Completeness Theorem, using the same ideas as in the proof of the completeness theorem. In the proof of the Completeness Theorem we started with a consistent set Γ of sentences, expanded it to a consistent and complete set Γ^* of sentences, and then showed that in the valuation $\mathfrak{v}(\Gamma^*)$ constructed from Γ^* , all sentences of Γ are true, so Γ is satisfiable.

We can use the same method to show that a finitely satisfiable set of sentences is satisfiable. We just have to prove the corresponding versions of the results leading to the truth lemma where we replace “consistent” with “finitely satisfiable.”

Proposition 7.10. *Suppose Γ is complete and finitely satisfiable. Then:*

*pl:com:cpd:
prop:fsat-ccs*

1. $(\varphi \wedge \psi) \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$.
2. $(\varphi \vee \psi) \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$.
3. $(\varphi \rightarrow \psi) \in \Gamma$ iff either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.

Problem 7.5. Prove Proposition 7.10. Avoid the use of \vdash .

Lemma 7.11. *Every finitely satisfiable set Γ can be extended to a complete and finitely satisfiable set Γ^* .*

*pl:com:cpd:
lem:fsat-lindenbaum*

Problem 7.6. Prove Lemma 7.11. (Hint: the crucial step is to show that if Γ_n is finitely satisfiable, then either $\Gamma_n \cup \{\varphi_n\}$ or $\Gamma_n \cup \{\neg\varphi_n\}$ is finitely satisfiable.)

Theorem 7.12 (Compactness). *Γ is satisfiable if and only if it is finitely satisfiable.*

*pl:com:cpd:
thm:compactness-direct*

Proof. If Γ is satisfiable, then there is a valuation \mathfrak{v} such that $pSat_{\mathfrak{v}}\varphi$ for all $\varphi \in \Gamma$. Of course, this \mathfrak{v} also satisfies every finite subset of Γ , so Γ is finitely satisfiable.

Now suppose that Γ is finitely satisfiable. By Lemma 7.11, Γ can be extended to a complete and finitely satisfiable set Γ^* . Construct the valuation $\mathfrak{v}(\Gamma^*)$ as in Definition 7.4. The proof of the Truth Lemma (Lemma 7.5) goes through if we replace references to Proposition 7.2. \square

Problem 7.7. Write out the complete proof of the Truth Lemma (Lemma 7.5) in the version required for the proof of Theorem 7.12.

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Bibliography