# Part I

# Propositional Logic

This part contains material on classical propositional logic. The first chapter is relatively rudimenatry and just lists definitions and results, many proofs are not carried out but are left as exercises. The material on proof systems and the completeness theorem is included from the part on first-order logic, with the "FOL" tag set to false. This leaves out everything related to predicates, terms, and quantifiers, and replaces talk of structures $\mathfrak{M}$ with talk about valuations $\mathfrak{v}$.

It is planned to expand this part to include more detail, and to add further topics and results, such as truth-functional completeness.

# Chapter 1

# Syntax and Semantics

This is a very quick summary of definitions only. It should be expanded to provide a gentle intro to proofs by induction on formulas, with lots more examples.

## 1.1 Introduction

pl:syn:int:sec Propositional logic deals with formulas that are built from propositional variables using the propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. Intuitively, a propositional variable $p$ stands for a sentence or proposition that is be true or false. Whenever the "truth value" of the propositional variable in a formula are determined, so is the truth value of any formulas formed from them using propositional connectives. We say that propositional logic is *truth functional*, because its semantics is given by functions of truth values. In particular, in propositional logic we leave out of consideration any further determination of truth and falsity, e.g., whether something is necessarily true rather than just contingently true, or whether something is known to be true, or whether something is true now rather than was true or will be true. We only consider two truth values true ($\mathbb{T}$) and false ($\mathbb{F}$), and so exclude from discussion the possibility that a statement may be neither true nor false, or only half true. We also concentrate only on connectives where the truth value of a formula built from them is completely determined by the truth values of its parts (and not, say, on its meaning). In particular, whether the truth value of conditionals in English is truth functional in this sense is contentious. The material conditional $\rightarrow$ is; other logics deal with conditionals that are not truth functional.

In order to develop the theory and metatheory of truth-functional propositional logic, we must first define the syntax and semantics of its expressions. We will describe one way of constructing formulas from propositional variables using the connectives. Alternative definitions are possible. Other systems will chose different symbols, will select different sets of connectives as primitive, will use parentheses differently (or even not at all, as in the case of so-called Polish notation). What all approaches have in common, though, is that the formation rules define the set of formulas *inductively*. If done properly, every expression can result essentially in only one way according to the formation rules. The inductive definition resulting in expressions that are *uniquely readable* means we can give meanings to these expressions using the same method—inductive definition.

Giving the meaning of expressions is the domain of semantics. The central concept in semantics for propositonal logic is that of satisfaction in a valuation. A valuation $\mathfrak{v}$ assigns truth values $\mathbb{T}$, $\mathbb{F}$ to the propositional variables. Any valuation determines a truth value $\overline{\mathfrak{v}}(\varphi)$ for any formula $\varphi$. A formula is satisfied in a valuation $\mathfrak{v}$ iff $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$—we write this as $\mathfrak{v} \vDash \varphi$. This relation can also be defined by induction on the structure of $\varphi$, using the truth functions for the logical connectives to define, say, satisfaction of $\varphi \wedge \psi$ in terms of satisfaction (or not) of $\varphi$ and $\psi$.

On the basis of the satisfaction relation $\mathfrak{v} \vDash \varphi$ for sentences we can then define the basic semantic notions of tautology, entailment, and satisfiability. A formula is a tautology, $\vDash \varphi$, if every valuation satisfies it, i.e., $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ for any $\mathfrak{v}$. It is entailed by a set of formulas, $\Gamma \vDash \varphi$, if every valuation that satisfies all the formulas in $\Gamma$ also satisfies $\varphi$. And a set of formulas is satisfiable if some valuation satisfies all formulas in it at the same time. Because formulas are inductively defined, and satisfaction is in turn defined by induction on the structure of formulas, we can use induction to prove properties of our semantics and to relate the semantic notions defined.

## 1.2 Propositional Formulas

Formulas of propositional logic are built up from *propositional variables*, the propositional constant $\bot$ and the propositional constant $\top$ using *logical connectives*. pl:syn:fml:sec

1. A denumerable set $\mathrm{At}_0$ of propositional variables $p_0$, $p_1$, …

2. The propositional constant for falsity $\bot$.

3. The propositional constant for truth $\top$.

4. The logical connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (conditional), $\leftrightarrow$ (biconditional)

5. Punctuation marks: (, ), and the comma.

intro You may be familiar with different terminology and symbols than the ones we use above. Logic texts (and teachers) commonly use either $\sim$, $\neg$, and ! for "negation", $\wedge$, $\cdot$, and & for "conjunction". Commonly used symbols for the "conditional" or "implication" are $\rightarrow$, $\Rightarrow$, and $\supset$. Symbols for "biconditional," "bi-implication," or "(material) equivalence" are $\leftrightarrow$, $\Leftrightarrow$, and $\equiv$. The $\bot$ symbol is variously called "falsity," "falsum,", "absurdity,", or "bottom." The $\top$ symbol is variously called "truth," "verum,", or "top."

pl:syn:fml: defn:formulas **Definition 1.1** (Formula)**.** The set $\mathrm{Frm}(\mathcal{L}_0)$ of *formulas* of propositional logic is defined inductively as follows:

1. $\bot$ is an atomic formula.
2. $\top$ is an atomic formula.
3. Every propositional variable $p_i$ is an atomic formula.
4. If $\varphi$ is a formula, then $\neg\varphi$ is formula.
5. If $\varphi$ and $\psi$ are formulas, then $(\varphi \wedge \psi)$ is a formula.
6. If $\varphi$ and $\psi$ are formulas, then $(\varphi \vee \psi)$ is a formula.
7. If $\varphi$ and $\psi$ are formulas, then $(\varphi \rightarrow \psi)$ is a formula.
8. If $\varphi$ and $\psi$ are formulas, then $(\varphi \leftrightarrow \psi)$ is a formula.
9. If $\varphi$ is a formula and $x$ is a variable, then $\forall x\, \varphi$ is a formula.
10. If $\varphi$ is a formula and $x$ is a variable, then $\exists x\, \varphi$ is a formula.
11. Nothing else is a formula.

The definitions of the set of terms and that of formulas are *inductive definitions*. Essentially, we construct the set of formulas in infinitely many stages. In the initial stage, we pronounce all atomic formulas to be formulas; this corresponds to the first few cases of the definition, i.e., the cases for $\top$, $\bot$, $p_i$. "Atomic formula" thus means any formula of this form. explanation

The other cases of the definition give rules for constructing new formulas out of formulas already constructed. At the second stage, we can use them to construct formulas out of atomic formulas. At the third stage, we construct new formulas from the atomic formulas and those obtained in the second stage, and so on. A formula is anything that is eventually constructed at such a stage, and nothing else.

**Definition 1.2** (Syntactic identity)**.** The symbol $\equiv$ expresses syntactic identity between strings of symbols, i.e., $\varphi \equiv \psi$ iff $\varphi$ and $\psi$ are strings of symbols of the same length and which contain the same symbol in each place.

The $\equiv$ symbol may be flanked by strings obtained by concatenation, e.g., $\varphi \equiv (\psi \vee \chi)$ means: the string of symbols $\varphi$ is the same string as the one obtained by concatenating an opening parenthesis, the string $\psi$, the $\vee$ symbol, the string $\chi$, and a closing parenthesis, in this order. If this is the case, then we know that the first symbol of $\varphi$ is an opening parenthesis, $\varphi$ contains $\psi$ as a substring (starting at the second symbol), that substring is followed by $\vee$, etc.

## 1.3 Preliminaries

pl:syn:pre:sec

**Theorem 1.3** (*Principle of induction on formulas*)**.** *If some property $P$ holds for all the atomic formulas and is such that* pl:syn:pre:thm:induction

1. *it holds for $\neg\varphi$ whenever it holds for $\varphi$;*
2. *it holds for $(\varphi \wedge \psi)$ whenever it holds for $\varphi$ and $\psi$;*
3. *it holds for $(\varphi \vee \psi)$ whenever it holds for $\varphi$ and $\psi$;*
4. *it holds for $(\varphi \rightarrow \psi)$ whenever it holds for $\varphi$ and $\psi$;*
5. *it holds for $(\varphi \leftrightarrow \psi)$ whenever it holds for $\varphi$ and $\psi$;*

*then $P$ holds for all formulas.*

*Proof.* Let $S$ be the collection of all formulas with property $P$. Clearly $S \subseteq \mathrm{Frm}(\mathcal{L}_0)$. $S$ satisfies all the conditions of Definition 1.1: it contains all atomic formulas and is closed under the logical operators. $\mathrm{Frm}(\mathcal{L}_0)$ is the smallest such class, so $\mathrm{Frm}(\mathcal{L}_0) \subseteq S$. So $\mathrm{Frm}(\mathcal{L}_0) = S$, and every formula has propery $P$. □

**Proposition 1.4.** *Any formula in* $\mathrm{Frm}(\mathcal{L}_0)$ *is* balanced*, in that it has as many left parentheses as right ones.* pl:syn:pre:prop:balanced

**Problem 1.1.** Prove Proposition 1.4

**Proposition 1.5.** *No proper initial segment of a formula is a formula.* pl:syn:pre:prop:noinit

**Problem 1.2.** Prove Proposition 1.5

**Proposition 1.6** (Unique Readability)**.** *Any formula $\varphi$ in* $\mathrm{Frm}(\mathcal{L}_0)$ *has exactly one parsing as one of the following*

1. $\bot$.
2. $\top$.
3. $p_n$ *for some* $p_n \in \mathrm{At}_0$.
4. $\neg\psi$ *for some formula* $\psi$.
5. $(\psi \wedge \chi)$ *for some formulas* $\psi$ *and* $\chi$.
6. $(\psi \vee \chi)$ *for some formulas* $\psi$ *and* $\chi$.
7. $(\psi \rightarrow \chi)$ *for some formulas* $\psi$ *and* $\chi$.
8. $(\psi \leftrightarrow \chi)$ *for some formulas* $\psi$ *and* $\chi$.

*Moreover, this parsing is* unique*.*

*Proof.* By induction on $\varphi$. For instance, suppose that $\varphi$ has two distinct readings as $(\psi \to \chi)$ and $(\psi' \to \chi')$. Then $\psi$ and $\psi'$ must be the same (or else one would be a proper initial segment of the other); so if the two readings of $\varphi$ are distinct it must be because $\chi$ and $\chi'$ are distinct readings of the same sequence of symbols, which is impossible by the inductive hypothesis. □

**Definition 1.7** (Uniform Substitution)**.** If $\varphi$ and $\psi$ are formulas, and $p_i$ is a propositional variable, then $\varphi[\psi/p_i]$ denotes the result of replacing each occurrence of $p_i$ by an occurrence of $\psi$ in $\varphi$; similarly, the simultaneous substitution of $p_1, \dots, p_n$ by formulas $\psi_1, \dots, B_n$ is denoted by $\varphi[\psi_1/p_1, \dots, \psi_n/p_n]$.

**Problem 1.3.** Give a mathematically rigorous definition of $\varphi[\psi/p]$ by induction.

## 1.4 Valuations and Satisfaction

pl:syn:val:sec

**Definition 1.8** (Valuations)**.** Let $\{\mathbb{T}, \mathbb{F}\}$ be the set of the two truth values, "true" and "false." A *valuation* for $\mathcal{L}_0$ is a function $\mathfrak{v}$ assigning either $\mathbb{T}$ or $\mathbb{F}$ to the propositional variables of the language, i.e., $\mathfrak{v}\colon \mathrm{At}_0 \to \{\mathbb{T}, \mathbb{F}\}$.

**Definition 1.9.** Given a valuation $v$, define the evaluation function $\overline{\mathfrak{v}}(:)\mathrm{Frm}(\mathcal{L}_0) \to \{\mathbb{T}, \mathbb{F}\}$ inductively by:

$$\begin{aligned}
\overline{\mathfrak{v}}(\bot) &= \mathbb{F};\\
\overline{\mathfrak{v}}(\top) &= \mathbb{T};\\
\overline{\mathfrak{v}}(p_n) &= \mathfrak{v}(p_n);\\
\overline{\mathfrak{v}}(\neg\varphi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F};\\ \mathbb{F} & \text{otherwise.}\end{cases}\\
\overline{\mathfrak{v}}(\varphi \land \psi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T} \text{ and } \overline{\mathfrak{v}}(\psi) = \mathbb{T};\\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F} \text{ or } \overline{\mathfrak{v}}(\psi) = \mathbb{F}.\end{cases}\\
\overline{\mathfrak{v}}(\varphi \lor \psi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T} \text{ or } \overline{\mathfrak{v}}(\psi) = \mathbb{T};\\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F} \text{ and } \overline{\mathfrak{v}}(\psi) = \mathbb{F}.\end{cases}\\
\overline{\mathfrak{v}}(\varphi \to \psi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{F} \text{ or } \overline{\mathfrak{v}}(\psi) = \mathbb{T};\\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) = \mathbb{T} \text{ and } \overline{\mathfrak{v}}(\psi) = \mathbb{F}.\end{cases}\\
\overline{\mathfrak{v}}(\varphi \leftrightarrow \psi) &= \begin{cases} \mathbb{T} & \text{if } \overline{\mathfrak{v}}(\varphi) = \overline{\mathfrak{v}}(\psi);\\ \mathbb{F} & \text{if } \overline{\mathfrak{v}}(\varphi) \neq \overline{\mathfrak{v}}(\psi).\end{cases}
\end{aligned}$$

The valuation clauses correspond to the following truth tables: explanation

| $\varphi$ | $\psi$ | $\varphi \wedge \psi$ | $\varphi \vee \psi$ | $\varphi \rightarrow \psi$ | $\varphi \leftrightarrow \psi$ |
|---|---|---|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ |

**Theorem 1.10** (Local Determination)**.** *Suppose that $\mathfrak{v}_1$ and $\mathfrak{v}_2$ are valuations that agree on the propositional letters occurring in $\varphi$, i.e., $\overline{\mathfrak{v}_1}(p_n) = \overline{\mathfrak{v}_2}(p_n)$ whenever $p_n$ occurs in $\varphi$. Then they also agree on any $\varphi$, i.e., $\overline{\mathfrak{v}_1}(\varphi) = \overline{\mathfrak{v}_2}(\varphi)$.* pl:syn:val: thm:LocalDetermination

*Proof.* By induction on $\varphi$. □

**Definition 1.11** (Satisfaction)**.** Using the evaluation function, we can define the notion of *satisfaction of a formula $\varphi$ by a valuation* $\mathfrak{v}$, $\mathfrak{v} \vDash \varphi$, inductively as follows. (We write $\mathfrak{v} \nvDash \varphi$ to mean "not $\mathfrak{v} \vDash \varphi$.") pl:syn:val: defn:satisfaction

1. $\varphi \equiv \bot$: $\mathfrak{v} \nvDash \varphi$.
2. $\varphi \equiv \top$: $\mathfrak{v} \vDash \varphi$.
3. $\varphi \equiv p_i$: $\mathfrak{M} \vDash \varphi$ iff $\overline{\mathfrak{v}}(p_i) = \mathbb{T}$.
4. $\varphi \equiv \neg\psi$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \nvDash \psi$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \vDash \psi$ and $\mathfrak{v} \vDash \chi$.
6. $\varphi \equiv (\psi \vee \chi)$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \vDash \varphi$ or $\mathfrak{v} \vDash \psi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathfrak{v} \vDash \varphi$ iff $\mathfrak{v} \nvDash \psi$ or $\mathfrak{v} \vDash \chi$ (or both).
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{v} \vDash \varphi$ iff either both $\mathfrak{v} \vDash \psi$ and $\mathfrak{v} \vDash \chi$, or neither $\mathfrak{v} \vDash \psi$ nor $\mathfrak{v} \vDash \chi$.

If $\Gamma$ is a set of formulas, $\mathfrak{v} \vDash \Gamma$ iff $\mathfrak{v} \vDash \varphi$ for every $\varphi \in \Gamma$.

**Proposition 1.12.** $\mathfrak{v} \vDash \varphi$ *iff* $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$. pl:syn:val: prop:sat-value

*Proof.* By induction on $\varphi$. □

**Problem 1.4.** Prove Proposition 1.12

## 1.5 Semantic Notions

We define the following semantic notions: pl:syn:sem: sec

**Definition 1.13.**

1. A formula $\varphi$ is *satisfiable* if for some $\mathfrak{v}$, $\mathfrak{v} \vDash \varphi$; it is *unsatisfiable* if for no $\mathfrak{v}$, $\mathfrak{v} \vDash \varphi$;
2. A formula $\varphi$ is a *tautology* if $\mathfrak{v} \vDash \varphi$ for all valuations $v$;
3. A formula $\varphi$ is *contingent* if it is satisfiable but not a tautology;

4. If $\Gamma$ is a set of formulas, $\Gamma \vDash \varphi$ ("$\Gamma$ entails $\varphi$") if and only if $\mathfrak{v} \vDash \varphi$ for every valuation $\mathfrak{v}$ for which $\mathfrak{v} \vDash \Gamma$.

5. If $\Gamma$ is a set of formulas, $\Gamma$ is *satisfiable* if there is a valuation $\mathfrak{v}$ for which $\mathfrak{v} \vDash \Gamma$, and $\Gamma$ is *unsatisfiable* otherwise.

*pl:syn:sem: prop:semanticalfacts* **Proposition 1.14.**

1. *$\varphi$ is a tautology if and only if $\emptyset \vDash \varphi$;*

2. *If $\Gamma \vDash \varphi$ and $\Gamma \vDash \varphi \to \psi$ then $\Gamma \vDash \psi$;*

3. *If $\Gamma$ is satisfiable then every finite subset of $\Gamma$ is also satisfiable;*

*pl:syn:sem: def:Monotony* 4. *Monotony: if $\Gamma \subseteq \Delta$ and $\Gamma \vDash \varphi$ then also $\Delta \vDash \varphi$;*

*pl:syn:sem: def:Cut* 5. *Transitivity: if $\Gamma \vDash \varphi$ and $\Delta \cup \{\varphi\} \vDash \psi$ then $\Gamma \cup \Delta \vDash \psi$;*

*Proof.* Exercise. □

**Problem 1.5.** Prove Proposition 1.14

*pl:syn:sem: prop:entails-unsat* **Proposition 1.15.** *$\Gamma \vDash \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable;*

*Proof.* Exercise. □

**Problem 1.6.** Prove Proposition 1.15

*pl:syn:sem: thm:sem-deduction* **Theorem 1.16** (Semantic Deduction Theorem). *$\Gamma \vDash \varphi \to \psi$ if and only if $\Gamma \cup \{\varphi\} \vDash \psi$.*

*Proof.* Exercise. □

**Problem 1.7.** Prove Theorem 1.16

# Chapter 2

# Derivation Systems

This chapter collects general material on derivation systems. A textbook using a specific system can insert the introduction section plus the relevant survey section at the beginning of the chapter introducing that system.

## 2.1 Introduction

Logics commonly have both a semantics and a derivation system. The semantics concerns concepts such as truth, satisfiability, validity, and entailment. pl:prf:int:sec The purpose of derivation systems is to provide a purely syntactic method of establishing entailment and validity. They are purely syntactic in the sense that a derivation in such a system is a finite syntactic object, usually a sequence (or other finite arrangement) of sentences or formulas. Good derivation systems have the property that any given sequence or arrangement of sentences or formulas can be verified mechanically to be "correct."

The simplest (and historically first) derivation systems for first-order logic were *axiomatic*. A sequence of formulas counts as a derivation in such a system if each individual formula in it is either among a fixed set of "axioms" or follows from formulas coming before it in the sequence by one of a fixed number of "inference rules"—and it can be mechanically verified if a formula is an axiom and whether it follows correctly from other formulas by one of the inference rules. Axiomatic proof systems are easy to describe—and also easy to handle meta-theoretically—but derivations in them are hard to read and understand, and are also hard to produce.

Other derivation systems have been developed with the aim of making it easier to construct derivations or easier to understand derivations once they are complete. Examples are natural deduction, truth trees, also known as tableaux proofs, and the sequent calculus. Some derivation systems are designed especially with mechanization in mind, e.g., the resolution method is easy to implement in software (but its derivations are essentially impossible to understand). Most of these other proof systems represent derivations as trees of formulas rather than sequences. This makes it easier to see which parts of a derivation depend on which other parts.

So for a given logic, such as first-order logic, the different derivation systems will give different explications of what it is for a sentence to be a *theorem* and what it means for a sentence to be derivable from some others. However that is done (via axiomatic derivations, natural deductions, sequent derivations, truth trees, resolution refutations), we want these relations to match the semantic notions of validity and entailment. Let's write $\vdash \varphi$ for "$\varphi$ is a theorem" and "$\Gamma \vdash \varphi$" for "$\varphi$ is derivable from $\Gamma$." However $\vdash$ is defined, we want it to match up with $\vDash$, that is:

1. $\vdash \varphi$ if and only if $\vDash \varphi$
2. $\Gamma \vdash \varphi$ if and only if $\Gamma \vDash \varphi$

The "only if" direction of the above is called *soundness*. A derivation system is sound if derivability guarantees entailment (or validity). Every decent derivation system has to be sound; unsound derivation systems are not useful at all. After all, the entire purpose of a derivation is to provide a syntactic guarantee of validity or entailment. We'll prove soundness for the derivation systems we present.

The converse "if" direction is also important: it is called *completeness*. A complete derivation system is strong enough to show that $\varphi$ is a theorem whenever $\varphi$ is valid, and that there $\Gamma \vdash \varphi$ whenever $\Gamma \vDash \varphi$. Completeness is harder to establish, and some logics have no complete derivation systems. First-order logic does. Kurt Gödel was the first one to prove completeness for a derivation system of first-order logic in his 1929 dissertation.

Another concept that is connected to derivation systems is that of *consistency*. A set of sentences is called inconsistent if anything whatsoever can be derived from it, and consistent otherwise. Inconsistency is the syntactic counterpart to unsatisfiablity: like unsatisfiable sets, inconsistent sets of sentences do not make good theories, they are defective in a fundamental way. Consistent sets of sentences may not be true or useful, but at least they pass that minimal threshold of logical usefulness. For different derivation systems the specific definition of consistency of sets of sentences might differ, but like $\vdash$, we want consistency to coincide with its semantic counterpart, satisfiability. We want it to always be the case that $\Gamma$ is consistent if and only if it is satisfiable. Here, the "if" direction amounts to completeness (consistency guarantees satisfiability), and the "only if" direction amounts to soundness (satisfiability guarantees consistency). In fact, for classical first-order logic, the two versions of soundness and completeness are equivalent.

## 2.2 The Sequent Calculus

pl:prf:seq:sec

While many derivation systems operate with arrangements of sentences, the sequent calculus operates with *sequents*. A sequent is an expression of the form

$$\varphi_1, \dots, \varphi_m \Rightarrow \psi_1, \dots, \psi_m,$$

that is a pair of sequences of sentences, separated by the sequent symbol $\Rightarrow$. Either sequence may be empty. A derivation in the sequent calculus is a tree of sequents, where the topmost sequents are of a special form (they are called "initial sequents" or "axioms") and every other sequent follows from the sequents immediately above it by one of the rules of inference. The rules of inference either manipulate the sentences in the sequents (adding, removing, or rearranging them on either the left or the right), or they introduce a complex formula in the conclusion of the rule. For instance, the $\land$L rule allows the inference from $\varphi, \Gamma \Rightarrow \Delta$ to $A \land \psi, \Gamma \Rightarrow \Delta$, and the $\to$R allows the inference from $\varphi, \Gamma \Rightarrow \Delta, \psi$ to $\Gamma \Rightarrow \Delta, \varphi \to \psi$, for any $\Gamma$, $\Delta$, $\varphi$, and $\psi$. (In particular, $\Gamma$ and $\Delta$ may be empty.)

The $\vdash$ relation based on the sequent calculus is defined as follows: $\Gamma \vdash \varphi$ iff there is some sequence $\Gamma_0$ such that every $\varphi$ in $\Gamma_0$ is in $\Gamma$ and there is a derivation with the sequent $\Gamma_0 \Rightarrow \varphi$ at its root. $\varphi$ is a theorem in the sequent calculus if the sequent $\Rightarrow \varphi$ has a derivation. For instance, here is a derivation that shows that $\vdash (\varphi \land \psi) \to \varphi$:

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \land \psi \Rightarrow \varphi}\land\text{L}}{\Rightarrow (\varphi \land \psi) \to \varphi}\to\text{R}$$

A set $\Gamma$ is inconsistent in the sequent calculus if there is a derivation of $\Gamma_0 \Rightarrow$ (where every $\varphi \in \Gamma_0$ is in $\Gamma$ and the right side of the sequent is empty). Using the rule WR, any sentence can be derived from an inconsistent set.

The sequent calculus was invented in the 1930s by Gerhard Gentzen. Because of its systematic and symmetric design, it is a very useful formalism for developing a theory of derivations. It is relatively easy to find derivations in the sequent calculus, but these derivations are often hard to read and their connection to proofs are sometimes not easy to see. It has proved to be a very elegant approach to derivation systems, however, and many logics have sequent calculus systems.

## 2.3 Natural Deduction

Natural deduction is a derivation system intended to mirror actual reasoning (especially the kind of regimented reasoning employed by mathematicians). pl:prf:ntd:sec
Actual reasoning proceeds by a number of "natural" patterns. For instance, proof by cases allows us to establish a conclusion on the basis of a disjunctive premise, by establishing that the conclusion follows from either of the disjuncts. Indirect proof allows us to establish a conclusion by showing that its negation leads to a contradiction. Conditional proof establishes a conditional claim "if ...then ..." by showing that the consequent follows from the antecedent.

Natural deduction is a formalization of some of these natural inferences. Each of the logical connectives and quantifiers comes with two rules, an introduction and an elimination rule, and they each correspond to one such natural inference pattern. For instance, $\to$Intro corresponds to conditional proof, and $\lor$Elim to proof by cases. A particularly simple rule is $\land$Elim which allows the inference from $\varphi \land \psi$ to $\varphi$ (or $\psi$).

One feature that distinguishes natural deduction from other derivation systems is its use of assumptions. A derivation in natural deduction is a tree of formulas. A single formula stands at the root of the tree of formulas, and the "leaves" of the tree are formulas from which the conclusion is derived. In natural deduction, some leaf formulas play a role inside the derivation but are "used up" by the time the derivation reaches the conclusion. This corresponds to the practice, in actual reasoning, of introducing hypotheses which only remain in effect for a short while. For instance, in a proof by cases, we assume the truth of each of the disjuncts; in conditional proof, we assume the truth of the antecedent; in indirect proof, we assume the truth of the negation of the conclusion. This way of introducing hypothetical assumptions and then doing away with them in the service of establishing an intermediate step is a hallmark of natural deduction. The formulas at the leaves of a natural deduction derivation are called assumptions, and some of the rules of inference may "discharge" them. For instance, if we have a derivation of $\psi$ from some assumptions which include $\varphi$, then the $\to$Intro rule allows us to infer $\varphi \to \psi$ and discharge any assumption of the form $\varphi$. (To keep track of which assumptions are discharged at which inferences, we label the inference and the assumptions it discharges with a number.) The assumptions that remain undischarged at the end of the derivation are together sufficient for the truth of the conclusion, and so a derivation establishes that its undischarged assumptions entail its conclusion.

The relation $\Gamma \vdash \varphi$ based on natural deduction holds iff there is a derivation in which $\varphi$ is the last sentence in the tree, and every leaf which is undischarged is in $\Gamma$. $\varphi$ is a theorem in natural deduction iff there is a derivation in which $\varphi$ is the last sentence and all assumptions are discharged. For instance, here is a derivation that shows that $\vdash (\varphi \land \psi) \to \varphi$:

$$1\;\dfrac{\dfrac{[\varphi \land \psi]^1}{\varphi}\,\land\text{Elim}}{(\varphi \land \psi) \to \varphi}\,\to\text{Intro}$$

The label 1 indicates that the assumption $\varphi \land \psi$ is discharged at the $\to$Intro inference.

A set $\Gamma$ is inconsistent iff $\Gamma \vdash \bot$ in natural deduction. The rule $\bot_I$ makes it so that from an inconsistent set, any sentence can be derived.

Natural deduction systems were developed by Gerhard Gentzen and Stanisław Jaśkowski in the 1930s, and later developed by Dag Prawitz and Frederic Fitch. Because its inferences mirror natural methods of proof, it is favored by philosophers. The versions developed by Fitch are often used in introductory logic textbooks. In the philosophy of logic, the rules of natural deduction have sometimes been taken to give the meanings of the logical operators ("proof-theoretic semantics").

## 2.4 Tableaux

While many derivation systems operate with arrangements of sentences, tableaux operate with signed formulas. A signed formula is a pair consisting of a truth value sign ($\mathbb{T}$ or $\mathbb{F}$) and a sentence pl:prf:tab:sec

$$\mathbb{T}\,\varphi \text{ or } \mathbb{F}\,\varphi.$$

A tableau consists of signed formulas arranged in a downward-branching tree. It begins with a number of *assumptions* and continues with signed formulas which result from one of the signed formulas above it by applying one of the rules of inference. Each rule allows us to add one or more signed formulas to the end of a branch, or two signed formulas side by side—in this case a branch splits into two, with the two added signed formulas forming the ends of the two branches.

A rule applied to a complex signed formula results in the addition of signed formulas which are immediate sub-formulas. They come in pairs, one rule for each of the two signs. For instance, the $\wedge\mathbb{T}$ rule applies to $\mathbb{T}\,\varphi \wedge \psi$, and allows the addition of both the two signed formulas $\mathbb{T}\,\varphi$ and $\mathbb{T}\,\psi$ to the end of any branch containing $\mathbb{T}\,\varphi \wedge \psi$, and the rule $\varphi \wedge \psi\mathbb{F}$ allows a branch to be split by adding $\mathbb{F}\,\varphi$ and $\mathbb{F}\,\psi$ side-by-side. A tableau is closed if every one of its branches contains a matching pair of signed formulas $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$.

The $\vdash$ relation based on tableaux is defined as follows: $\Gamma \vdash \varphi$ iff there is some finite set $\Gamma_0 = \{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ such that there is a closed tableau for the assumptions

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

For instance, here is a closed tableau that shows that $\vdash (\varphi \wedge \psi) \to \varphi$:

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,(\varphi \wedge \psi) \to \varphi$ | Assumption |
| 2. | $\mathbb{T}\,\varphi \wedge \psi$ | $\to\mathbb{F}\,1$ |
| 3. | $\mathbb{F}\,\varphi$ | $\to\mathbb{F}\,1$ |
| 4. | $\mathbb{T}\,\varphi$ | $\to\mathbb{T}\,2$ |
| 5. | $\mathbb{T}\,\psi$ | $\to\mathbb{T}\,2$ |
| | $\otimes$ | |

A set $\Gamma$ is inconsistent in the tableau calculus if there is a closed tableau for assumptions

$$\{\mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

for some $\psi_i \in \Gamma$.

The sequent calculus was invented in the 1950s independently by Evert Beth and Jaakko Hintikka, and simplified and popularized by Raymond Smullyan. It is very easy to use, since constructing a tableau is a very systematic procedure. Because of the systematic nature of tableaux, they also lend themselves to implementation by computer. However, tableau is often hard to read and their connection to proofs are sometimes not easy to see. The approach is also quite general, and many different logics have tableau systems. Tableaux also help us to find structures that satisfy given (sets of) sentences: if the set is satisfiable, it won't have a closed tableau, i.e., any tableau will have an open branch. The satisfying structure can be "read off" an open branch, provided all rules it is possible to apply have been applied on that branch. There is also a very close connection to the sequent calculus: essentially, a closed tableau is a condensed derivation in the sequent calculus, written upside-down.

## 2.5 Axiomatic Derivations

pl:prf:axd:sec Axiomatic derivations are the oldest and simplest logical derivation systems. Its derivations are simply sequences of sentences. A sequence of sentences conunts as a correct derivation if every sentence $\varphi$ in it satisfies one of the following conditions:

1. $\varphi$ is an axiom, or
2. $\varphi$ is an element of a given set $\Gamma$ of sentences, or
3. $\varphi$ is justified by a rule of inference.

To be an axiom, $\varphi$ has to have the form of on of a number of fixed sentence schemas. There are many sets of axiom schemas that provide a satisfactory (sound and complete) derivation system for first-order logic. Some are organized according to the connectives they govern, e.g., the schemas

$$\varphi \to (\psi \to \varphi) \qquad \psi \to (\psi \lor \chi) \qquad (\psi \land \chi) \to \psi$$

are common axioms that govern $\to$, $\lor$ and $\land$. Some axiom systems aim at a minimal number of axioms. Depending on the connectives that are taken as primitives, it is even possible to find axiom systems that consist of a single axiom.

A rule of inference is a conditional statement that gives a sufficient condition for a sentence in a derivation to be justified. Modus ponens is one very common such rule: it says that if $\varphi$ and $\varphi \to \psi$ are already justified, then $\psi$ is justified. This means that a line in a derivation containing the sentence $\psi$ is justified, provided that both $\varphi$ and $\varphi \to \psi$ (for some sentence $\varphi$) appear in the derivation before $\psi$.

The $\vdash$ relation based on axiomatic derivations is defined as follows: $\Gamma \vdash \varphi$ iff there is a derivation with the sentence $\varphi$ as its last formula (and $\Gamma$ is taken as the set of sentences in that derivation which are justified by (2) above). $\varphi$ is a theorem if $\varphi$ has a derivation where $\Gamma$ is empty, i.e., every sentence in the derivation is justfied either by (1) or (3). For instance, here is a derivation that shows that $\vdash \varphi \to (\psi \to (\psi \lor \varphi))$:

1. $\psi \to (\psi \lor \varphi)$
2. $(\psi \to (\psi \lor \varphi)) \to (\varphi \to (\psi \to (\psi \lor \varphi)))$
3. $\varphi \to (\psi \to (\psi \lor \varphi))$

The sentence on line 1 is of the form of the axiom $\varphi \to (\varphi \lor \psi)$ (with the roles of $\varphi$ and $\psi$ reversed). The sentence on line 2 is of the form of the axiom $\varphi \to (\psi \to \varphi)$. Thus, both lines are justified. Line 3 is justified by modus ponens: if we abbreviate it as $\theta$, then line 2 has the form $\chi \to \theta$, where $\chi$ is $\psi \to (\psi \lor \varphi)$, i.e., line 1.

A set $\Gamma$ is inconsistent if $\Gamma \vdash \bot$. A complete axiom system will also prove that $\bot \to \varphi$ for any $\varphi$, and so if $\Gamma$ is inconsistent, then $\Gamma \vdash \varphi$ for any $\varphi$.

Systems of axiomatic derivations for logic were first given by Gottlob Frege in his 1879 *Begriffsschrift*, which for this reason is often considered the first work of modern logic. They were perfected in Alfred North Whitehead and Bertrand Russell's *Principia Mathematica* and by David Hilbert and his students in the 1920s. They are thus often called "Frege systems" or "Hilbert systems." They are very versatile in that it is often easy to find an axiomatic system for a logic. Because derivations have a very simple structure and only one or two inference rules, it is also relatively easy to prove things *about* them. However, they are very hard to use in practice, i.e., it is difficult to find and write proofs.

# Chapter 3

# The Sequent Calculus

This chapter presents Gentzen's standard sequent calculus LK for classical first-order logic. It could use more examples and exercises. To include or exclude material relevant to the sequent calculus as a proof system, use the "prfLK" tag.

## 3.1 Rules and Derivations

pl:seq:rul:sec For the following, let $\Gamma, \Delta, \Pi, \Lambda$ represent finite sequences of sentences.

**Definition 3.1** (Sequent). A *sequent* is an expression of the form

$$\Gamma \Rightarrow \Delta$$

where $\Gamma$ and $\Delta$ are finite (possibly empty) sequences of sentences of the language $\mathcal{L}$. $\Gamma$ is called the *antecedent*, while $\Delta$ is the *succedent*.

The intuitive idea behind a sequent is: if all of the sentences in the antecedent hold, then at least one of the sentences in the succedent holds. That is, if $\Gamma = \langle \varphi_1, \ldots, \varphi_m \rangle$ and $\Delta = \langle \psi_1, \ldots, \psi_n \rangle$, then $\Gamma \Rightarrow \Delta$ holds iff explanation

$$(\varphi_1 \wedge \cdots \wedge \varphi_m) \to (\psi_1 \vee \cdots \vee \psi_n)$$

holds. There are two special cases: where $\Gamma$ is empty and when $\Delta$ is empty. When $\Gamma$ is empty, i.e., $m = 0$, $\Rightarrow \Delta$ holds iff $\psi_1 \vee \cdots \vee \psi_n$ holds. When $\Delta$ is empty, i.e., $n = 0$, $\Gamma \Rightarrow$ holds iff $\neg(\varphi_1 \wedge \cdots \wedge \varphi_m)$ does. We say a sequent is valid iff the corresponding sentence is valid.

If $\Gamma$ is a sequence of sentences, we write $\Gamma, \varphi$ for the result of appending $\varphi$ to the right end of $\Gamma$ (and $\varphi, \Gamma$ for the result of appending $\varphi$ to the left end of $\Gamma$). If $\Delta$ is a sequence of sentences also, then $\Gamma, \Delta$ is the concatenation of the two sequences.

**Definition 3.2** (Initial Sequent)**.** An *initial sequent* is a sequent of one of the following forms:

1. $\varphi \Rightarrow \varphi$
2. $\Rightarrow \top$
3. $\bot \Rightarrow$

for any sentence $\varphi$ in the language.

Derivations in the sequent calculus are certain trees of sequents, where the topmost sequents are initial sequents, and if a sequent stands below one or two other sequents, it must follow correctly by a rule of inference. The rules for **LK** are divided into two main types: *logical* rules and *structural* rules. The logical rules are named for the main operator of the sentence containing $\varphi$ and/or $\psi$ in the lower sequent. Each one comes in two versions, one for inferring a sequent with the sentence containg the logical operator on the left, and one with the sentence on the right.

## 3.2 Propositional Rules

### Rules for ¬ pl:seq:prl:sec

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\ \neg\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ \neg\mathrm{R}$$

### Rules for ∧

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \land \psi, \Gamma \Rightarrow \Delta}\ \land\mathrm{L}$$

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \land \psi, \Gamma \Rightarrow \Delta}\ \land\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \land \psi}\ \land\mathrm{R}$$

### Rules for ∨

$$\frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \lor \psi, \Gamma \Rightarrow \Delta}\ \lor\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \lor \psi}\ \lor\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \lor \psi}\ \lor\mathrm{R}$$

### Rules for $\rightarrow$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Pi \Rightarrow \Lambda}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \rightarrow\text{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \rightarrow\text{R}$$

## 3.3 Structural Rules

pl:seq:srl:sec We also need a few rules that allow us to rearrange sentences in the left and right side of a sequent. Since the logical rules require that the sentences in the premise which the rule acts upon stand either to the far left or to the far right, we need an "exchange" rule that allows us to move sentences to the right position. It's also important sometimes to be able to combine two identical sentences into one, and to add a sentence on either side.

### Weakening

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{WL} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \text{WR}$$

### Contraction

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{CL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \text{CR}$$

### Exchange

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \text{XL} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda} \text{XR}$$

A series of weakening, contraction, and exchange inferences will often be indicated by double inference lines.

The following rule, called "cut," is not strictly speaking necessary, but makes it a lot easier to reuse and combine derivations.

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \text{Cut}$$

## 3.4 Derivations

explanation We've said what an initial sequent looks like, and we've given the rules of inference. Derivations in the sequent calculus are inductively generated from these: each derivation either is an initial sequent on its own, or consists of one or two derivations followed by an inference. pl:seq:der:sec

**Definition 3.3** (**LK** derivation)**.** An **LK**-*derivation* of a sequent $S$ is a tree of sequents satisfying the following conditions:

1. The topmost sequents of the tree are initial sequents.
2. The bottommost sequent of the tree is $S$.
3. Every sequent in the tree except $S$ is a premise of a correct application of an inference rule whose conclusion stands directly below that sequent in the tree.

We then say that $S$ is the *end-sequent* of the derivation and that $S$ is *derivable in* **LK** (or **LK**-derivable).

**Example 3.4.** Every initial sequent, e.g., $\chi \Rightarrow \chi$ is a derivation. We can obtain a new derivation from this by applying, say, the WL rule,

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{WL}$$

The rule, however, is meant to be general: we can replace the $\varphi$ in the rule with any sentence, e.g., also with $\theta$. If the premise matches our initial sequent $\chi \Rightarrow \chi$, that means that both $\Gamma$ and $\Delta$ are just $\chi$, and the conclusion would then be $\theta, \chi \Rightarrow \chi$. So, the following is a derivation:

$$\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\,\mathrm{WL}$$

We can now apply another rule, say XL, which allows us to switch two sentences on the left. So, the following is also a correct derivation:

$$\cfrac{\cfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\,\mathrm{WL}}{\chi, \theta \Rightarrow \chi}\,\mathrm{XL}$$

In this application of the rule, which was given as

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta,}\,\mathrm{XL}$$

both $\Gamma$ and $\Pi$ were empty, $\Delta$ is $\chi$, and the roles of $\varphi$ and $\psi$ are played by $\theta$ and $\chi$, respectively. In much the same way, we also see that

$$\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta}\,\mathrm{WL}$$

is a derivation. Now we can take these two derivations, and combine them using $\land$R. That rule was

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \land \psi} \land\text{R}$$

In our case, the premises must match the last sequents of the derivations ending in the premises. That means that $\Gamma$ is $\chi, \theta$, $\Delta$ is empty, $\varphi$ is $\chi$ and $\psi$ is $\theta$. So the conclusion, if the inference should be correct, is $\chi, \theta \Rightarrow \chi \land \theta$. Of course, we can also reverse the premises, then $\varphi$ would be $\theta$ and $\psi$ would be $\chi$. So both of the following are correct derivations.

$$\dfrac{\dfrac{\dfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\text{WL}}{\chi, \theta \Rightarrow \chi}\text{XL} \qquad \dfrac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta}\text{WL}}{\chi, \theta \Rightarrow \chi \land \theta}\land\text{R} \qquad \dfrac{\dfrac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta}\text{WL} \qquad \dfrac{\dfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\text{WL}}{\chi, \theta \Rightarrow \chi}\text{XL}}{\chi, \theta \Rightarrow \theta \land \chi}\land\text{R}$$

## 3.5 Examples of Derivations

pl:seq:pro: sec

**Example 3.5.** Give an **LK**-derivation for the sequent $\varphi \land \psi \Rightarrow \varphi$.

We begin by writing the desired end-sequent at the bottom of the derivation.

$$\dfrac{}{\varphi \land \psi \Rightarrow \varphi}$$

Next, we need to figure out what kind of inference could have a lower sequent of this form. This could be a structural rule, but it is a good idea to start by looking for a logical rule. The only logical connective occurring in the lower sequent is $\land$, so we're looking for an $\land$ rule, and since the $\land$ symbol occurs in the antecedent, we're looking at the $\land$L rule.

$$\dfrac{}{\varphi \land \psi \Rightarrow \varphi}\land\text{L}$$

There are two options for what could have been the upper sequent of the $\land$L inference: we could have an upper sequent of $\varphi \Rightarrow \varphi$, or of $\psi \Rightarrow \varphi$. Clearly, $\varphi \Rightarrow \varphi$ is an initial sequent (which is a good thing), while $\psi \Rightarrow \varphi$ is not derivable in general. We fill in the upper sequent:

$$\dfrac{\varphi \Rightarrow \varphi}{\varphi \land \psi \Rightarrow \varphi}\land\text{L}$$

We now have a correct **LK**-derivation of the sequent $\varphi \land \psi \Rightarrow \varphi$.

**Example 3.6.** Give an **LK**-derivation for the sequent $\neg\varphi \lor \psi \Rightarrow \varphi \rightarrow \psi$.

Begin by writing the desired end-sequent at the bottom of the derivation.

$$\dfrac{}{\neg\varphi \lor \psi \Rightarrow \varphi \rightarrow \psi}$$

To find a logical rule that could give us this end-sequent, we look at the logical connectives in the end-sequent: $\neg$, $\lor$, and $\to$. We only care at the moment about $\lor$ and $\to$ because they are main operators of sentences in the end-sequent, while $\neg$ is inside the scope of another connective, so we will take care of it later. Our options for logical rules for the final inference are therefore the $\lor$L rule and the $\to$R rule. We could pick either rule, really, but let's pick the $\to$R rule (if for no reason other than it allows us to put off splitting into two branches). According to the form of $\to$R inferences which can yield the lower sequent, this must look like:

$$\dfrac{\dfrac{}{\varphi, \neg\varphi \lor \psi \Rightarrow \psi}}{\neg\varphi \lor \psi \Rightarrow \varphi \to \psi}\,{\to}\mathrm{R}$$

If we move $\neg\varphi \lor \psi$ to the outside of the antecedent, we can apply the $\lor$L rule. According to the schema, this must split into two upper sequents as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\varphi, \varphi \Rightarrow \psi} \qquad \dfrac{}{\psi, \varphi \Rightarrow \psi}}{\neg\varphi \lor \psi, \varphi \Rightarrow \psi}\,{\lor}\mathrm{L}}{\varphi, \neg\varphi \lor \psi \Rightarrow \psi}\,\mathrm{XR}}{\neg\varphi \lor \psi \Rightarrow \varphi \to \psi}\,{\to}\mathrm{R}$$

Remember that we are trying to wind our way up to initial sequents; we seem to be pretty close! The right branch is just one weakening and one exchange away from an initial sequent and then it is done:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\varphi, \varphi \Rightarrow \psi} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi}\,\mathrm{WL}}{\psi, \varphi \Rightarrow \psi}\,\mathrm{XL}}{\neg\varphi \lor \psi, \varphi \Rightarrow \psi}\,{\lor}\mathrm{L}}{\varphi, \neg\varphi \lor \psi \Rightarrow \psi}\,\mathrm{XR}}{\neg\varphi \lor \psi \Rightarrow \varphi \to \psi}\,{\to}\mathrm{R}$$

Now looking at the left branch, the only logical connective in any sentence is the $\neg$ symbol in the antecedent sentences, so we're looking at an instance of the $\neg$L rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \psi, \varphi}}{\neg\varphi, \varphi \Rightarrow \psi}\,\neg\mathrm{L} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi}\,\mathrm{WL}}{\psi, \varphi \Rightarrow \psi}\,\mathrm{XL}}{\neg\varphi \lor \psi, \varphi \Rightarrow \psi}\,{\lor}\mathrm{L}}{\varphi, \neg\varphi \lor \psi \Rightarrow \psi}\,\mathrm{XR}}{\neg\varphi \lor \psi \Rightarrow \varphi \to \psi}\,{\to}\mathrm{R}$$

Similarly to how we finished off the right branch, we are just one weakening and one exchange away from finishing off this left branch as well.

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{WR}
\UnaryInfC{$\varphi \Rightarrow \varphi, \psi$}
\RightLabel{XR}
\UnaryInfC{$\varphi \Rightarrow \psi, \varphi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\varphi, \varphi \Rightarrow \psi$}
\AxiomC{$\psi \Rightarrow \psi$}
\RightLabel{WL}
\UnaryInfC{$\varphi, \psi \Rightarrow \psi$}
\RightLabel{XL}
\UnaryInfC{$\psi, \varphi \Rightarrow \psi$}
\RightLabel{$\lor$L}
\BinaryInfC{$\neg\varphi \lor \psi, \varphi \Rightarrow \psi$}
\RightLabel{XR}
\UnaryInfC{$\varphi, \neg\varphi \lor \psi \Rightarrow \psi$}
\RightLabel{$\rightarrow$R}
\UnaryInfC{$\neg\varphi \lor \psi \Rightarrow \varphi \rightarrow \psi$}
\end{prooftree}
$$

**Example 3.7.** Give an **LK**-derivation of the sequent $\neg\varphi \lor \neg\psi \Rightarrow \neg(\varphi \land \psi)$

Using the techniques from above, we start by writing the desired end-sequent at the bottom.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\neg\varphi \lor \neg\psi \Rightarrow \neg(\varphi \land \psi)$}
\end{prooftree}
$$

The available main connectives of sentences in the end-sequent are the $\lor$ symbol and the $\neg$ symbol. It would work to apply either the $\lor$L or the $\neg$R rule here, but we start with the $\neg$R rule because it avoids splitting up into two branches for a moment:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\varphi \land \psi, \neg\varphi \lor \neg\psi \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\neg\varphi \lor \neg\psi \Rightarrow \neg(\varphi \land \psi)$}
\end{prooftree}
$$

Now we have a choice of whether to look at the $\land$L or the $\lor$L rule. Let's see what happens when we apply the $\land$L rule: we have a choice to start with either the sequent $\varphi, \neg\varphi \lor \psi \Rightarrow$ or the sequent $\psi, \neg\varphi \lor \psi \Rightarrow$ . Since the proof is symmetric with regards to $\varphi$ and $\psi$, let's go with the former:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\varphi, \neg\varphi \lor \neg\psi \Rightarrow$}
\RightLabel{$\land$L}
\UnaryInfC{$\varphi \land \psi, \neg\varphi \lor \neg\psi \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\neg\varphi \lor \neg\psi \Rightarrow \neg(\varphi \land \psi)$}
\end{prooftree}
$$

Continuing to fill in the derivation, we see that we run into a problem:

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\varphi, \varphi \Rightarrow$}
\AxiomC{}
\RightLabel{?}
\UnaryInfC{$\varphi \Rightarrow \psi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\psi, \varphi \Rightarrow$}
\RightLabel{$\lor$L}
\BinaryInfC{$\neg\varphi \lor \neg\psi, \varphi \Rightarrow$}
\RightLabel{XL}
\UnaryInfC{$\varphi, \neg\varphi \lor \neg\psi \Rightarrow$}
\RightLabel{$\land$L}
\UnaryInfC{$\varphi \land \psi, \neg\varphi \lor \neg\psi \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\neg\varphi \lor \neg\psi \Rightarrow \neg(\varphi \land \psi)$}
\end{prooftree}
$$

The top of the right branch cannot be reduced any further, and it cannot be brought by way of structural inferences to an initial sequent, so this is not the right path to take. So clearly, it was a mistake to apply the $\land$L rule above. Going back to what we had before and carrying out the $\lor$L rule instead, we get

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\neg\varphi, \varphi \wedge \psi \Rightarrow$}
\AxiomC{}
\UnaryInfC{$\neg\psi, \varphi \wedge \psi \Rightarrow$}
\RightLabel{$\vee$L}
\BinaryInfC{$\neg\varphi \vee \neg\psi, \varphi \wedge \psi \Rightarrow$}
\RightLabel{XL}
\UnaryInfC{$\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$}
\end{prooftree}
$$

Completing each branch as we've done before, we get

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$\wedge$L}
\UnaryInfC{$\varphi \wedge \psi \Rightarrow \varphi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\varphi, \varphi \wedge \psi \Rightarrow$}
\AxiomC{$\psi \Rightarrow \psi$}
\RightLabel{$\wedge$L}
\UnaryInfC{$\varphi \wedge \psi \Rightarrow \psi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\psi, \varphi \wedge \psi \Rightarrow$}
\RightLabel{$\vee$L}
\BinaryInfC{$\neg\varphi \vee \neg\psi, \varphi \wedge \psi \Rightarrow$}
\RightLabel{XL}
\UnaryInfC{$\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$}
\end{prooftree}
$$

(We could have carried out the $\wedge$ rules lower than the $\neg$ rules in these steps and still obtained a correct derivation).

**Example 3.8.** So far we haven't used the contraction rule, but it is sometimes required. Here's an example where that happens. Suppose we want to prove $\Rightarrow A \vee \neg\varphi$. Applying $\vee$R backwards would give us one of these two derivations:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\Rightarrow \varphi$}
\RightLabel{$\vee$R}
\UnaryInfC{$\Rightarrow \varphi \vee \neg\varphi$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\varphi \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\Rightarrow \neg\varphi$}
\RightLabel{$\vee$R}
\UnaryInfC{$\Rightarrow \varphi \vee \neg\varphi$}
\end{prooftree}
$$

Neither of these of course ends in an initial sequent. The trick is to realize that the contraction rule allows us to combine two copies of a sentence into one—and when we're searching for a proof, i.e., going from bottom to top, we can keep a copy of $\varphi \vee \neg\varphi$ in the premise, e.g.,

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\Rightarrow \varphi \vee \neg\varphi, \varphi$}
\RightLabel{$\vee$R}
\UnaryInfC{$\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi$}
\RightLabel{CR}
\UnaryInfC{$\Rightarrow \varphi \vee \neg\varphi$}
\end{prooftree}
$$

Now we can apply $\vee$R a second time, and also get $\neg\varphi$, which leads to a complete derivation.

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$\neg$R}
\UnaryInfC{$\Rightarrow \varphi, \neg\varphi$}
\RightLabel{$\vee$R}
\UnaryInfC{$\Rightarrow \varphi, \varphi \vee \neg\varphi$}
\RightLabel{XR}
\UnaryInfC{$\Rightarrow \varphi \vee \neg\varphi, \varphi$}
\RightLabel{$\vee$R}
\UnaryInfC{$\Rightarrow \varphi \vee \neg\varphi, \varphi \vee \neg\varphi$}
\RightLabel{CR}
\UnaryInfC{$\Rightarrow \varphi \vee \neg\varphi$}
\end{prooftree}
$$

**Problem 3.1.** Give derivations of the following sequents:

1. $\Rightarrow \neg(\varphi \to \psi) \to (\varphi \wedge \neg\psi)$

2. $(\varphi \wedge \psi) \rightarrow \chi \Rightarrow (\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)$

This section collects the definitions of the provability relation and consistency for natural deduction.

## 3.6 Proof-Theoretic Notions

pl:seq:ptn:sec Just as we've defined a number of important semantic notions (validity, entailment, satisfiabilty), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sequents. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorem*. explanation

**Definition 3.9** (Theorems)**.** A sentence $\varphi$ is a *theorem* if there is a derivation in **LK** of the sequent $\Rightarrow \varphi$. We write $\vdash \varphi$ if $\varphi$ is a theorem and $\nvdash \varphi$ if it is not.

**Definition 3.10** (Derivability)**.** A sentence $\varphi$ is *derivable from* a set of sentences $\Gamma$, $\Gamma \vdash \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence $\Gamma_0'$ of the sentences in $\Gamma_0$ such that **LK** derives $\Gamma_0' \Rightarrow \varphi$. If $\varphi$ is not derivable from $\Gamma$ we write $\Gamma \nvdash \varphi$.

Because of the contraction, weakening, and exchange rules, the order and number of sentences in $\Gamma_0'$ does not matter: if a sequent $\Gamma_0' \Rightarrow \varphi$ is derivable, then so is $\Gamma_0'' \Rightarrow \varphi$ for any $\Gamma_0''$ that contains the same sentences as $\Gamma_0'$. For instance, if $\Gamma_0 = \{\psi, \chi\}$ then both $\Gamma_0' = \langle \psi, \psi, \chi \rangle$ and $\Gamma_0'' = \langle \chi, \chi, \psi \rangle$ are sequences containing just the sentences in $\Gamma_0$. If a sequent containing one is derivable, so is the other, e.g.:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \psi, \psi, \chi \Rightarrow \varphi\end{matrix}}{\psi, \chi \Rightarrow \varphi}\,\text{CL}}{\chi, \psi \Rightarrow \varphi}\,\text{XL}}{\chi, \chi, \psi \Rightarrow \varphi}\,\text{WL}$$

From now on we'll say that if $\Gamma_0$ is a finite set of sentences then $\Gamma_0 \Rightarrow \varphi$ is any sequent where the antecedent is a sequence of sentences in $\Gamma_0$ and tacitly include contractions, exchanges, and weakenings if necessary.

**Definition 3.11** (Consistency)**.** A set of sentences $\Gamma$ is *inconsistent* iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow$ . If $\Gamma$ is not inconsistent, i.e., if for every finite $\Gamma_0 \subseteq \Gamma$, **LK** does not derive $\Gamma_0 \Rightarrow$ , we say it is *consistent*.

**Proposition 3.12** (Reflexivity)**.** *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.* *pl:seq:ptn: prop:reflexivity*

*Proof.* The initial sequent $\varphi \Rightarrow \varphi$ is derivable, and $\{\varphi\} \subseteq \Gamma$. □

**Proposition 3.13** (Monotony)**.** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.* *pl:seq:ptn: prop:monotony*

*Proof.* Suppose $\Gamma \vdash \varphi$, i.e., there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \Rightarrow \varphi$ is derivable. Since $\Gamma \subseteq \Delta$, then $\Gamma_0$ is also a finite subset of $\Delta$. The derivation of $\Gamma_0 \Rightarrow \varphi$ thus also shows $\Delta \vdash \varphi$. □

**Proposition 3.14** (Transitivity)**.** *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.* *pl:seq:ptn: prop:transitivity*

*Proof.* If $\Gamma \vdash \varphi$, there is a finite $\Gamma_0 \subseteq \Gamma$ and a derivation $\pi_0$ of $\Gamma_0 \Rightarrow \varphi$. If $\{\varphi\} \cup \Delta \vdash \psi$, then for some finite subset $\Delta_0 \subseteq \Delta$, there is a derivation $\pi_1$ of $\varphi, \Delta_0 \Rightarrow \psi$. Consider the following derivation:

$$\dfrac{\begin{matrix}\vdots\,\pi_0 \\ \Gamma_0 \Rightarrow \varphi\end{matrix} \qquad \begin{matrix}\vdots\,\pi_1 \\ \varphi, \Delta_0 \Rightarrow \psi\end{matrix}}{\Gamma_0, \Delta_0 \Rightarrow \psi}\ \text{Cut}$$

Since $\Gamma_0 \cup \Delta_0 \subseteq \Gamma \cup \Delta$, this shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \ldots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each $i$, then $\Gamma \vdash \psi$.

**Proposition 3.15.** *$\Gamma$ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence $\varphi$.* *pl:seq:ptn: prop:incons*

*Proof.* Exercise. □

**Problem 3.2.** Prove **??**

**Proposition 3.16** (Compactness)**.** *pl:seq:ptn: prop:proves-compact*

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*

2. *If every finite subset of $\Gamma$ is consistent, then $\Gamma$ is consistent.*

*Proof.*
1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Rightarrow \varphi$ has a derivation. Consequently, $\Gamma_0 \vdash \varphi$.

2. If $\Gamma$ is inconsistent, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow$ . But then $\Gamma_0$ is a finite subset of $\Gamma$ that is inconsistent.

□

## 3.7 Derivability and Consistency

pl:seq:prv: sec

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

pl:seq:prv: prop:provability-contr

**Proposition 3.17.** *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.*

*Proof.* There are finite $\Gamma_0$ and $\Gamma_1 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow \varphi$ and $\varphi, \Gamma_1 \Rightarrow$ . Let the **LK**-derivation of $\Gamma_0 \Rightarrow \varphi$ be $\pi_0$ and the **LK**-derivation of $\Gamma_1, \varphi \Rightarrow$ be $\pi_1$. We can then derive

$$\dfrac{\begin{array}{c}\vdots\,\pi_0\\ \Gamma_0 \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\vdots\,\pi_1\\ \varphi, \Gamma_1 \Rightarrow\end{array}}{\Gamma_0, \Gamma_1 \Rightarrow}\ \text{Cut}$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence $\Gamma$ is inconsistent. □

pl:seq:prv: prop:prov-incons

**Proposition 3.18.** *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

*Proof.* First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation $\pi_0$ of $\Gamma \Rightarrow \varphi$. By adding a $\neg$L rule, we obtain a derivation of $\neg\varphi, \Gamma \Rightarrow$ , i.e., $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

If $\Gamma \cup \{\neg A\}$ is inconsistent, there is a derivation $\pi_1$ of $\neg\varphi, \Gamma \Rightarrow$ . The following is a derivation of $\Gamma \Rightarrow \varphi$:

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\Rightarrow \varphi, \neg\varphi}\ \neg\text{R} \qquad \begin{array}{c}\vdots\,\pi_1\\ \neg\varphi, \Gamma \Rightarrow\end{array}}{\Gamma \Rightarrow \varphi}\ \text{Cut}$$

□

**Problem 3.3.** Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

pl:seq:prv: prop:explicit-inc

**Proposition 3.19.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then $\Gamma$ is inconsistent.*

*Proof.* Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a derivation $\pi$ of a sequent $\Gamma_0 \Rightarrow \varphi$. The sequent $\neg\varphi, \Gamma_0 \Rightarrow$ is also derivable:

$$\dfrac{\begin{array}{c}\vdots\,\pi\\ \Gamma_0 \Rightarrow \varphi\end{array} \qquad \dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\ \neg\text{L}}{\varphi, \neg\varphi \Rightarrow}\ \text{XL}}{\Gamma, \neg\varphi \Rightarrow}\ \text{Cut}$$

Since $\neg\varphi \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, this shows that $\Gamma$ is inconsistent. □

**Proposition 3.20.** *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.* pl:seq:prv: prop:provability-exhaustive

*Proof.* There are finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ and **LK**-derivations $\pi_0$ and $\pi_1$ of $\varphi, \Gamma_0 \Rightarrow$ and $\neg\varphi, \Gamma_1 \Rightarrow$, respectively. We can then derive

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_0\\ \varphi, \Gamma_0 \Rightarrow\end{matrix}}{\Gamma_0 \Rightarrow \neg\varphi}\,\neg\text{R} \qquad \begin{matrix}\vdots\,\pi_1\\ \neg\varphi, \Gamma_1 \Rightarrow\end{matrix}}{\Gamma_0, \Gamma_1 \Rightarrow}\,\text{Cut}$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$. Hence $\Gamma$ is inconsistent. □

## 3.8 Derivability and the Propositional Connectives

pl:seq:ppr: sec

**Proposition 3.21.** pl:seq:ppr: prop:provability-land

1. *Both $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$.* pl:seq:ppr: prop:provability-land-left

2. *$\varphi, \psi \vdash \varphi \wedge \psi$.* pl:seq:ppr: prop:provability-land-right

*Proof.* 1. Both sequents $\varphi \wedge \psi \Rightarrow \varphi$ and $\varphi \wedge \psi \Rightarrow \psi$ are derivable:

$$\dfrac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi}\,\wedge\text{L} \qquad\qquad \dfrac{\psi \Rightarrow \psi}{\varphi \wedge \psi \Rightarrow \psi}\,\wedge\text{L}$$

2. Here is a derivation of the sequent $\varphi, \psi \Rightarrow \varphi \wedge \psi$:

$$\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \varphi \wedge \psi}\,\wedge\text{R}$$

□

**Proposition 3.22.** pl:seq:ppr: prop:provability-lor

1. *$\varphi \vee \psi, \neg\varphi, \neg\psi$ is inconsistent.*

2. *Both $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$.*

*Proof.* 1. We give a derivation of the sequent $\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow$:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg\text{L}}{\varphi, \neg\varphi, \neg\psi \Rightarrow} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\neg\psi, \psi \Rightarrow}\,\neg\text{L}}{\psi, \neg\varphi, \neg\psi \Rightarrow}}{\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow}\,\vee\text{L}$$

(Recall that double inference lines indicate several weakening, contraction, and exchange inferences.)

2. Both sequents $\varphi \Rightarrow \varphi \lor \psi$ and $\psi \Rightarrow \varphi \lor \psi$ have derivations:

$$\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \lor \psi}\,\lor\mathrm{R} \qquad\qquad \dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \lor \psi}\,\lor\mathrm{R}$$

□

*pl:seq:ppr: prop:provability-lif* **Proposition 3.23.**

*pl:seq:ppr: prop:provability-lif-left* 1. $\varphi, \varphi \to \psi \vdash \psi$.

*pl:seq:ppr: prop:provability-lif-right* 2. *Both* $\neg\varphi \vdash \varphi \to \psi$ *and* $\psi \vdash \varphi \to \psi$.

*Proof.* 1. The sequent $\varphi \to \psi, \varphi \Rightarrow \psi$ is derivable:

$$\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi \to \psi, \varphi \Rightarrow \psi}\,\to\mathrm{L}$$

2. Both sequents $\neg\varphi \Rightarrow \varphi \to \psi$ and $\psi \Rightarrow \varphi \to \psi$ are derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg\mathrm{L}}{\varphi, \neg\varphi \Rightarrow}\,\mathrm{XL}}{\varphi, \neg\varphi \Rightarrow \psi}\,\mathrm{WR}}{\neg\varphi \Rightarrow \varphi \to \psi}\,\to\mathrm{R} \qquad\qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi}\,\mathrm{WL}}{\psi \Rightarrow \varphi \to \psi}\,\to\mathrm{R}$$

□

## 3.9 Soundness

pl:seq:sou: sec A derivation system, such as the sequent calculus, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that explanation

1. every derivable $\varphi$ is a tautology;

2. if a sentence is derivable from some others, it is also a consequence of them;

3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Because all these proof-theoretic properties are defined via derivability in the sequent calculus of certain sequents, proving (1)–(3) above requires proving something about the semantic properties of derivable sequents. We will first define what it means for a sequent to be *valid*, and then show that every derivable sequent is valid. (1)–(3) then follow as corollaries from this result.

**Definition 3.24.** A valuation $\mathfrak{v}$ *satisfies* a sequent $\Gamma \Rightarrow \Delta$ iff either $\mathfrak{v} \nvDash \varphi$ for some $\varphi \in \Gamma$ or $\mathfrak{v} \vDash \varphi$ for some $\varphi \in \Delta$.

A sequent is *valid* iff every valuation $\mathfrak{v}$ satisfies it.

**Theorem 3.25** (Soundness)**.** *If* **LK** *derives* $\Theta \Rightarrow \Xi$*, then* $\Theta \Rightarrow \Xi$ *is valid.* pl:seq:sou: thm:sequent-soundness

*Proof.* Let $\pi$ be a derivation of $\Theta \Rightarrow \Xi$. We proceed by induction on the number of inferences $n$ in $\pi$.

If the number of inferences is 0, then $\pi$ consists only of an initial sequent. Every initial sequent $\varphi \Rightarrow \varphi$ is obviously valid, since for every $\mathfrak{v}$, either $\mathfrak{v} \nvDash \varphi$ or $\mathfrak{v} \vDash \varphi$.

If the number of inferences is greater than 0, we distinguish cases according to the type of the lowermost inference. By induction hypothesis, we can assume that the premises of that inference are valid, since the number of inferences in the proof of any premise is smaller than $n$.

First, we consider the possible inferences with only one premise.

1. The last inference is a weakening. Then $\Theta \Rightarrow \Xi$ is either $A, \Gamma \Rightarrow \Delta$ (if the last inference is WL) or $\Gamma \Rightarrow \Delta, \varphi$ (if it's WR), and the derivation ends in one of

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta\end{matrix}}{\varphi, \Gamma \Rightarrow \Delta}\,\mathrm{WL} \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta, \varphi}\,\mathrm{WR}$$

   By induction hypothesis, $\Gamma \Rightarrow \Delta$ is valid, i.e., for every valuation $\mathfrak{v}$, either there is some $\chi \in \Gamma$ such that $\mathfrak{v} \nvDash \chi$ or there is some $\chi \in \Delta$ such that $\mathfrak{v} \vDash \chi$.

   If $\mathfrak{v} \nvDash \chi$ for some $\chi \in \Gamma$, then $\chi \in \Theta$ as well since $\Theta = \varphi, \Gamma$, and so $\mathfrak{v} \nvDash \chi$ for some $\chi \in \Theta$. Similarly, if $\mathfrak{v} \vDash \chi$ for some $\chi \in \Delta$, as $\chi \in \Xi$, $\mathfrak{v} \vDash \chi$ for some $\chi \in \Xi$. Consequently, $\Theta \Rightarrow \Xi$ is valid.

2. The last inference is $\neg$L: Then the premise of the last inference is $\Gamma \Rightarrow \Delta, \varphi$ and the conclusion is $\neg\varphi, \Gamma \Rightarrow \Delta$, i.e., the derivation ends in

$$\cfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\ \neg\mathrm{L}$$

and $\Theta = \neg\varphi, \Gamma$ while $\Xi = \Delta$.

The induction hypothesis tells us that $\Gamma \Rightarrow \Delta, \varphi$ is valid, i.e., for every $\mathfrak{v}$, either (a) for some $\chi \in \Gamma$, $\mathfrak{v} \nvDash \chi$, or (b) for some $\chi \in \Delta$, $\mathfrak{v} \vDash \chi$, or (c) $\mathfrak{v} \vDash \varphi$. We want to show that $\Theta \Rightarrow \Xi$ is also valid. Let $\mathfrak{v}$ be a valuation. If (a) holds, then there is $\chi \in \Gamma$ so that $\mathfrak{v} \nvDash \varphi$, but $\varphi \in \Theta$ as well. If (b) holds, there is $\chi \in \Delta$ such that $\mathfrak{v} \vDash \chi$, but $\chi \in \Xi$ as well. Finally, if $\mathfrak{v} \vDash \varphi$, then $\mathfrak{v} \nvDash \neg\varphi$. Since $\neg\varphi \in \Theta$, there is $\chi \in \Theta$ such that $\mathfrak{v} \nvDash \chi$. Consequently, $\Theta \Rightarrow \Xi$ is valid.

3. The last inference is $\neg$R: Exercise.

4. The last inference is $\wedge$L: There are two variants: $\varphi \wedge \psi$ may be inferred on the left from $\varphi$ or from $\psi$ on the left side of the premise. In the first case, the $\pi$ ends in

$$\cfrac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ \wedge\mathrm{L}$$

and $\Theta = \varphi \wedge \psi, \Gamma$ while $\Xi = \Delta$. Consider a valuation $\mathfrak{v}$. Since by induction hypothesis, $\varphi, \Gamma \Rightarrow \Delta$ is valid, (a) $\mathfrak{v} \nvDash \varphi$, (b) $\mathfrak{v} \nvDash \chi$ for some $\chi \in \Gamma$, or (c) $\mathfrak{v} \vDash \chi$ for some $\chi \in \Delta$. In case (a), $\mathfrak{v} \nvDash \varphi \wedge \psi$, so there is $\chi \in \Theta$ (namely, $\varphi \wedge \psi$) such that $\mathfrak{v} \nvDash \chi$. In case (b), there is $\chi \in \Gamma$ such that $\mathfrak{v} \nvDash \chi$, and $\chi \in \Theta$ as well. In case (c), there is $\chi \in \Delta$ such that $\mathfrak{v} \vDash \chi$, and $\chi \in \Xi$ as well since $\Xi = \Delta$. So in each case, $\mathfrak{v}$ satisfies $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$. Since $\mathfrak{v}$ was arbitrary, $\Gamma \Rightarrow \Delta$ is valid. The case where $\varphi \wedge \psi$ is inferred from $\psi$ is handled the same, changing $\varphi$ to $\psi$.

5. The last inference is $\vee$R: There are two variants: $\varphi \vee \psi$ may be inferred on the right from $\varphi$ or from $\psi$ on the right side of the premise. In the first case, $\pi$ ends in

$$\cfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ \vee\mathrm{R}$$

Now $\Theta = \Gamma$ and $\Xi = \Delta, \varphi \vee \psi$. Consider a valuation $\mathfrak{v}$. Since $\Gamma \Rightarrow \Delta, \varphi$ is valid, (a) $\mathfrak{v} \vDash \varphi$, (b) $\mathfrak{v} \nvDash \chi$ for some $\chi \in \Gamma$, or (c) $\mathfrak{v} \vDash \chi$ for some $\chi \in \Delta$. In case (a), $\mathfrak{v} \vDash \varphi \vee \psi$. In case (b), there is $\chi \in \Gamma$ such that $\mathfrak{v} \nvDash \chi$. In case (c), there is $\chi \in \Delta$ such that $\mathfrak{v} \vDash \chi$. So in each case, $\mathfrak{v}$ satisfies $\Gamma \Rightarrow \Delta, \varphi \vee \psi$, i.e., $\Theta \Rightarrow \Xi$. Since $\mathfrak{v}$ was arbitrary, $\Theta \Rightarrow \Xi$ is valid. The case where $\varphi \vee \psi$ is inferred from $\psi$ is handled the same, changing $\varphi$ to $\psi$.

6. The last inference is $\rightarrow$R: Then $\pi$ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \varphi, \Gamma \Rightarrow \Delta, \varphi\end{array}}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}\ \rightarrow\text{R}$$

Again, the induction hypothesis says that the premise is valid; we want to show that the conclusion is valid as well. Let $\mathfrak{v}$ be arbitrary. Since $\varphi, \Gamma \Rightarrow \Delta, \psi$ is valid, at least one of the following cases obtains: (a) $\mathfrak{v} \nvDash \varphi$, (b) $\mathfrak{v} \vDash \psi$, (c) $\mathfrak{v} \nvDash \chi$ for some $\chi \in \Gamma$, or (c) $\mathfrak{v} \vDash \chi$ for some $\chi \in \Delta$. In cases (a) and (b), $\mathfrak{v} \vDash \varphi \rightarrow \psi$ and so there is a $\chi \in \Delta, \varphi \rightarrow \psi$ such that $\mathfrak{v} \vDash \chi$. In case (c), for some $\chi \in \Gamma$, $\mathfrak{v} \nvDash \chi$. In case (d), for some $\chi \in \Delta$, $\mathfrak{v} \vDash \chi$. In each case, $\mathfrak{v}$ satisfies $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$. Since $\mathfrak{v}$ was arbitrary, $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$ is valid.

Now let's consider the possible inferences with two premises.

1. The last inference is a cut: then $\pi$ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \varphi, \Pi \Rightarrow \Lambda\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ \text{Cut}$$

Let $\mathfrak{v}$ be a valuation. By induction hypothesis, the premises are valid, so $\mathfrak{v}$ satisfies both premises. We distinguish two cases: (a) $\mathfrak{v} \nvDash \varphi$ and (b) $\mathfrak{v} \vDash \varphi$. In case (a), in order for $\mathfrak{v}$ to satisfy the left premise, it must satisfy $\Gamma \Rightarrow \Delta$. But then it also satisfies the conclusion. In case (b), in order for $\mathfrak{v}$ to satisfy the right premise, it must satisfy $\Pi \setminus \Lambda$. Again, $\mathfrak{v}$ satisfies the conclusion.

2. The last inference is $\wedge$R. Then $\pi$ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \psi\end{array}}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ \wedge\text{R}$$

Consider a valuation $\mathfrak{v}$. If $\mathfrak{v}$ satisfies $\Gamma \Rightarrow \Delta$, we are done. So suppose it doesn't. Since $\Gamma \Rightarrow \Delta, \varphi$ is valid by induction hypothesis, $\mathfrak{v} \vDash \varphi$. Similarly, since $\Gamma \Rightarrow \Delta, \psi$ is valid, $\mathfrak{v} \vDash \psi$. But then $\mathfrak{v} \vDash \varphi \wedge \psi$.

3. The last inference is $\vee$L: Exercise.

4. The last inference is $\to$L. Then $\pi$ ends in

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \psi, \Pi \Rightarrow \Lambda\end{array}}{\varphi \to \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \to\text{L}$$

Again, consider a valuation $\mathfrak{v}$ and suppose $\mathfrak{v}$ doesn't satisfy $\Gamma, \Pi \Rightarrow \Lambda, \Pi$. We have to show that $\mathfrak{v} \nvDash \varphi \to \psi$. If $\mathfrak{v}$ doesn't satisfy $\Gamma, \Pi \Rightarrow \Lambda, \Pi$, it satisfies neither $\Gamma \Rightarrow \Delta$ nor $\Pi \Rightarrow \Lambda$. Since, $\Gamma \Rightarrow \Delta, \varphi$ is valid, we have $\mathfrak{v} \vDash \varphi$. Since $\psi, \Pi \Rightarrow \Lambda$ is valid, we have $\mathfrak{v} \nvDash \psi$. But then $\mathfrak{v} \nvDash \varphi \to \psi$, which is what we wanted to show.

□

**Problem 3.4.** Complete the proof of Theorem 3.25.

*pl:seq:sou: cor:weak-soundness* **Corollary 3.26.** *If $\vdash \varphi$ then $\varphi$ is a tautology.*

*pl:seq:sou: cor:entailment-soundness* **Corollary 3.27.** *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

*Proof.* If $\Gamma \vdash \varphi$ then for some finite subset $\Gamma_0 \subseteq \Gamma$, there is a derivation of $\Gamma_0 \Rightarrow \varphi$. By Theorem 3.25, every valuation $\mathfrak{v}$ either makes some $\psi \in \Gamma_0$ false or makes $\varphi$ true. Hence, if $\mathfrak{v} \vDash \Gamma$ then also $\mathfrak{v} \vDash \varphi$. □

*pl:seq:sou: cor:consistency-soundness* **Corollary 3.28.** *If $\Gamma$ is satisfiable, then it is consistent.*

*Proof.* We prove the contrapositive. Suppose that $\Gamma$ is not consistent. Then there is a finite $\Gamma_0 \subseteq \Gamma$ and a derivation of $\Gamma_0 \Rightarrow$ . By Theorem 3.25, $\Gamma_0 \Rightarrow$ is valid. In other words, for every valuation $\mathfrak{v}$, there is $\chi \in \Gamma_0$ so that $\mathfrak{v} \nvDash \chi$, and since $\Gamma_0 \subseteq \Gamma$, that $\chi$ is also in $\Gamma$. Thus, no $\mathfrak{v}$ satisfies $\Gamma$, and $\Gamma$ is not satisfiable. □

# Chapter 4

# Natural Deduction

This chapter presents a natural deduction system in the style of Gentzen/Prawitz.

To include or exclude material relevant to natural deduction as a proof system, use the "prfND" tag.

## 4.1 Rules and Derivations

explanation Natural deduction systems are meant to closely parallel the informal reasoning used in mathematical proof (hence it is somewhat "natural"). Natural deduction proofs begin with assumptions. Inference rules are then applied. Assumptions are "discharged" by the $\lnot$Intro, $\to$Intro, and $\lor$Elim inference rules, and the label of the discharged assumption is placed beside the inference for clarity. pl:ntd:rul:sec

**Definition 4.1** (Initial Formula)**.** An *initial formula* or *assumption* is any formula in the topmost position of any branch.

Derivations in natural deduction are certain trees of sentences, where the topmost sentences are assumptions, and if a sentence stands below one, two, or three other sequents, it must follow correctly by a rule of inference. The sentences at the top of the inference are called the *premises* and the sentence below the *conclusion* of the inference. The rules come in pairs, an introduction and an elimination rule for each logical operator. They introduce a logical operator in the conclusion or remove a logical operator from a premise of the rule. Some of the rules allow an assumption of a certain type to be *discharged*. To indicate which assumption is discharged by which inference, we also assign labels to both the assumption and the inference. This is indicated by writing the assumption as "$[\varphi]^n$".

It is customary to consider rules for all logical operators, even for those (if any) that we consider as defined.

## 4.2 Propositional Rules

pl:ntd:prl:sec **Rules for $\land$**

$$\frac{\varphi \qquad \psi}{\varphi \land \psi}\land\text{Intro} \qquad\qquad \frac{\varphi \land \psi}{\varphi}\land\text{Elim} \qquad \frac{\varphi \land \psi}{\psi}\land\text{Elim}$$

**Rules for $\lor$**

$$\frac{\varphi}{\varphi \lor \psi}\lor\text{Intro} \qquad \frac{\psi}{\varphi \lor \psi}\lor\text{Intro}$$

$$\begin{array}{c}
 \qquad\qquad [\varphi]^n \qquad\quad [\psi]^n \\
 \qquad\qquad \vdots \qquad\qquad \vdots \\
 {}_n\,\dfrac{\varphi \lor \psi \qquad \chi \qquad\quad \chi}{\chi}\lor\text{Elim}
\end{array}$$

**Rules for $\to$**

$$\begin{array}{c}
 [\varphi]^n \\
 \vdots \\
 {}_n\,\dfrac{\psi}{\varphi \to \psi}\to\text{Intro}
\end{array} \qquad\qquad \frac{\varphi \to \psi \qquad \varphi}{\psi}\to\text{Elim}$$

**Rules for $\neg$**

$$\begin{array}{c}
 [\varphi]^n \\
 \vdots \\
 {}_n\,\dfrac{\bot}{\neg\varphi}\neg\text{Intro}
\end{array} \qquad\qquad \frac{\neg\varphi \qquad \varphi}{\bot}\neg\text{Elim}$$

### Rules for $\bot$

$$\frac{\bot}{\varphi}\;\bot_I \qquad\qquad \begin{array}{c}[\neg\varphi]^n\\ \vdots\\ n\;\dfrac{\bot}{\varphi}\;\bot_C\end{array}$$

Note that ¬Intro and $\bot_C$ are very similar: The difference is that ¬Intro derives a negated sentence $\neg\varphi$ but $\bot_C$ a positive sentence $\varphi$.

## 4.3 Derivations

explanation We've said what an assumption is, and we've given the rules of inference. Derivations in natural deduction are inductively generated from these: each derivation either is an assumption on its own, or consists of one, two, or three derivations followed by a correct inference. pl:ntd:der:sec

**Definition 4.2** (Derivation)**.** A *derivation* of a sentence $\varphi$ from assumptions $\Gamma$ is a tree of sentences satisfying the following conditions:

1. The topmost sentences of the tree are either in $\Gamma$ or are discharged by an inference in the tree.

2. The bottommost sentence of the tree is $\varphi$.

3. Every sentence in the tree except $\varphi$ is a premise of a correct application of an inference rule whose conclusion stands directly below that sentence in the tree.

We then say that $\varphi$ is the *conclusion* of the derivation and that $\varphi$ is *derivable* from $\Gamma$.

**Example 4.3.** Every assumption on its own is a derivation. So, e.g., $\chi$ by itself is a derivation, and so is $\theta$ by itself. We can obtain a new derivation from these by applying, say, the ∧Intro rule,

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\;\wedge\mathrm{Intro}$$

These rules are meant to be general: we can replace the $\varphi$ and $\psi$ in it with any sentences, e.g., by $\chi$ and $\theta$. Then the conclusion would be $\chi \wedge \theta$, and so

$$\frac{\chi \qquad \theta}{\chi \wedge \theta}\;\wedge\mathrm{Intro}$$

is a correct derivation. Of course, we can also switch the assumptions, so that $\theta$ plays the role of $\varphi$ and $\chi$ that of $\psi$. Thus,

$$\dfrac{\theta \qquad \chi}{\theta \wedge \chi}\ \wedge\text{Intro}$$

is also a correct derivation.

We can now apply another rule, say, $\rightarrow$Intro, which allows us to conclude a conditional and allows us to discharge any assumption that is identical to the conclusion of that conditional. So both of the following would be correct derivations:

$$1\ \dfrac{\dfrac{[\chi]^1 \qquad \theta}{\chi \wedge \theta}\ \wedge\text{Intro}}{\chi \rightarrow (\chi \wedge \theta)}\ \rightarrow\text{Intro} \qquad 1\ \dfrac{\dfrac{\chi \qquad [\theta]^1}{\chi \wedge \theta}\ \wedge\text{Intro}}{\theta \rightarrow (\chi \wedge \theta)}\ \rightarrow\text{Intro}$$

## 4.4 Examples of Derivations

pl:ntd:pro:
sec

**Example 4.4.** Let's give a derivation of the sentence $(\varphi \wedge \psi) \rightarrow \varphi$.

We begin by writing the desired conclusion at the bottom of the derivation.

$$\dfrac{}{(\varphi \wedge \psi) \rightarrow \varphi}$$

Next, we need to figure out what kind of inference could result in a sentence of this form. The main operator of the conclusion is $\rightarrow$, so we'll try to arrive at the conclusion using the $\rightarrow$Intro rule. It is best to write down the assumptions involved and label the inference rules as you progress, so it is easy to see whether all assumptions have been discharged at the end of the proof.

$$1\ \dfrac{\begin{array}{c}[\varphi \wedge \psi]^1 \\ \vdots \\ \varphi\end{array}}{(\varphi \wedge \psi) \rightarrow \varphi}\ \rightarrow\text{Intro}$$

We now need to fill in the steps from the assumption $\varphi \wedge \psi$ to $\varphi$. Since we only have one connective to deal with, $\wedge$, we must use the $\wedge$ elim rule. This gives us the following proof:

$$1\ \dfrac{\dfrac{[\varphi \wedge \psi]^1}{\varphi}\ \wedge\text{Elim}}{(\varphi \wedge \psi) \rightarrow \varphi}\ \rightarrow\text{Intro}$$

We now have a correct derivation of $(\varphi \wedge \psi) \rightarrow \varphi$.

**Example 4.5.** Now let's give a derivation of $(\neg\varphi \lor \psi) \to (\varphi \to \psi)$.

We begin by writing the desired conclusion at the bottom of the derivation.

$$\frac{}{(\neg\varphi \lor \psi) \to (\varphi \to \psi)}$$

To find a logical rule that could give us this conclusion, we look at the logical connectives in the conclusion: $\neg$, $\lor$, and $\to$. We only care at the moment about the first occurence of $\to$ because it is the main operator of the sentence in the end-sequent, while $\neg$, $\lor$ and the second occurence of $\to$ are inside the scope of another connective, so we will take care of those later. We therefore start with the $\to$Intro rule. A correct application must look like this:

$$1\ \frac{\begin{array}{c}[\neg\varphi \lor \psi]^1\\ \vdots\\ \varphi \to \psi\end{array}}{(\neg\varphi \lor \psi) \to (\varphi \to \psi)}\ {\to}\text{Intro}$$

This leaves us with two possibilities to continue. Either we can keep working from the bottom up and look for another application of the $\to$Intro rule, or we can work from the top down and apply a $\lor$Elim rule. Let us apply the latter. We will use the assumption $\neg\varphi \lor \psi$ as the leftmost premise of $\lor$Elim. For a valid application of $\lor$Elim, the other two premises must be identical to the conclusion $\varphi \to \psi$, but each may be derived in turn from another assumption, namely the two disjuncts of $\neg\varphi \lor \psi$. So our derivation will look like this:

$$1\ \frac{2\ \dfrac{[\neg\varphi \lor \psi]^1 \qquad \begin{array}{c}[\neg\varphi]^2\\ \vdots\\ \varphi \to \psi\end{array} \qquad \begin{array}{c}[\psi]^2\\ \vdots\\ \varphi \to \psi\end{array}}{\varphi \to \psi}\ {\lor}\text{Elim}}{(\neg\varphi \lor \psi) \to (\varphi \to \psi)}\ {\to}\text{Intro}$$

In each of the two branches on the right, we want to derive $\varphi \to \psi$, which is best done using $\to$Intro.

$$1\ \frac{2\ \dfrac{[\neg\varphi \lor \psi]^1 \qquad 3\ \dfrac{\begin{array}{c}[\neg\varphi]^2, [\varphi]^3\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}\ {\to}\text{Intro} \qquad 4\ \dfrac{\begin{array}{c}[\psi]^2, [\varphi]^4\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}\ {\to}\text{Intro}}{\varphi \to \psi}\ {\lor}\text{Elim}}{(\neg\varphi \lor \psi) \to (\varphi \to \psi)}\ {\to}\text{Intro}$$

For the two missing parts of the derivation, we need derivations of $\psi$ from $\neg\varphi$ and $\varphi$ in the middle, and from $\varphi$ and $\psi$ on the left. Let's take the former first. $\neg\varphi$ and $\varphi$ are the two premises of $\neg$Elim:

$$
\begin{prooftree}
\AxiomC{$[\neg\varphi]^2$}
\AxiomC{$[\varphi]^3$}
\RightLabel{$\neg$Elim}
\BinaryInfC{$\bot$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\psi$}
\end{prooftree}
$$

By using $\bot_I$, we can obtain $\psi$ as a conclusion and complete the branch.

$$
\begin{prooftree}
\AxiomC{$[\neg\varphi\lor\psi]^1$}
\AxiomC{$[\neg\varphi]^2$}
\AxiomC{$[\varphi]^3$}
\RightLabel{$\bot$Intro}
\BinaryInfC{$\bot$}
\RightLabel{$\bot_I$}
\UnaryInfC{$\psi$}
\LeftLabel{3}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi\to\psi$}
\AxiomC{$[\psi]^2, [\varphi]^4$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\psi$}
\LeftLabel{4}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi\to\psi$}
\LeftLabel{2}
\RightLabel{$\lor$Elim}
\TrinaryInfC{$\varphi\to\psi$}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$(\neg\varphi\lor\psi)\to(\varphi\to\psi)$}
\end{prooftree}
$$

Let's now look at the rightmost branch. Here it's important to realize that the definition of derivation *allows assumptions to be discharged* but *does not require* them to be. In other words, if we can derive $\psi$ from one of the assumptions $\varphi$ and $\psi$ without using the other, that's ok. And to derive $\psi$ from $\psi$ is trivial: $\psi$ by itself is such a derivation, and no inferences are needed. So we can simply delete the assumtion $\varphi$.

$$
\begin{prooftree}
\AxiomC{$[\neg\varphi\lor\psi]^1$}
\AxiomC{$[\neg\varphi]^2$}
\AxiomC{$[\varphi]^3$}
\RightLabel{$\neg$Elim}
\BinaryInfC{$\bot$}
\RightLabel{$\bot_I$}
\UnaryInfC{$\psi$}
\LeftLabel{3}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi\to\psi$}
\AxiomC{$[\psi]^2$}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi\to\psi$}
\LeftLabel{2}
\RightLabel{$\lor$Elim}
\TrinaryInfC{$\varphi\to\psi$}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$(\neg\varphi\lor\psi)\to(\varphi\to\psi)$}
\end{prooftree}
$$

Note that in the finished derivation, the rightmost $\to$Intro inference does not actually discharge any assumptions.

**Example 4.6.** So far we have not needed the $\bot_C$ rule. It is special in that it allows us to discharge an assumption that isn't a sub-formula of the conclusion of the rule. It is closely related to the $\bot_I$ rule. In fact, the $\bot_I$ rule is a special case of the $\bot_C$ rule—there is a logic called "intuitionistic logic" in which only $\bot_I$ is allowed. The $\bot_C$ rule is a last resort when nothing else works. For instance, suppose we want to derive $\varphi\lor\neg\varphi$. Our usual strategy would be to attempt to derive $\varphi\lor\neg\varphi$ using $\lor$Intro. But this would require us to derive either $\varphi$ or $\neg\varphi$ from no assumptions, and this can't be done. $\bot_C$ to the rescue!

$$
{}^{1}\dfrac{\begin{array}{c}[\neg(\varphi\lor\neg\varphi)]^1\\ \vdots\\ \bot\end{array}}{\varphi\lor\neg\varphi}\,\bot_C
$$

Now we're looking for a derivation of $\bot$ from $\neg(\varphi\lor\neg\varphi)$. Since $\bot$ is the conclusion of $\neg$Elim we might try that:

$$
{}^{1}\dfrac{\dfrac{\begin{array}{c}[\neg(\varphi\lor\neg\varphi)]^1\\ \vdots\\ \neg\varphi\end{array}\qquad\begin{array}{c}[\neg(\varphi\lor\neg\varphi)]^1\\ \vdots\\ \varphi\end{array}}{\bot}\,\neg\mathrm{Elim}}{\varphi\lor\neg\varphi}\,\bot_C
$$

Our strategy for finding a derivation of $\neg\varphi$ calls for an application of $\neg$Intro:

$$
{}^{1}\dfrac{\dfrac{{}^{2}\dfrac{\begin{array}{c}[\neg(\varphi\lor\neg\varphi)]^1, [\varphi]^2\\ \vdots\\ \bot\end{array}}{\neg\varphi}\,\neg\mathrm{Intro}\qquad\begin{array}{c}[\neg(\varphi\lor\neg\varphi)]^1\\ \vdots\\ \varphi\end{array}}{\bot}\,\neg\mathrm{Elim}}{\varphi\lor\neg\varphi}\,\bot_C
$$

Here, we can get $\bot$ easily by applying $\neg$Elim to the assumption $\neg(\varphi\lor\neg\varphi)$ and $\varphi\lor\neg\varphi$ which follows from our new assumption $\varphi$ by $\lor$Intro:

$$
{}^{1}\dfrac{\dfrac{{}^{2}\dfrac{\dfrac{[\neg(\varphi\lor\neg\varphi)]^1\qquad\dfrac{[\varphi]^2}{\varphi\lor\neg\varphi}\,\lor\mathrm{Intro}}{\bot}\,\neg\mathrm{Elim}}{\neg\varphi}\,\neg\mathrm{Intro}\qquad\begin{array}{c}[\neg(\varphi\lor\neg\varphi)]^1\\ \vdots\\ \varphi\end{array}}{\bot}\,\neg\mathrm{Elim}}{\varphi\lor\neg\varphi}\,\bot_C
$$

On the right side we use the same strategy, except we get $\varphi$ by $\bot_C$:

$$
{}^{1}\dfrac{\dfrac{{}^{2}\dfrac{\dfrac{[\neg(\varphi\lor\neg\varphi)]^1\qquad\dfrac{[\varphi]^2}{\varphi\lor\neg\varphi}\,\lor\mathrm{Intro}}{\bot}\,\neg\mathrm{Elim}}{\neg\varphi}\,\neg\mathrm{Intro}\qquad{}^{3}\dfrac{\dfrac{[\neg(\varphi\lor\neg\varphi)]^1\qquad\dfrac{[\neg\varphi]^3}{\varphi\lor\neg\varphi}\,\lor\mathrm{Intro}}{\bot}\,\neg\mathrm{Elim}}{\varphi}\,\bot_C}{\bot}\,\neg\mathrm{Elim}}{\varphi\lor\neg\varphi}\,\bot_C
$$

**Problem 4.1.** Give derivations of the following:

1. $\neg(\varphi\to\psi)\to(\varphi\land\neg\psi)$
2. $(\varphi\to\chi)\lor(\psi\to\chi)$ from the assumption $(\varphi\land\psi)\to\chi$

## 4.5 Proof-Theoretic Notions

pl:ntd:ptn: sec

> This section collects the definitions the provability relation and consistency for natural deduction.

Just as we've defined a number of important semantic notions (validity, entailment, satisfiabilty), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sentences from others. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*. explanation

**Definition 4.7** (Theorems)**.** A sentence $\varphi$ is a *theorem* if there is a derivation of $\varphi$ in natural deduction in which all assumptions are discharged. We write $\vdash \varphi$ if $\varphi$ is a theorem and $\nvdash \varphi$ if it is not.

**Definition 4.8** (Derivability)**.** A sentence $\varphi$ is *derivable from* a set of sentences $\Gamma$, $\Gamma \vdash \varphi$, if there is a derivation with conclusion $\varphi$ and in which every assumption is either discharged or is in $\Gamma$. If $\varphi$ is not derivable from $\Gamma$ we write $\Gamma \nvdash \varphi$.

**Definition 4.9** (Consistency)**.** A set of sentences $\Gamma$ is *inconsistent* iff $\Gamma \vdash \bot$. If $\Gamma$ is not inconsistent, i.e., if $\Gamma \nvdash \bot$, we say it is *consistent*.

*pl:ntd:ptn: prop:reflexivity* **Proposition 4.10** (Reflexivity)**.** *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

*Proof.* The assumption $\varphi$ by itself is a derivation of $\varphi$ where every undischarged assumption (i.e., $\varphi$) is in $\Gamma$. □

*pl:ntd:ptn: prop:monotony* **Proposition 4.11** (Monotony)**.** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

*Proof.* Any derivation of $\varphi$ from $\Gamma$ is also a derivation of $\varphi$ from $\Delta$. □

*pl:ntd:ptn: prop:transitivity* **Proposition 4.12** (Transitivity)**.** *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

*Proof.* If $\Gamma \vdash \varphi$, there is a derivation $\delta_0$ of $\varphi$ with all undischarged assumptions in $\Gamma$. If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a derivation $\delta_1$ of $\psi$ with all undischarged assumptions in $\{\varphi\} \cup \Delta$. Now consider:

$$\dfrac{{}^{1}\dfrac{\begin{matrix}\Delta, [\varphi]^1\\ \vdots\ \delta_1\\ \psi\end{matrix}}{\varphi \to \psi}\ {\to}\mathrm{Intro} \qquad \begin{matrix}\Gamma\\ \vdots\ \delta_0\\ \varphi\end{matrix}}{\psi}\ {\to}\mathrm{Elim}$$

The undischarged assumptions are now all among $\Gamma \cup \Delta$, so this shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each $i$, then $\Gamma \vdash \psi$.

**Proposition 4.13.** *$\Gamma$ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence $\varphi$.* *pl:ntd:ptn: prop:incons*

*Proof.* Exercise. □

**Problem 4.2.** Prove Proposition 4.13

**Proposition 4.14** (Compactness)**.** *pl:ntd:ptn: prop:proves-compact*

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*
2. *If every finite subset of $\Gamma$ is consistent, then $\Gamma$ is consistent.*

*Proof.*
1. If $\Gamma \vdash \varphi$, then there is a derivation $\delta$ of $\varphi$ from $\Gamma$. Let $\Gamma_0$ be the set of undischarged assumptions of $\delta$. Since any derivation is finite, $\Gamma_0$ can only contain finitely many sentences. So, $\delta$ is a derivation of $\varphi$ from a finite $\Gamma_0 \subseteq \Gamma$.
2. This is the contrapositive of (1) for the special case $\varphi \equiv \bot$.

□

## 4.6 Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem. *pl:ntd:prv: sec*

**Proposition 4.15.** *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.* *pl:ntd:prv: prop:provability-contr*

*Proof.* Let the derivation of $\varphi$ from $\Gamma$ be $\delta_1$ and the derivation of $\bot$ from $\Gamma \cup \{\varphi\}$ be $\delta_2$. We can then derive:

$$\dfrac{{}^{1}\,\dfrac{\begin{array}{c}\Gamma, [\varphi]^1 \\ \vdots\ \delta_2 \\ \bot\end{array}}{\neg\varphi}\ \neg\text{Intro} \qquad \begin{array}{c}\Gamma \\ \vdots\ \delta_1 \\ \varphi\end{array}}{\bot}\ \neg\text{Elim}$$

In the new derivation, the assumption $\varphi$ is discharged, so it is a derivation from $\Gamma$. □

**Proposition 4.16.** *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.* *pl:ntd:prv: prop:prov-incons*

*Proof.* First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation $\delta_0$ of $\varphi$ from undischarged assumptions $\Gamma$. We obtain a derivation of $\bot$ from $\Gamma \cup \{\neg\varphi\}$ as follows:

$$\dfrac{\neg\varphi \qquad \begin{matrix}\Gamma \\ \vdots\ \delta_0 \\ \varphi\end{matrix}}{\bot}\ \neg\text{Elim}$$

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, and let $\delta_1$ be the corresponding derivation of $\bot$ from undischarged assumptions in $\Gamma \cup \{\neg\varphi\}$. We obtain a derivation of $\varphi$ from $\Gamma$ alone by using $\bot_C$:

$$\dfrac{\begin{matrix}\Gamma, [\neg\varphi]^1 \\ \vdots\ \delta_1 \\ \bot\end{matrix}}{\varphi}\ \bot_C$$

□

**Problem 4.3.** Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

*pl:ntd:prv: prop:explicit-inc* **Proposition 4.17.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then $\Gamma$ is inconsistent.*

*Proof.* Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a derivation $\delta$ of $\varphi$ from $\Gamma$. Consider this simple application of the $\neg$Elim rule:

$$\dfrac{\neg\varphi \qquad \begin{matrix}\Gamma \\ \vdots\ \delta \\ \varphi\end{matrix}}{\bot}\ \neg\text{Elim}$$

Since $\neg\varphi \in \Gamma$, all undischarged assumptions are in $\Gamma$, this shows that $\Gamma \vdash \bot$. □

*pl:ntd:prv: prop:provability-exhaustive* **Proposition 4.18.** *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.*

*Proof.* There are derivations $\delta_1$ and $\delta_2$ of $\bot$ from $\Gamma \cup \{\varphi\}$ and $\bot$ from $\Gamma \cup \{\neg\varphi\}$, respectively. We can then derive

$$\dfrac{{}_2\dfrac{\begin{matrix}\Gamma, [\neg\varphi]^2 \\ \vdots\ \delta_2 \\ \bot\end{matrix}}{\neg\neg\varphi}\ \neg\text{Intro} \qquad {}_1\dfrac{\begin{matrix}\Gamma, [\varphi]^1 \\ \vdots\ \delta_1 \\ \bot\end{matrix}}{\neg\varphi}\ \neg\text{Intro}}{\bot}\ \neg\text{Elim}$$

Since the assumptions $\varphi$ and $\neg\varphi$ are discharged, this is a derivation of $\bot$ from $\Gamma$ alone. Hence $\Gamma$ is inconsistent. □

## 4.7 Derivability and the Propositional Connectives

pl:ntd:ppr: sec

**Proposition 4.19.**

*pl:ntd:ppr: prop:provability-land*

1. *Both* $\varphi \land \psi \vdash \varphi$ *and* $\varphi \land \psi \vdash \psi$ *pl:ntd:ppr: prop:provability-land-left*
2. $\varphi, \psi \vdash \varphi \land \psi$. *pl:ntd:ppr: prop:provability-land-right*

*Proof.* 1. We can derive both

$$\frac{\varphi \land \psi}{\varphi}\land\text{Elim} \qquad\qquad \frac{\varphi \land \psi}{\psi}\land\text{Elim}$$

2. We can derive:

$$\frac{\varphi \qquad \psi}{\varphi \land \psi}\land\text{Intro}$$

□

**Proposition 4.20.** *pl:ntd:ppr: prop:provability-lor*

1. $\varphi \lor \psi, \neg\varphi, \neg\psi$ *is inconsistent.*
2. *Both* $\varphi \vdash \varphi \lor \psi$ *and* $\psi \vdash \varphi \lor \psi$.

*Proof.* 1. Consider the following derivation:

$$1\,\frac{\varphi \lor \psi \qquad \dfrac{\neg\varphi \qquad [\varphi]^1}{\bot}\neg\text{Elim} \qquad \dfrac{\neg\psi \qquad [\psi]^1}{\bot}\neg\text{Elim}}{\bot}\lor\text{Elim}$$

This is a derivation of $\bot$ from undischarged assumptions $\varphi \lor \psi$, $\neg\varphi$, and $\neg\psi$.

2. We can derive both

$$\frac{\varphi}{\varphi \lor \psi}\lor\text{Intro} \qquad\qquad \frac{\psi}{\varphi \lor \psi}\lor\text{Intro}$$

□

**Proposition 4.21.** *pl:ntd:ppr: prop:provability-lif*

1. $\varphi, \varphi \to \psi \vdash \psi$. *pl:ntd:ppr: prop:provability-lif-left*

pl:ntd:ppr: prop:provability-lif-right

2. *Both* $\neg\varphi \vdash \varphi \rightarrow \psi$ *and* $\psi \vdash \varphi \rightarrow \psi$.

*Proof.* 1. We can derive:

$$\dfrac{\varphi \rightarrow \psi \qquad \psi}{\psi} \rightarrow\text{Elim}$$

2. This is shown by the following two derivations:

$$\dfrac{\dfrac{\dfrac{\neg\varphi \qquad [\varphi]^1}{\bot}\neg\text{Elim}}{\psi}\bot_I}{\varphi \rightarrow \psi}{\scriptstyle 1}\rightarrow\text{Intro} \qquad \dfrac{\psi}{\varphi \rightarrow \psi}\rightarrow\text{Intro}$$

Note that $\rightarrow$Intro may, but does not have to, discharge the assumption $\varphi$. □

## 4.8 Soundness

pl:ntd:sou: sec

A derivation system, such as natural deduction, is *sound* if it cannot derive things that do not actually follow. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that explanation

1. every derivable sentence is a tautology;
2. if a sentence is derivable from some others, it is also a consequence of them;
3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

pl:ntd:sou: thm:soundness

**Theorem 4.22** (Soundness). *If* $\varphi$ *is derivable from the undischarged assumptions* $\Gamma$, *then* $\Gamma \vDash \varphi$.

*Proof.* Let $\delta$ be a derivation of $\varphi$. We proceed by induction on the number of inferences in $\delta$.

For the induction basis we show the claim if the number of inferences is 0. In this case, $\delta$ consists only of an initial formula. Every initial formula $\varphi$ is an undischarged assumption, and as such, any valuation $\mathfrak{v}$ that satisfies all of the undischarged assumptions of the proof also satisfies $\varphi$.

Now for the inductive step. Suppose that $\delta$ contains $n$ inferences. The premise(s) of the lowermost inference are derived using sub-derivations, each of which contains fewer than $n$ inferences. We assume the induction hypothesis:

The premises of the last inference follow from the undischarged assumptions of the sub-derivations ending in those premises. We have to show that $\varphi$ follows from the undischarged assumptions of the entire proof.

We distinguish cases according to the type of the lowermost inference. First, we consider the possible inferences with only one premise.

1. Suppose that the last inference is $\lnot$Intro: The derivation has the form

$$\begin{array}{c} \Gamma, [\varphi]^n \\ \vdots\ \delta_1 \\ n\ \dfrac{\bot}{\lnot\varphi}\ \lnot\text{Intro} \end{array}$$

   By inductive hypothesis, $\bot$ follows from the undischarged assumptions $\Gamma \cup \{\varphi\}$ of $\delta_1$. Consider a valuation $\mathfrak{v}$. We need to show that, if $\mathfrak{v} \vDash \Gamma$, then $\mathfrak{v} \vDash \lnot\varphi$. Suppose for reductio that $\mathfrak{v} \vDash \Gamma$, but $\mathfrak{v} \nvDash \lnot\varphi$, i.e., $\mathfrak{v} \vDash \varphi$. This would mean that $\mathfrak{v} \vDash \Gamma \cup \{\varphi\}$. This is contrary to our inductive hypothesis. So, $\mathfrak{v} \vDash \lnot\varphi$.

2. The last inference is $\land$Elim: There are two variants: $\varphi$ or $\psi$ may be inferred from the premise $\varphi \land \psi$. Consider the first case. The derivation $\delta$ looks like this:

$$\begin{array}{c} \Gamma \\ \vdots\ \delta_1 \\ \dfrac{\varphi \land \psi}{\varphi}\ \land\text{Elim} \end{array}$$

   By inductive hypothesis, $\varphi \land \psi$ follows from the undischarged assumptions $\Gamma$ of $\delta_1$. Consider a structure $\mathfrak{v}$. We need to show that, if $\mathfrak{v} \vDash \Gamma$, then $\mathfrak{v} \vDash \varphi$. Suppose $\mathfrak{v} \vDash \Gamma$. By our inductive hypothesis ($\Gamma \vDash \varphi \lor \psi$), we know that $\mathfrak{v} \vDash \varphi \land \psi$. By definition, $\mathfrak{v} \vDash \varphi \land \psi$ iff $\mathfrak{v} \vDash \varphi$ and $\mathfrak{v} \vDash \psi$. (The case where $\psi$ is inferred from $\varphi \land \psi$ is handled similarly.)

3. The last inference is $\lor$Intro: There are two variants: $\varphi \lor \psi$ may be inferred from the premise $\varphi$ or the premise $\psi$. Consider the first case. The derivation has the form

$$\begin{array}{c} \Gamma \\ \vdots\ \delta_1 \\ \dfrac{\varphi}{\varphi \lor \psi}\ \lor\text{Intro} \end{array}$$

By inductive hypothesis, $\varphi$ follows from the undischarged assumptions $\Gamma$ of $\delta_1$. Consider a valuation $\mathfrak{v}$. We need to show that, if $\mathfrak{v} \vDash \Gamma$, then $\mathfrak{v} \vDash \varphi \lor \psi$. Suppose $\mathfrak{v} \vDash \Gamma$; then $\mathfrak{v} \vDash \varphi$ since $\Gamma \vDash \varphi$ (the inductive hypothesis). So it must also be the case that $\mathfrak{v} \vDash \varphi \lor \psi$. (The case where $\varphi \lor \psi$ is inferred from $\psi$ is handled similarly.)

4. The last inference is $\to$Intro: $\varphi \to \psi$ is inferred from a subproof with assumption $\varphi$ and conclusion $\psi$, i.e.,

$$\dfrac{\begin{array}{c}\Gamma, [\varphi]^n \\ \vdots\ \delta_1 \\ \psi\end{array}}{\varphi \to \psi}\ {\to}\mathrm{Intro}\ (n)$$

By inductive hypothesis, $\psi$ follows from the undischarged assumptions of $\delta_1$, i.e., $\Gamma \cup \{\varphi\} \vDash \psi$. Consider a valuation $\mathfrak{v}$. The undischarged assumptions of $\delta$ are just $\Gamma$, since $\varphi$ is discharged at the last inference. So we need to show that $\Gamma \vDash \varphi \to \psi$. For reductio, suppose that for some valuation $\mathfrak{v}$, $\mathfrak{v} \vDash \Gamma$ but $\mathfrak{v} \nvDash \varphi \to \psi$. So, $\mathfrak{v} \vDash \varphi$ and $\mathfrak{v} \nvDash \psi$. But by hypothesis, $\psi$ is a consequence of $\Gamma \cup \{\varphi\}$, i.e., $\mathfrak{v} \vDash \psi$, which is a contradiction. So, $\Gamma \vDash \varphi \to \psi$.

5. The last inference is $\bot_I$: Here, $\delta$ ends in

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots\ \delta_1 \\ \bot\end{array}}{\varphi}\ \bot_I$$

By induction hypothesis, $\Gamma \vDash \bot$. We have to show that $\Gamma \vDash \varphi$. Suppose not; then for some $\mathfrak{v}$ we have $\mathfrak{v} \vDash \Gamma$ and $\mathfrak{v} \nvDash \varphi$. But we always have $\mathfrak{v} \nvDash \bot$, so this would mean that $\Gamma \nvDash \bot$, contrary to the induction hypothesis.

6. The last inference is $\bot_C$: Exercise.

Now let's consider the possible inferences with several premises: $\lor$Elim, $\land$Intro, and $\to$Elim.

1. The last inference is $\land$Intro. $\varphi \land \psi$ is inferred from the premises $\varphi$ and $\psi$ and $\delta$ has the form

$$\dfrac{\begin{array}{c}\Gamma_1 \\ \vdots\ \delta_1 \\ \varphi\end{array} \qquad \begin{array}{c}\Gamma_2 \\ \vdots\ \delta_2 \\ \psi\end{array}}{\varphi \land \psi}\ \land\mathrm{Intro}$$

By induction hypothesis, $\varphi$ follows from the undischarged assumptions $\Gamma_1$ of $\delta_1$ and $\psi$ follows from the undischarged assumptions $\Gamma_2$ of $\delta_2$. The undischarged assumptions of $\delta$ are $\Gamma_1 \cup \gamma_2$, so we have to show that $\Gamma_1 \cup \Gamma_2 \vDash \varphi \wedge \psi$. Consider a valuation $\mathfrak{v}$ with $\mathfrak{v} \vDash \Gamma_1 \cup \Gamma_2$. Since $\mathfrak{v} \vDash \Gamma_1$, it must be the case that $\mathfrak{v} \vDash \varphi$ as $\Gamma_1 \vDash \varphi$, and since $\mathfrak{v} \vDash \Gamma_2$, $\mathfrak{v} \vDash \psi$ since $\Gamma_2 \vDash \psi$. Together, $\mathfrak{v} \vDash \varphi \wedge \psi$.

2. The last inference is $\vee$Elim: Exercise.

3. The last inference is $\rightarrow$Elim. $\psi$ is inferred from the premises $\varphi \rightarrow \psi$ and $\varphi$. The derivation $\delta$ looks like this:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\,\delta_1\\ \varphi \rightarrow \psi\end{array} \qquad \begin{array}{c}\Gamma_2\\ \vdots\,\delta_2\\ \varphi\end{array}}{\psi}\ \rightarrow\text{Elim}$$

By induction hypothesis, $\varphi \rightarrow \psi$ follows from the undischarged assumptions $\Gamma_1$ of $\delta_1$ and $\varphi$ follows from the undischarged assumptions $\Gamma_2$ of $\delta_2$. Consider a valuation $\mathfrak{v}$. We need to show that, if $\mathfrak{v} \vDash \Gamma_1 \cup \Gamma_2$, then $\mathfrak{v} \vDash \psi$. Suppose $\mathfrak{v} \vDash \Gamma_1 \cup \Gamma_2$. Since $\Gamma_1 \vDash \varphi \rightarrow \psi$, $\mathfrak{v} \vDash \varphi \rightarrow \psi$. Since $\Gamma_2 \vDash \varphi$, we have $\mathfrak{v} \vDash \varphi$. This means that $\mathfrak{v} \vDash \psi$ (For if $\mathfrak{v} \nvDash \psi$, since $\mathfrak{v} \vDash \varphi$, we'd have $\mathfrak{v} \nvDash \varphi \rightarrow \psi$, contradicting $\mathfrak{v} \vDash \varphi \rightarrow \psi$).

4. The last inference is $\neg$Elim: Exercise.

□

**Problem 4.4.** Complete the proof of Theorem 4.22.

**Corollary 4.23.** *If $\vdash \varphi$, then $\varphi$ is a tautology.* pl:ntd:sou: cor:weak-soundness

**Corollary 4.24.** *If $\Gamma$ is satisfiable, then it is consistent.* pl:ntd:sou: cor:consistency-soundness

*Proof.* We prove the contrapositive. Suppose that $\Gamma$ is not consistent. Then $\Gamma \vdash \bot$, i.e., there is a derivation of $\bot$ from undischarged assumptions in $\Gamma$. By Theorem 4.22, any valuation $\mathfrak{v}$ that satisfies $\Gamma$ must satisfy $\bot$. Since $\mathfrak{v} \nvDash \bot$ for every valuation $\mathfrak{v}$, no $\mathfrak{v}$ can satisfy $\Gamma$, i.e., $\Gamma$ is not satisfiable. □

# Chapter 5

# Tableaux

> This chapter presents a signed analytic tableaux system.
>
> To include or exclude material relevant to natural deduction as a proof system, use the "prfTab" tag.

## 5.1 Rules and Tableaux

pl:tab:rul:sec A tableau is a systematic survey of the possible ways a sentence can be true or false in a structure. The bulding blocks of a tableau are signed formulas: sentences plus a truth value "sign," either $\mathbb{T}$ or $\mathbb{F}$. These signed formulas are arranged in a (downward growing) tree.

**Definition 5.1.** A *signed formula* is a pair consisting of a truth value and a sentence, i.e., either:

$$\mathbb{T}\,\varphi \text{ or } \mathbb{F}\,\varphi.$$

Intuitively, we might read $\mathbb{T}\,\varphi$ as "$\varphi$ might be true" and $\mathbb{F}\,\varphi$ as "$\varphi$ might be false" (in some structure).

Each signed formula in the tree is either an *assumption* (which are listed at the very top of the tree), or it is obtained from a signed formula above it by one of a number of rules of inference. There are two rules for each possible main operator of the preceding formula, one for the case when the sign is $\mathbb{T}$, and one for the case where the sign is $\mathbb{F}$. Some rules allow the tree to branch, and some only add signed formulas to the branch. A rule may be (and often must be) applied not to the immediately preceding signed formula, but to any signed formula in the branch from the root to the place the rule is applied.

A branch is *closed* when it contains both $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$. A closed tableau is one where every branch is closed. Under the intuitive interpretation, any branch describes a joint possibility, but $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$ are not jointly possible. In other words, if a branch is closed, the possibility it describes has been ruled out. In particular, that means that a closed tableau rules out all possibilities of simultaneously making every assumption of the form $\mathbb{T}\,\varphi$ true and every assumption of the form $\mathbb{F}\,\varphi$ false.

A closed tableau *for* $\varphi$ is a closed tableau with root $\mathbb{F}\,\varphi$. If such a closed tableau exists, all possibilities for $\varphi$ being false have been ruled out; i.e., $\varphi$ must be true in every structure.

## 5.2 Propositional Rules

### Rules for $\neg$

pl:tab:prl:sec

$$\frac{\mathbb{T}\,\neg\varphi}{\mathbb{F}\,\varphi}\;\neg\mathbb{T} \qquad\qquad \frac{\mathbb{F}\,\neg\varphi}{\mathbb{T}\,\varphi}\;\neg\mathbb{F}$$

### Rules for $\wedge$

$$\frac{\mathbb{T}\,\varphi\wedge\psi}{\begin{array}{c}\mathbb{T}\,\varphi\\ \mathbb{T}\,\psi\end{array}}\;\wedge\mathbb{T} \qquad\qquad \frac{\mathbb{F}\,\varphi\wedge\psi}{\mathbb{F}\,\varphi \quad\mid\quad \mathbb{F}\,\psi}\;\wedge\mathbb{F}$$

### Rules for $\vee$

$$\frac{\mathbb{T}\,\varphi\vee\psi}{\mathbb{T}\,\varphi \quad\mid\quad \mathbb{T}\,\psi}\;\vee\mathbb{T} \qquad\qquad \frac{\mathbb{F}\,\varphi\vee\psi}{\begin{array}{c}\mathbb{F}\,\varphi\\ \mathbb{F}\,\psi\end{array}}\;\vee\mathbb{F}$$

### Rules for $\rightarrow$

$$\frac{\mathbb{T}\,\varphi\rightarrow\psi}{\mathbb{F}\,\varphi \quad\mid\quad \mathbb{T}\,\psi}\;\rightarrow\mathbb{T} \qquad\qquad \frac{\mathbb{F}\,\varphi\rightarrow\psi}{\begin{array}{c}\mathbb{T}\,\varphi\\ \mathbb{F}\,\psi\end{array}}\;\rightarrow\mathbb{F}$$

### The Cut Rule

$$\frac{}{\mathbb{T}\,\varphi \quad\mid\quad \mathbb{F}\,\varphi}\;\text{Cut}$$

The Cut rule is not applied "to" a previous signed formula; rather, it allows every branch in a tableau to be split in two, one branch containing $\mathbb{T}\,\varphi$, the other $\mathbb{F}\,\varphi$. It is not necessary—any set of signed formulas with a closed tableau has one not using Cut—but it allows us to combine tableaux in a convenient way.

## 5.3 Tableaux

pl:tab:der:sec We've said what an assumption is, and we've given the rules of inference. explanation
Tableaux are inductively generated from these: each tableau either is a single branch consisting of one or more assumptions, or it results from a tableau by applying one of the rules of inference on a branch.

**Definition 5.2** (Tableau)**.** A tableau for assumptions $S_1\varphi_1, \ldots, S_n\varphi_n$ (where each $S_i$ is either $\mathbb{T}$ or $\mathbb{F}$) is a tree of signed formulas satisfying the following conditions:

1. The $n$ topmost signed formulas of the tree are $S_i\varphi_i$, one below the other.

2. Every signed formula in the tree that is not one of the assumptions results from a correct application of an inference rule to a signed formula in the branch above it.

A branch of a tableau is *closed* iff it contains both $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$, and *open* otherwise. A tableau in which every branch is closed is a *closed tableau* (for its set of assumptions). If a tableau is not closed, i.e., if it contains at least one open branch, it is *open*.

**Example 5.3.** Every set of assumptions on its own is a tableau, but it will generally not be closed. (Obviously, it is closed only if the assumptions already contain a pair of signed formulas $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$.)

From a tableau (open or closed) we can obtain a new, larger one by applying one of the rules of inference to a signed formula $\varphi$ in it. The rule will append one or more signed formulas to the end of any branch containing the occurrence of $\varphi$ to which we apply the rule.

For instance, consider the assumption $\mathbb{T}\,\varphi \wedge \neg\varphi$. Here is the (open) tableau consisting of just that assumption:

| | | |
|---|---|---|
| 1. | $\mathbb{T}\,\varphi \wedge \neg\varphi$ | Assumption |

We obtain a new tableau from it by applying the $\wedge\mathbb{T}$ rule to the assumption. That rule allows us to add two new lines to the tableau, $\mathbb{T}\,\varphi$ and $\mathbb{T}\,\neg\varphi$:

| | | |
|---|---|---|
| 1. | $\mathbb{T}\,\varphi \wedge \neg\varphi$ | Assumption |
| 2. | $\mathbb{T}\,\varphi$ | $\wedge\mathbb{T}\,1$ |
| 3. | $\mathbb{T}\,\neg\varphi$ | $\wedge\mathbb{T}\,1$ |

When we write down tableaux, we record the rules we've applied on the right (e.g., $\wedge\mathbb{T}1$ means that the signed formula on that line is the result of applying the $\wedge\mathbb{T}$ rule to the signed formula on line 1). This new tableau now contains additional signed formulas, but to only one ($\mathbb{T}\,\neg\varphi$) can we apply a rule (in this case, the $\neg\mathbb{T}$ rule). This results in the closed tableau

| | | |
|---|---|---|
| 1. | $\mathbb{T}\,\varphi \wedge \neg\varphi$ | Assumption |
| 2. | $\mathbb{T}\,\varphi$ | $\wedge\mathbb{T}\,1$ |
| 3. | $\mathbb{T}\,\neg\varphi$ | $\wedge\mathbb{T}\,1$ |
| 4. | $\mathbb{F}\,\varphi$ | $\neg\mathbb{T}\,3$ |
| | $\otimes$ | |

## 5.4 Examples of Tableaux

**Example 5.4.** Let's find a closed tableau for the sentence $(\varphi \wedge \psi) \to \varphi$. pl:tab:pro:sec

We begin by writing the corresponding assumption at the top of the tableau.

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,(\varphi \wedge \psi) \to \varphi$ | Assumption |

There is only one assumption, so only one signed formula to which we can apply a rule. (For every signed formula, there is always at most one rule that can be applied: it's the rule for the corresponding sign and main operator of the sentence.) In this case, this means, we must apply $\to\mathbb{F}$.

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,(\varphi \wedge \psi) \to \varphi$ ✓ | Assumption |
| 2. | $\mathbb{T}\,\varphi \wedge \psi$ | $\to\mathbb{F}\,1$ |
| 3. | $\mathbb{F}\,\varphi$ | $\to\mathbb{F}\,1$ |

To keep track of which signed formulas we have applied their corresponding rules to, we write a checkmark next to the sentence. However, *only* write a checkmark if the rule has been applied to all open branches. Once a signed formula has had the corresponding rule applied in every open branch, we will not have to return to it and apply the rule again. In this case, there is only one branch, so the rule only has to be applied once. (Note that checkmarks are only a convenience for constructing tableaux and are not officially part of the syntax of tableaux.)

There is one new signed formula to which we can apply a rule: the $\mathbb{T}\,\varphi \wedge \psi$ on line 3. Applying the $\wedge\mathbb{T}$ rule results in:

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,(\varphi \wedge \psi) \to \varphi$ ✓ | Assumption |
| 2. | $\mathbb{T}\,\varphi \wedge \psi$ ✓ | $\to\mathbb{F}\,1$ |
| 3. | $\mathbb{F}\,\varphi$ | $\to\mathbb{F}\,1$ |
| 4. | $\mathbb{T}\,\varphi$ | $\wedge\mathbb{T}\,2$ |
| 5. | $\mathbb{T}\,\psi$ | $\wedge\mathbb{T}\,2$ |
| | $\otimes$ | |

Since the branch now contains both $\mathbb{T}\,\varphi$ (on line 4) and $\mathbb{F}\,\varphi$ (on line 3), the branch is closed. Since it is the only branch, the tableau is closed. We have found a closed tableau for $(\varphi \wedge \psi) \to \varphi$.

**Example 5.5.** Now let's find a closed tableau for $(\neg\varphi \vee \psi) \to (\varphi \to \psi)$.

We begin with the corresponding assumption:

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,(\neg\varphi \vee \psi) \to (\varphi \to \psi)$ | Assumption |

The one signed formula in this tableau has main operator $\to$ and sign $\mathbb{F}$, so we apply the $\to\mathbb{F}$ rule to it to obtain:

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,(\neg\varphi \vee \psi) \to (\varphi \to \psi)$ ✓ | Assumption |
| 2. | $\mathbb{T}\,\neg\varphi \vee \psi$ | $\to\mathbb{F}\,1$ |
| 3. | $\mathbb{F}\,(\varphi \to \psi)$ | $\to\mathbb{F}\,1$ |

We now have a choice as to whether to apply $\vee\mathbb{T}$ to line 2 or $\to\mathbb{F}$ to line 3. It actually doesn't matter which order we pick, as long as each signed formula has its corresponding rule applied in every branch. So let's pick the first one. The $\vee\mathbb{T}$ rule allows the tableau to branch, and the two conclusions of the rule will be the new signed formulas added to the two new branches. This results in:

| | | | |
|---|---|---|---|
| 1. | $\mathbb{F}\,(\neg\varphi \vee \psi) \to (\varphi \to \psi)$ ✓ | | Assumption |
| 2. | $\mathbb{T}\,\neg\varphi \vee \psi$ ✓ | | $\to\mathbb{F}\,1$ |
| 3. | $\mathbb{F}\,(\varphi \to \psi)$ | | $\to\mathbb{F}\,1$ |
| 4. | $\mathbb{T}\,\neg\varphi$ | $\mathbb{T}\,\psi$ | $\vee\mathbb{T}\,2$ |

We have not applied the $\to\mathbb{F}$ rule to line 3 yet: let's do that now. To save time, we apply it to both branches. Recall that we write a checkmark next to a signed formula only if we have applied the corresponding rule in every open branch. So it's a good idea to apply a rule at the end of every branch that contains the signed formula the rule applies to. That way we won't have to return to that signed formula lower down in the various branches.

| | | | |
|---|---|---|---|
| 1. | $\mathbb{F}\,(\neg\varphi \vee \psi) \to (\varphi \to \psi)$ ✓ | | Assumption |
| 2. | $\mathbb{T}\,\neg\varphi \vee \psi$ ✓ | | $\to\mathbb{F}\,1$ |
| 3. | $\mathbb{F}\,(\varphi \to \psi)$ ✓ | | $\to\mathbb{F}\,1$ |
| 4. | $\mathbb{T}\,\neg\varphi$ | $\mathbb{T}\,\psi$ | $\vee\mathbb{T}\,2$ |
| 5. | $\mathbb{T}\,\varphi$ | $\mathbb{T}\,\varphi$ | $\to\mathbb{F}\,3$ |
| 6. | $\mathbb{F}\,\psi$ | $\mathbb{F}\,\psi$ | $\to\mathbb{F}\,3$ |
| | | $\otimes$ | |

The right branch is now closed. On the left branch, we can still apply the $\neg\mathbb{T}$ rule to line 4. This results in $\mathbb{F}\,\varphi$ and closes the left branch:

| | | | |
|---|---|---|---|
| 1. | $\mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi)$ ✓ | | Assumption |
| 2. | $\mathbb{T}\,\neg\varphi \lor \psi$ ✓ | | $\to\mathbb{F}\,1$ |
| 3. | $\mathbb{F}\,(\varphi \to \psi)$ ✓ | | $\to\mathbb{F}\,1$ |
| 4. | $\mathbb{T}\,\neg\varphi$ | $\mathbb{T}\,\psi$ | $\lor\mathbb{T}\,2$ |
| 5. | $\mathbb{T}\,\varphi$ | $\mathbb{T}\,\varphi$ | $\to\mathbb{F}\,3$ |
| 6. | $\mathbb{F}\,\psi$ | $\mathbb{F}\,\psi$ | $\to\mathbb{F}\,3$ |
| 7. | $\mathbb{F}\,\varphi$ | $\otimes$ | $\neg\mathbb{T}\,4$ |
| | $\otimes$ | | |

**Example 5.6.** We can give tableaux for any number of signed formulas as assumptions. Often it is also necessary to apply more than one rule that allows branching; and in general a tableau can have any number of branches. For instance, consider a tableau for $\{\mathbb{T}\,\varphi \lor (\psi \land \chi), \mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)\}$. We start by applying the $\lor\mathbb{T}$ to the first assumption:

| | | | |
|---|---|---|---|
| 1. | $\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓ | | Assumption |
| 2. | $\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ | | Assumption |
| 3. | $\mathbb{T}\,\varphi$ | $\mathbb{T}\,\psi \land \chi$ | $\lor\mathbb{T}\,1$ |

Now we can apply the $\land\mathbb{F}$ rule to line 2. We do this on both branches simultaneously, and can therefore check off line 2:

| | | | | | |
|---|---|---|---|---|---|
| 1. | $\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓ | | | | Assumption |
| 2. | $\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓ | | | | Assumption |
| 3. | $\mathbb{T}\,\varphi$ | | $\mathbb{T}\,\psi \land \chi$ | | $\lor\mathbb{T}\,1$ |
| 4. | $\mathbb{F}\,\varphi \lor \psi$ | $\mathbb{F}\,\varphi \lor \chi$ | $\mathbb{F}\,\varphi \lor \psi$ | $\mathbb{F}\,\varphi \lor \chi$ | $\land\mathbb{F}\,2$ |

Now we can apply $\lor\mathbb{F}$ to all the branches containing $\varphi \lor \psi$:

| | | | | | |
|---|---|---|---|---|---|
| 1. | $\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓ | | | | Assumption |
| 2. | $\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓ | | | | Assumption |
| 3. | $\mathbb{T}\,\varphi$ | | $\mathbb{T}\,\psi \land \chi$ | | $\lor\mathbb{T}\,1$ |
| 4. | $\mathbb{F}\,\varphi \lor \psi$ ✓ | $\mathbb{F}\,\varphi \lor \chi$ | $\mathbb{F}\,\varphi \lor \psi$ ✓ | $\mathbb{F}\,\varphi \lor \chi$ | $\land\mathbb{F}\,2$ |
| 5. | $\mathbb{F}\,\varphi$ | | $\mathbb{F}\,\varphi$ | | $\lor\mathbb{F}\,4$ |
| 6. | $\mathbb{F}\,\psi$ | | $\mathbb{F}\,\psi$ | | $\lor\mathbb{F}\,4$ |
| | $\otimes$ | | | | |

The leftmost branch is now closed. Let's now apply $\lor\mathbb{F}$ to $\varphi \lor \chi$:

| | | | | | |
|---|---|---|---|---|---|
| 1. | $\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓ | | | | Assumption |
| 2. | $\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓ | | | | Assumption |
| 3. | $\mathbb{T}\,\varphi$ | | $\mathbb{T}\,\psi \land \chi$ | | $\lor\mathbb{T}\,1$ |
| 4. | $\mathbb{F}\,\varphi \lor \psi$ ✓ | $\mathbb{F}\,\varphi \lor \chi$ ✓ | $\mathbb{F}\,\varphi \lor \psi$ ✓ | $\mathbb{F}\,\varphi \lor \chi$ ✓ | $\land\mathbb{F}\,2$ |
| 5. | $\mathbb{F}\,\varphi$ | | $\mathbb{F}\,\varphi$ | | $\lor\mathbb{F}\,4$ |
| 6. | $\mathbb{F}\,\psi$ | | $\mathbb{F}\,\psi$ | | $\lor\mathbb{F}\,4$ |
| 7. | $\otimes$ | $\mathbb{F}\,\varphi$ | | $\mathbb{F}\,\varphi$ | $\lor\mathbb{F}\,4$ |
| 8. | | $\mathbb{F}\,\chi$ | | $\mathbb{F}\,\chi$ | $\lor\mathbb{F}\,4$ |
| | | $\otimes$ | | | |

Note that we moved the result of applying $\lor\mathbb{F}$ a second time below for clarity. In this instance it would not have been needed, since the justifications would have been the same.

Two branches remain open, and $\mathbb{T}\,\psi \land \chi$ on line 3 remains unchecked. We apply $\land\mathbb{T}$ to it to obtain a closed tableau:

| | | | | | |
|---|---|---|---|---|---|
| 1. | $\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓ | | | | Assumption |
| 2. | $\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓ | | | | Assumption |
| 3. | $\mathbb{T}\,\varphi$ | | $\mathbb{T}\,\psi \land \chi$ ✓ | | $\lor\mathbb{T}\,1$ |
| 4. | $\mathbb{F}\,\varphi \lor \psi$ ✓ | $\mathbb{F}\,\varphi \lor \chi$ ✓ | $\mathbb{F}\,\varphi \lor \psi$ ✓ | $\mathbb{F}\,\varphi \lor \chi$ ✓ | $\land\mathbb{F}\,2$ |
| 5. | $\mathbb{F}\,\varphi$ | $\mathbb{F}\,\varphi$ | $\mathbb{F}\,\varphi$ | $\mathbb{F}\,\varphi$ | $\lor\mathbb{F}\,4$ |
| 6. | $\mathbb{F}\,\psi$ | $\mathbb{F}\,\chi$ | $\mathbb{F}\,\psi$ | $\mathbb{F}\,\chi$ | $\lor\mathbb{F}\,4$ |
| 7. | $\otimes$ | $\otimes$ | $\mathbb{T}\,\psi$ | $\mathbb{T}\,\psi$ | $\land\mathbb{T}\,3$ |
| 8. | | | $\mathbb{T}\,\chi$ | $\mathbb{T}\,\chi$ | $\land\mathbb{T}\,3$ |
| | | | $\otimes$ | $\otimes$ | |

For comparison, here's a closed tableau for the same set of assumptions in which the rules are applied in a different order:

$$
\begin{array}{llllll}
1. & & \mathbb{T}\,\varphi \lor (\psi \land \chi) \checkmark & & & \text{Assumption} \\
2. & & \mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi) \checkmark & & & \text{Assumption} \\
3. & \mathbb{F}\,\varphi \lor \psi \checkmark & & \mathbb{F}\,\varphi \lor \chi \checkmark & & \land\mathbb{F}\,2 \\
4. & \mathbb{F}\,\varphi & & \mathbb{F}\,\varphi & & \lor\mathbb{F}\,3 \\
5. & \mathbb{F}\,\psi & & \mathbb{F}\,\chi & & \lor\mathbb{F}\,3 \\
6. & \mathbb{T}\,\varphi & \mathbb{T}\,\psi \land \chi \checkmark & \mathbb{T}\,\varphi & \mathbb{T}\,\psi \land \chi \checkmark & \lor\mathbb{T}\,1 \\
7. & \otimes & \mathbb{T}\,\psi & \otimes & \mathbb{T}\,\psi & \land\mathbb{T}\,3 \\
8. & & \mathbb{T}\,\chi & & \mathbb{T}\,\chi & \land\mathbb{T}\,3 \\
 & & \otimes & & \otimes &
\end{array}
$$

**Problem 5.1.** Give closed tableaux of the following:

1. $\mathbb{F}\,\neg(\varphi \to \psi) \to (\varphi \land \neg\psi)$
2. $\mathbb{F}\,(\varphi \to \chi) \lor (\psi \to \chi), \mathbb{T}\,(\varphi \land \psi) \to \chi$

## 5.5 Proof-Theoretic Notions

pl:tab:ptn: sec

This section collects the definitions of the provability relation and consistency for tableaux.

explanation

Just as we've defined a number of important semantic notions (validity, entailment, satisfiabilty), we now define corresponding *proof-theoretic notions.* These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the existence of certain closed tableaux. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems.*

**Definition 5.7** (Theorems)**.** A sentence $\varphi$ is a *theorem* if there is a closed tableau for $\mathbb{F}\,\varphi$. We write $\vdash \varphi$ if $\varphi$ is a theorem and $\nvdash \varphi$ if it is not.

**Definition 5.8** (Derivability)**.** A sentence $\varphi$ is *derivable from* a set of sentences $\Gamma$, $\Gamma \vdash \varphi$, iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed tableau for the set

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n,\}$$

If $\varphi$ is not derivable from $\Gamma$ we write $\Gamma \nvdash \varphi$.

**Definition 5.9** (Consistency)**.** A set of sentences $\Gamma$ is *inconsistent* iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed tableau for the set

$$\{\mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n,\}.$$

If $\Gamma$ is not inconsistent, we say it is *consistent.*

pl:tab:ptn: prop:reflexivity **Proposition 5.10** (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

*Proof.* If $\varphi \in \Gamma$, $\{\varphi\}$ is a finite subset of $\Gamma$ and the tableau

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,\varphi$ | Assumption |
| 2. | $\mathbb{T}\,\varphi$ | Assumption |
| | $\otimes$ | |

is closed. □

pl:tab:ptn: prop:monotony **Proposition 5.11** (Monotony). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

*Proof.* Any finite subset of $\Gamma$ is also a finite subset of $\Delta$. □

pl:tab:ptn: prop:transitivity **Proposition 5.12** (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

*Proof.* If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a finite subset $\Delta_0 = \{\chi_1, \dots, \chi_n\} \subseteq \Delta$ such that

$$\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_n\}$$

has a closed tableau. If $\Gamma \vdash \varphi$ then there are $\theta_1$, ..., $\theta_m$ such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\theta_1, \dots, \mathbb{T}\,\theta_m\}$$

has a closed tableau.

Now consider the tableau with assumptions

$$\mathbb{F}\,\psi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_n, \mathbb{T}\,\theta_1, \dots, \mathbb{T}\,\theta_m.$$

Apply the Cut rule on $\varphi$. This generates two branches, one has $\mathbb{T}\,\varphi$ in it, the other $\mathbb{F}\,\varphi$. Thus, on the one branch, all of

$$\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_n\}$$

are available. Since there is a closed tableau for these assumptions, we can attach it to that branch; every branch through $\mathbb{T}\,\varphi_1$ closes. On the other branch, all of

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\theta_1, \dots, \mathbb{T}\,\theta_m\}$$

are available, so we can also complete the other side to obtain a closed tableau. This shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each $i$, then $\Gamma \vdash \psi$.

pl:tab:ptn: prop:incons **Proposition 5.13.** *$\Gamma$ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence $\varphi$.*

*Proof.* Exercise. □

**Problem 5.2.** Prove Proposition 5.13

**Proposition 5.14** (Compactness). pl:tab:ptn: prop:proves-compact

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*
2. *If every finite subset of $\Gamma$ is consistent, then $\Gamma$ is consistent.*

*Proof.*

1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and a closed tableau for
$$\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \cdots \mathbb{T}\,\psi_n$$
This tableau also shows $\Gamma_0 \vdash \varphi$.

2. If $\Gamma$ is inconsistent, then for some finite subset $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ there is a closed tableau for
$$\mathbb{T}\,\psi_1, \cdots \mathbb{T}\,\psi_n$$
This closed tableau shows that $\Gamma_0$ is inconsistent.

□

## 5.6 Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem. pl:tab:prv: sec

**Proposition 5.15.** *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.* pl:tab:prv: prop:provability-contr

*Proof.* There are finite $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and $\Gamma_1 = \{\chi_1, \dots, \chi_n\} \subseteq \Gamma$ such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$
$$\{\mathbb{T}\,\neg\varphi, \mathbb{T}\,\chi_1, \dots, \mathbb{T}\,\chi_m\}$$

have closed tableaux. Using the Cut rule on $\varphi$ we can combine these into a single closed tableau that shows $\Gamma_0 \cup \Gamma_1$ is inconsistent. Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence $\Gamma$ is inconsistent. □

**Proposition 5.16.** *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.* pl:tab:prv: prop:prov-incons

*Proof.* First suppose $\Gamma \vdash \varphi$, i.e., there is a closed tableau for

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

Using the $\neg\mathbb{T}$ rule, this can be turned into a closed tableau for

$$\{\mathbb{T}\,\neg\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

On the other hand, if there is a closed tableau for the latter, we can turn it into a closed tableau of the former by removing every formula that results from

$\neg\mathbb{T}$applied to the first assumption $\mathbb{T}\neg\varphi$ as well as that assumption, and adding the assumption $\mathbb{F}\varphi$. For if a branch was closed before because it contained the conclusion of $\neg\mathbb{T}$applied to $\mathbb{T}\neg\varphi$, i.e., $\mathbb{F}\varphi$, the corresponding branch in the new tableau is also closed. If a branch in the old tableau was closed because it contained the assumption $\mathbb{T}\neg\varphi$ as well as $\mathbb{F}\neg\varphi$ we can turn it into a closed branch by applying $\neg\mathbb{F}$ to $\mathbb{F}\neg\varphi$ to obtain $\mathbb{T}\varphi$. This closes the branch since we added $\mathbb{F}\varphi$ as an assumption. □

**Problem 5.3.** Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

*pl:tab:prv: prop:explicit-inc*

**Proposition 5.17.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then $\Gamma$ is inconsistent.*

*Proof.* Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ such that

$$\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$$

has a closed tableau. Replace the assumption $\mathbb{F}\varphi$by $\mathbb{T}\neg\varphi$, and insert the conclusion of $\neg\mathbb{T}$applied to $\mathbb{F}\varphi$after the assumptions. Any sentence in the tableau justified by appeal to line 1 in the old tableau is now justified by appeal to line $n+1$. So if the old tableau was closed, the new one is. It shows that $\Gamma$ is inconsistent, since all assumptions are in $\Gamma$. □

*pl:tab:prv: prop:provability-exhaustive*

**Proposition 5.18.** *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.*

*Proof.* If there are $\psi_1, \dots, \psi_n \in \Gamma$ and $\chi_1, \dots, \chi_m \in \Gamma$ such that

$$\{\mathbb{T}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$$
$$\{\mathbb{T}\neg\varphi, \mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m\}$$

both have closed tableaux, we can construct a tableau that shows that $\Gamma$ is inconsistent by using as assumptions $\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n$ together with $\mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m$, followed by an application of the Cut rule, yielding two branches, one starting with $\mathbb{T}\varphi$, the other with $\mathbb{F}\varphi$. Add on the part below the assumptions of the first tableau on the left side. Here, every rule application is still correct, and every branch closes. On the right side, add the part below the assumptions of the seond tableau, with the results of any applications of $\neg\mathbb{T}$ to $\mathbb{T}\neg\varphi$ removed.

For if a branch was closed before because it contained the conclusion of $\neg\mathbb{T}$applied to $\mathbb{T}\neg\varphi$, i.e., $\mathbb{F}\varphi$, as well as $\mathbb{F}\varphi$, the corresponding branch in the new tableau is also closed. If a branch in the old tableau was closed because it contained the assumption $\mathbb{T}\neg\varphi$ as well as $\mathbb{F}\neg\varphi$ we can turn it into a closed branch by applying $\neg\mathbb{F}$ to $\mathbb{F}\neg\varphi$ to obtain $\mathbb{T}\varphi$. □

## 5.7 Derivability and the Propositional Connectives

pl:tab:ppr: sec

*pl:tab:ppr: prop:provability-land*

**Proposition 5.19.**

1. *Both* $\varphi \land \psi \vdash \varphi$ *and* $\varphi \land \psi \vdash \psi$. *pl:tab:ppr: prop:provability-land-left*

2. $\varphi, \psi \vdash \varphi \land \psi$. *pl:tab:ppr: prop:provability-land-right*

*Proof.* 1. Both $\{\mathbb{F}\,\varphi, \mathbb{T}\,\varphi \land \psi\}$ and $\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi \land \psi\}$ have closed tableaux

$$\begin{array}{llc l}
1. & & \mathbb{F}\,\varphi & \text{Assumption} \\
2. & & \mathbb{T}\,\varphi \land \psi & \text{Assumption} \\
3. & & \mathbb{T}\varphi & \land\mathbb{T}\,2 \\
4. & & \mathbb{T}\psi & \land\mathbb{T}\,2 \\
 & & \otimes &
\end{array}$$

$$\begin{array}{llc l}
1. & & \mathbb{F}\,\psi & \text{Assumption} \\
2. & & \mathbb{T}\,\varphi \land \psi & \text{Assumption} \\
3. & & \mathbb{T}\varphi & \land\mathbb{T}\,2 \\
4. & & \mathbb{T}\psi & \land\mathbb{T}\,2 \\
 & & \otimes &
\end{array}$$

2. Here is a closed tableau for $\{\mathbb{T}\,\varphi, \mathbb{T}\,\psi, \mathbb{F}\,\varphi \land \psi\}$:

$$\begin{array}{l cc l}
1. & \multicolumn{2}{c}{\mathbb{F}\,\varphi \land \psi} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\varphi} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\psi} & \text{Assumption} \\
4. & \mathbb{F}\,\varphi & \mathbb{F}\,\psi & \land\mathbb{F}\,1 \\
 & \otimes & \otimes &
\end{array}$$

□

**Proposition 5.20.** *pl:tab:ppr: prop:provability-lor*

1. $\varphi \lor \psi, \neg\varphi, \neg\psi$ *is inconsistent.*

2. *Both* $\varphi \vdash \varphi \lor \psi$ *and* $\psi \vdash \varphi \lor \psi$.

*Proof.* 1. We give a closed tableau of $\{\mathbb{T}\,\varphi \lor \psi, \mathbb{T}\,\neg\varphi, \mathbb{T}\,\neg\psi\}$:

$$\begin{array}{l cc l}
1. & \multicolumn{2}{c}{\mathbb{T}\,\varphi \lor \psi} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\neg\varphi} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\neg\psi} & \text{Assumption} \\
4. & \multicolumn{2}{c}{\mathbb{F}\,\varphi} & \neg\mathbb{T}\,2 \\
5. & \multicolumn{2}{c}{\mathbb{F}\,\psi} & \neg\mathbb{T}\,3 \\
6. & \mathbb{T}\,\varphi & \mathbb{T}\,\psi & \lor\mathbb{T}\,1 \\
 & \otimes & \otimes &
\end{array}$$

2. Both $\{\mathbb{F}\,\varphi \lor \psi, \mathbb{T}\,\varphi\}$ and $\{\mathbb{F}\,\varphi \lor \psi, \mathbb{T}\,\psi\}$ have closed tableaux:

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,\varphi \lor \psi$ | Assumption |
| 2. | $\mathbb{T}\,\varphi$ | Assumption |
| 3. | $\mathbb{F}\varphi$ | $\lor\mathbb{F}\,1$ |
| 4. | $\mathbb{F}\psi$ | $\lor\mathbb{F}\,1$ |
| | $\otimes$ | |

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,\varphi \lor \psi$ | Assumption |
| 2. | $\mathbb{T}\,\psi$ | Assumption |
| 3. | $\mathbb{F}\varphi$ | $\lor\mathbb{F}\,1$ |
| 4. | $\mathbb{F}\psi$ | $\lor\mathbb{F}\,1$ |
| | $\otimes$ | |

□

pl:tab:ppr: prop:provability-lif

**Proposition 5.21.**

pl:tab:ppr: prop:provability-lif-left

1. *$\varphi, \varphi \to \psi \vdash \psi$.*

pl:tab:ppr: prop:provability-lif-right

2. *Both $\neg\varphi \vdash \varphi \to \psi$ and $\psi \vdash \varphi \to \psi$.*

*Proof.* 1. $\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi \to \psi, \mathbb{T}\,\varphi\}$ has a closed tableau:

| | | | |
|---|---|---|---|
| 1. | $\mathbb{F}\,\psi$ | | Assumption |
| 2. | $\mathbb{T}\,\varphi \to \psi$ | | Assumption |
| 3. | $\mathbb{T}\,\varphi$ | | Assumption |
| 4. | $\mathbb{F}\,\varphi$ | $\mathbb{T}\,\psi$ | $\to\mathbb{T}\,2$ |
| | $\otimes$ | $\otimes$ | |

2. Both s$\{\mathbb{F}\,\varphi \to \psi, \mathbb{T}\,\neg\varphi\}$ and $\{\mathbb{F}\,\varphi \to \psi, \mathbb{T}\,\neg\psi\}$ have closed tableaux:

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,\varphi \to \psi$ | Assumption |
| 2. | $\mathbb{T}\,\neg\varphi$ | Assumption |
| 3. | $\mathbb{T}\,\varphi$ | $\to\mathbb{F}\,1$ |
| 4. | $\mathbb{F}\,\psi$ | $\to\mathbb{F}\,1$ |
| 5. | $\mathbb{F}\,\varphi$ | $\neg\mathbb{T}\,2$ |
| | $\otimes$ | |

| | | |
|---|---|---|
| 1. | $\mathbb{F}\,\varphi \to \psi$ | Assumption |
| 2. | $\mathbb{T}\,\neg\psi$ | Assumption |
| 3. | $\mathbb{T}\,\varphi$ | $\to\mathbb{F}\,1$ |
| 4. | $\mathbb{F}\,\psi$ | $\to\mathbb{F}\,1$ |
| 5. | $\mathbb{F}\,\psi$ | $\neg\mathbb{T}\,2$ |
| | $\otimes$ | |

□

## 5.8 Soundness

explanation A derivation system, such as tableaux, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that pl:tab:sou:sec

1. every derivable $\varphi$ is a tautology;
2. if a sentence is derivable from some others, it is also a consequence of them;
3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Because all these proof-theoretic properties are defined via closed tableaux of some kind or other, proving (1)–(3) above requires proving something about the semantic properties of closed tableaux. We will first define what it means for a signed formula to be satisfied in a structure, and then show that if a tableau is closed, no structure satisfies all its assumptions. (1)–(3) then follow as corollaries from this result.

**Definition 5.22.** A valuation $\mathfrak{v}$ *satisfies* a signed formula $\mathbb{T}\,\varphi$ iff $\mathfrak{v} \vDash \varphi$, and it satisfies $\mathbb{F}\,\varphi$ iff $\mathfrak{v} \nvDash \varphi$. $\mathfrak{v}$ satisfies a set of signed formulas $\Gamma$ iff it satisfies every $S\,\varphi \in \Gamma$. $\Gamma$ is *satisfiable* if there is a valuation that satisfies it, and *unsatisfiable* otherwise.

**Theorem 5.23** (Soundness)**.** *If $\Gamma$ has a closed tableau, $\Gamma$ is unsatisfiable.* pl:tab:sou:thm:tableau-soundness

*Proof.* Let's call a branch of a tableau satisfiable iff the set of signed formulas on it is satisfiable, and let's call a tableau satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable tableau by one of the rules of inference always results in a satisfiable tableau. This will prove the theorem: any closed tableau results by applying rules of inference to the tableau consisting only of assumptions from $\Gamma$. So if $\Gamma$ were satisfiable, any tableau for it would be satisfiable. A closed tableau, however, is clearly not satisfiable: every branch contains both $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$, and no structure can both satisfy and not satisfy $\varphi$.

Suppose we have a satisfiable tableau, i.e., a tableau with at least one satisfiable branch. Applying a rule of inference either adds signed formulas to a branch, or splits a branch in two. If the tableau has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended tableau, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let $\Gamma$ be the set of signed formulas on that branch, and let $S\,\varphi \in \Gamma$ be the signed formula to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, i.e., $\Gamma$ together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable.

First, we consider the possible inferences with only one premise.

1. The branch is expanded by applying $\neg\mathbb{T}$ to $\mathbb{T}\,\neg\psi \in \Gamma$. Then the extended branch contains the signed formulas $\Gamma \cup \{\mathbb{F}\,\psi\}$. Suppose $\mathfrak{v} \vDash \Gamma$. In particular, $\mathfrak{v} \vDash \neg\psi$. Thus, $\mathfrak{v} \nvDash \psi$, i.e., $\mathfrak{v}$ satisfies $\mathbb{F}\,\psi$.

2. The branch is expanded by applying $\neg\mathbb{F}$ to $\mathbb{F}\,\neg\psi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\wedge\mathbb{T}$ to $\mathbb{T}\,\psi \wedge \chi \in \Gamma$, which results in two new signed formulas on the branch: $\mathbb{T}\,\psi$ and $\mathbb{T}\,\chi$. Suppose $\mathfrak{v} \vDash \Gamma$, in particular $\mathfrak{v} \vDash \psi \wedge \chi$. Then $\mathfrak{v} \vDash \psi$ and $\mathfrak{v} \vDash \chi$. This means that $\mathfrak{v}$ satisfies both $\mathbb{T}\,\psi$ and $\mathbb{T}\,\chi$.

4. The branch is expanded by applying $\vee\mathbb{F}$ to $\mathbb{T}\,\psi \vee \chi \in \Gamma$: Exercise.

5. The branch is expanded by applying $\rightarrow\mathbb{F}$ to $\mathbb{T}\,\psi \rightarrow \chi \in \Gamma$: This results in two new signed formulas on the branch: $\mathbb{T}\,\psi$ and $\mathbb{F}\,\chi$. Suppose $\mathfrak{v} \vDash \Gamma$, in particular $\mathfrak{v} \nvDash \psi \rightarrow \chi$. Then $\mathfrak{v} \vDash \psi$ and $\mathfrak{v} \nvDash \chi$. This means that $\mathfrak{v}$ satisfies both $\mathbb{T}\,\psi$ and $\mathbb{F}\,\chi$.

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying $\wedge\mathbb{F}$ to $\mathbb{F}\,\psi \wedge \chi \in \Gamma$, which results in two branches, a left one continuing through $\mathbb{F}\,\psi$ and a right one through $\mathbb{F}\,\chi$. Suppose $\mathfrak{v} \vDash \Gamma$, in particular $\mathfrak{v} \nvDash \psi \wedge \chi$. Then $\mathfrak{v} \nvDash \psi$ or $\mathfrak{v} \nvDash \chi$. In the former case, $\mathfrak{v}$ satisfies $\mathbb{F}\,\psi$, i.e., $\mathfrak{v}$ satisfies the formulas on the left branch. In the latter, $\mathfrak{v}$ satisfies $\mathbb{F}\,\chi$, i.e., $\mathfrak{v}$ satisfies the formulas on the right branch.

2. The branch is expanded by applying $\vee\mathbb{T}$ to $\mathbb{T}\,\psi \vee \chi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\rightarrow\mathbb{T}$ to $\mathbb{T}\,\psi \rightarrow \chi \in \Gamma$: Exercise.

4. The branch is expanded by Cut: This results in two branches, one containing $\mathbb{T}\,\psi$, the other containing $\mathbb{F}\,\psi$. Since $\mathfrak{v} \vDash \Gamma$ and either $\mathfrak{v} \vDash \psi$ or $\mathfrak{v} \nvDash \psi$, $\mathfrak{v}$ satisfies either the left or the right branch.

□

**Problem 5.4.** Complete the proof of Theorem 5.23.

**Corollary 5.24.** *If $\vdash \varphi$ then $\varphi$ is a tautology.* pl:tab:sou: cor:weak-soundness

**Corollary 5.25.** *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.* pl:tab:sou: cor:entailment-soundness

*Proof.* If $\Gamma \vdash \varphi$ then for some $\psi_1, \dots, \psi_n \in \Gamma$, $\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$ has a closed tableau. By Theorem 5.23, every valuation $\mathfrak{v}$ either makes some $\psi_i$ false or makes $\varphi$ true. Hence, if $\mathfrak{v} \vDash \Gamma$ then also $\mathfrak{v} \vDash \varphi$. □

**Corollary 5.26.** *If $\Gamma$ is satisfiable, then it is consistent.* pl:tab:sou: cor:consistency-soundness

*Proof.* We prove the contrapositive. Suppose that $\Gamma$ is not consistent. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ and a closed tableau for $\{\mathbb{T}\,\psi, \dots, \mathbb{T}\,\psi\}$. By Theorem 5.23, there is no $\mathfrak{v}$ such that $\mathfrak{v} \vDash \psi_i$ for all $i = 1, \dots, n$. But then $\Gamma$ is not satisfiable. □

# Chapter 6

# Axiomatic Derivations

No effort has been made yet to ensure that the material in this chapter respects various tags indicating which connectives and quantifiers are primitive or defined: all are assumed to be primitive. If the FOL tag is true, we produce a version with quantifiers, otherwise without.

## 6.1 Rules and Derivations

pl:axd:rul:sec Axiomatic derivations are perhaps the simplest proof system for logic. A derivation is just a sequence of formulas. To count as a derivation, every formula in the sequence must either be an instance of an axiom, or must follow from one or more formulas that precede it in the sequence by a rule of inference. A derivation derives its last formula. explanation

**Definition 6.1** (Derivability)**.** If $\Gamma$ is a set of formulas of $\mathcal{L}$ then a *derivation* from $\Gamma$ is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. $\varphi_i$ is an axiom; or
3. $\varphi_i$ follows from some $\varphi_j$ (and $\varphi_k$) with $j < i$ (and $k < i$) by a rule of inference.

What counts as a correct derivation depends on which inference rules we allow (and of course what we take to be axioms). And an inference rule is an if-then statement that tells us that, under certain conditions, a step $A_i$ in is a correct inference step.

**Definition 6.2** (Rule of inference)**.** A *rule of inference* gives a sufficient condition for what counts as a correct inference step in a derivation from $\Gamma$.

For instance, since any one-element sequence $\varphi$ with $\varphi \in \Gamma$ trivially counts as a derivation, the following might be a very simple rule of inference:

> If $\varphi \in \Gamma$, then $\varphi$ is always a correct inference step in any derivation from $\Gamma$.

Similarly, if $\varphi$ is one of the axioms, then $\varphi$ by itself is a derivation, and so this is also a rule of inference:

> If $\varphi$ is an axiom, then $\varphi$ is a correct inference step.

It gets more interesting if the rule of inference appeals to formulas that appear before the step considered. The following rule is called *modus ponens:*

> If $\psi \to \varphi$ and $\psi$ occur higher up in the derivation, then $\varphi$ is a correct inference step.

If this is the only rule of inference, then our definition of derivation above amounts to this: $\varphi_1, \dots, \varphi_n$ is a derivation iff for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. $\varphi_i$ is an axiom; or
3. for some $j < i$, $\varphi_j$ is $\psi \to \varphi_i$, and for some $k < i$, $\varphi_k$ is $\psi$.

The last clause says that $\varphi_i$ follows from $\varphi_j$ ($\psi$) and $\varphi_k$ ($\psi \to \varphi_i$) by modus ponens. If we can go from 1 to $n$, and each time we find a formula $\varphi_i$ that is either in $\Gamma$, an axiom, or which a rule of inference tells us that it is a correct inference step, then the entire sequence counts as a correct derivation.

**Definition 6.3** (Derivability)**.** A formula $\varphi$ is *derivable* from $\Gamma$, written $\Gamma \vdash \varphi$, if there is a derivation from $\Gamma$ ending in $\varphi$.

**Definition 6.4** (Theorems)**.** A formula $\varphi$ is a *theorem* if there is a derivation of $\varphi$ from the empty set. We write $\vdash \varphi$ if $\varphi$ is a theorem and $\nvdash \varphi$ if it is not.

## 6.2 Axiom and Rules for the Propositional Connectives

pl:axd:prp:sec

**Definition 6.5** (Axioms)**.** The set of $\mathrm{Ax}_0$ of *axioms* for the propositional connectives comprises all formulas of the following forms:

pl:axd:prp:ax:land1
$$(\varphi \land \psi) \to \varphi \tag{6.1}$$
pl:axd:prp:ax:land2
$$(\varphi \land \psi) \to \psi \tag{6.2}$$
pl:axd:prp:ax:land3
$$\varphi \to (\psi \to (\varphi \land \psi)) \tag{6.3}$$
pl:axd:prp:ax:lor1
$$\varphi \to (\varphi \lor \psi) \tag{6.4}$$
pl:axd:prp:ax:lor2
$$\varphi \to (\psi \lor \varphi) \tag{6.5}$$
pl:axd:prp:ax:lor3
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \lor \psi) \to \chi)) \tag{6.6}$$
pl:axd:prp:ax:lif1
$$\varphi \to (\psi \to \varphi) \tag{6.7}$$
pl:axd:prp:ax:lif2
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \tag{6.8}$$
pl:axd:prp:ax:lnot1
$$(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi) \tag{6.9}$$
pl:axd:prp:ax:lnot2
$$\neg\varphi \to (\varphi \to \psi) \tag{6.10}$$
pl:axd:prp:ax:ltrue
$$\top \tag{6.11}$$
pl:axd:prp:ax:lfalse1
$$\bot \to \varphi \tag{6.12}$$
pl:axd:prp:ax:lfalse2
$$(\varphi \to \bot) \to \neg\varphi \tag{6.13}$$
pl:axd:prp:ax:dne
$$\neg\neg\varphi \to \varphi \tag{6.14}$$

**Definition 6.6** (Modus ponens)**.** If $\psi$ and $\psi \to \varphi$ already occur in a derivation, then $\varphi$ is a correct inference step.

We'll abbreviate the rule modus ponens as "MP."

## 6.3 Examples of Derivations

pl:axd:pro:sec

**Example 6.7.** Suppose we want to prove $(\neg\theta \lor \alpha) \to (\theta \to \alpha)$. Clearly, this is not an instance of any of our axioms, so we have to use the MP rule to derive it. Our only rule is MP, which given $\varphi$ and $\varphi \to \psi$ allows us to justify $\psi$. One strategy would be to use eq. (6.6) with $\varphi$ being $\neg\theta$, $\psi$ being $\alpha$, and $\chi$ being $\theta \to \alpha$, i.e., the instance

$$(\neg\theta \to (\theta \to \alpha)) \to ((\alpha \to (\theta \to \alpha)) \to ((\neg\theta \lor \alpha) \to (\theta \to \alpha))).$$

Why? Two applications of MP yield the last part, which is what we want. And we easily see that $\neg\theta \to (\theta \to \alpha)$ is an instance of eq. (6.10), and $\alpha \to (\theta \to \alpha)$ is an instance of eq. (6.7). So our derivation is:

| | | |
|---|---|---|
| 1. | $\neg\theta \to (\theta \to \alpha)$ | eq. (6.7) |
| 2. | $(\neg\theta \to (\theta \to \alpha)) \to$ $\quad((\alpha \to (\theta \to \alpha)) \to ((\neg\theta \lor \alpha) \to (\theta \to \alpha)))$ | eq. (6.6) |
| 3. | $((\alpha \to (\theta \to \alpha)) \to ((\neg\theta \lor \alpha) \to (\theta \to \alpha))$ | 1, 2, MP |
| 4. | $\alpha \to (\theta \to \alpha)$ | eq. (6.7) |
| 5. | $(\neg\theta \lor \alpha) \to (\theta \to \alpha)$ | 3, 4, MP |

**Example 6.8.** Let's try to find a derivation of $\theta \to \theta$. It is not an instance of an axiom, so we have to use MP to derive it. eq. (6.7) is an axiom of the form $\varphi \to \psi$ to which we could apply MP. To be useful, of course, the $\psi$ which MP would justify as a correct step in this case would have to be $\theta \to \theta$, since this is what we want to derive. That means $\varphi$ would also have to be $\theta$, i.e., we might look at this instance of eq. (6.7): pl:axd:pro: ex:identity

$$\theta \to (\theta \to \theta)$$

In order to apply MP, we would also need to justify the corresponding second premise, namely $\varphi$. But in our case, that would be $\theta$, and we won't be able to derive $\theta$ by itself. So we need a different strategy.

The other axiom involving just $\to$ is eq. (6.8), i.e.,

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

We could get to the last nested conditional by applying MP twice. Again, that would mean that we want an instance of eq. (6.8) where $\varphi \to \chi$ is $\theta \to \theta$, the formula we are aiming for. Then of course, $\varphi$ and $\chi$ are both $\theta$. How should we pick $\psi$ so that both $\varphi \to (\psi \to \chi)$ and $\varphi \to \psi$, i.e., in our case $\theta \to (\psi \to \theta)$ and $\theta \to \psi$, are also derivable? Well, the first of these is already an instance of eq. (6.7), whatever we decide $\psi$ to be. And $\theta \to \psi$ would be another instance of eq. (6.7) if $\psi$ were $(\theta \to \theta)$. So, our derivation is:

| | | |
|---|---|---|
| 1. | $\theta \to ((\theta \to \theta) \to \theta)$ | eq. (6.7) |
| 2. | $(\theta \to ((\theta \to \theta) \to \theta)) \to$ <br> $((\theta \to (\theta \to \theta)) \to (\theta \to \theta))$ | eq. (6.8) |
| 3. | $(\theta \to (\theta \to \theta)) \to (\theta \to \theta)$ | 1, 2, MP |
| 4. | $\theta \to (\theta \to \theta)$ | eq. (6.7) |
| 5. | $\theta \to \theta$ | 3, 4, MP |

**Example 6.9.** Sometimes we want to show that there is a derivation of some formula from some other formulas $\Gamma$. For instance, let's show that we can derive $\varphi \to \chi$ from $\Gamma = \{\varphi \to \psi, \psi \to \chi\}$. pl:axd:pro: ex:chain

| | | |
|---|---|---|
| 1. | $\varphi \to \psi$ | Hyp |
| 2. | $\psi \to \chi$ | Hyp |
| 3. | $(\psi \to \chi) \to (\varphi \to (\psi \to \chi))$ | eq. (6.7) |
| 4. | $\varphi \to (\psi \to \chi)$ | 2, 3, MP |
| 5. | $(\varphi \to (\psi \to \chi)) \to$ <br> $((\varphi \to \psi) \to (\varphi \to \chi))$ | eq. (6.8) |
| 6. | $((\varphi \to \psi) \to (\varphi \to \chi))$ | 4, 5, MP |
| 7. | $\varphi \to \chi$ | 1, 6, MP |

The lines labelled "Hyp" (for "hypothesis") indicate that the formula on that line is an element of $\Gamma$.

**Proposition 6.10.** *If $\Gamma \vdash \varphi \to \psi$ and $\Gamma \vdash \psi \to \chi$, then $\Gamma \vdash \varphi \to \chi$* *pl:axd:pro: prop:chain*

*Proof.* Suppose $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \vdash \psi \rightarrow \chi$. Then there is a derivation of $\varphi \rightarrow \psi$ from $\Gamma$; and a derivation of $\psi \rightarrow \chi$ from $\Gamma$ as well. Combine these into a single derivation by concatenating them. Now add lines 3–7 of the derivation in the preceding example. This is a derivation of $\varphi \rightarrow \chi$—which is the last line of the new derivation—from $\Gamma$. Note that the justifications of lines 4 and 7 remain valid if the reference to line number 2 is replaced by reference to the last line of the derivation of $\varphi \rightarrow \psi$, and reference to line number 1 by reference to the last line of the derivation of $B \rightarrow \chi$. □

**Problem 6.1.** Show that the following hold by exhibiting derivations from the axioms:

1. $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
2. $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
3. $\neg(\varphi \vee \psi) \rightarrow \neg\varphi$

## 6.4 Proof-Theoretic Notions

pl:axd:ptn:sec Just as we've defined a number of important semantic notions (tautology, entailment, satisfiabilty), we now define corresponding *proof-theoretic notions.* These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain formulas. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems.* explanation

**Definition 6.11** (Derivability)**.** A formula $\varphi$ is *derivable* from $\Gamma$, written $\Gamma \vdash \varphi$, if there is a derivation from $\Gamma$ ending in $\varphi$.

**Definition 6.12** (Theorems)**.** A formula $\varphi$ is a *theorem* if there is a derivation of $\varphi$ from the empty set. We write $\vdash \varphi$ if $\varphi$ is a theorem and $\nvdash \varphi$ if it is not.

**Definition 6.13** (Consistency)**.** A set $\Gamma$ of formulas is *consistent* if and only if $\Gamma \nvdash \bot$; it is *inconsistent* otherwise.

*pl:axd:ptn:prop:reflexivity* **Proposition 6.14** (Reflexivity)**.** *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

*Proof.* The formula $\varphi$ by itself is a derivation of $\varphi$ from $\Gamma$. □

*pl:axd:ptn:prop:monotony* **Proposition 6.15** (Monotony)**.** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

*Proof.* Any derivation of $\varphi$ from $\Gamma$ is also a derivation of $\varphi$ from $\Delta$. □

*pl:axd:ptn:prop:transitivity* **Proposition 6.16** (Transitivity)**.** *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

*Proof.* Suppose $\{\varphi\} \cup \Delta \vdash \psi$. Then there is a derivation $\psi_1, \dots, \psi_l = \psi$ from $\{\varphi\} \cup \Delta$. Some of the steps in that derivation will be correct because of a rule which refers to a prior line $\psi_i = \varphi$. By hypothesis, there is a derivation of $\varphi$ from $\Gamma$, i.e., a derivation $\varphi_1, \dots, \varphi_k = \varphi$ where every $\varphi_i$ is an axiom, an element of $\Gamma$, or correct by a rule of inference. Now consider the sequence

$$\varphi_1, \dots, \varphi_k = \varphi, \psi_1, \dots, \psi_l = \psi.$$

This is a correct derivation of $\psi$ from $\Gamma \cup \Delta$ since every $B_i = \varphi$ is now justified by the same rule which justifies $\varphi_k = \varphi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each $i$, then $\Gamma \vdash \psi$.

**Proposition 6.17.** *$\Gamma$ is inconsistent iff $\Gamma \vdash \varphi$ for every $\varphi$.* *pl:axd:ptn:prop:incons*

*Proof.* Exercise. □

**Problem 6.2.** Prove Proposition 6.17.

**Proposition 6.18** (Compactness)**.** *pl:axd:ptn:prop:proves-compact*

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*
2. *If every finite subset of $\Gamma$ is consistent, then $\Gamma$ is consistent.*

*Proof.*
1. If $\Gamma \vdash \varphi$, then there is a finite sequence of formulas $\varphi_1, \dots, \varphi_n$ so that $\varphi \equiv \varphi_n$ and each $\varphi_i$ is either a logical axiom, an element of $\Gamma$ or follows from previous formulas by modus ponens. Take $\Gamma_0$ to be those $\varphi_i$ which are in $\Gamma$. Then the derivation is likewise a derivation from $\Gamma_0$, and so $\Gamma_0 \vdash \varphi$.
2. This is the contrapositive of (1) for the special case $\varphi \equiv \bot$.

□

## 6.5 The Deduction Theorem

*pl:axd:ded:sec* As we've seen, giving derivations in an axiomatic system is cumbersome, and derivations may be hard to find. Rather than actually write out long lists of formulas, it is generally easier to argue that such derivations exist, by making use of a few simple results. We've already established three such results: Proposition 6.14 says we can always assert that $\Gamma \vdash \varphi$ when we know that $\varphi \in \Gamma$. Proposition 6.15 says that if $\Gamma \vdash \varphi$ then also $\Gamma \cup \{\psi\} \vdash \varphi$. And Proposition 6.16 implies that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. Here's another simple result, a "meta"-version of modus ponens:

**Proposition 6.19.** *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.* *pl:axd:ded:prop:mp*

*Proof.* We have that $\{\varphi, \varphi \to \psi\} \vdash \psi$:

| | | |
|---|---|---|
| 1. | $\varphi$ | Hyp. |
| 2. | $\varphi \to \psi$ | Hyp. |
| 3. | $\psi$ | 1, 2, MP |

By Proposition 6.16, $\Gamma \vdash \psi$. □

The most important result we'll use in this context is the deduction theorem:

pl:axd:ded: thm:deduction-thm **Theorem 6.20** (Deduction Theorem). *$\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \to \psi$.*

*Proof.* The "if" direction is immediate. If $\Gamma \vdash \varphi \to \psi$ then also $\Gamma \cup \{\varphi\} \vdash \varphi \to \psi$ by Proposition 6.15. Also, $\Gamma \cup \{\varphi\} \vdash \varphi$ by Proposition 6.14. So, by Proposition 6.19, $\Gamma \cup \{\varphi\} \vdash \psi$.

For the "only if" direction, we proceed by induction on the length of the derivation of $\psi$ from $\Gamma \cup \{\varphi\}$.

For the induction basis, we prove the claim for every derivation of length 1. A derivation of $\psi$ from $\Gamma \cup \{\varphi\}$ of length 1 consists of $\psi$ by itself; and if it is correct $\psi$ is either $\in \Gamma \cup \{\varphi\}$ or is an axiom. If $\psi \in \Gamma$ or is an axiom, then $\Gamma \vdash \psi$. We also have that $\Gamma \vdash \psi \to (\varphi \to \psi)$ by eq. (6.7), and Proposition 6.19 gives $\Gamma \vdash \varphi \to \psi$. If $\psi \in \{\varphi\}$ then $\Gamma \vdash \varphi \to \psi$ because then last sentence $\varphi \to \psi$ is the same as $\varphi \to \varphi$, and we have derived that in Example 6.8.

For the inductive step, suppose a derivation of $\psi$ from $\Gamma \cup \{\varphi\}$ ends with a step $\psi$ which is justified by modus ponens. (If it is not justified by modus ponens, $\psi \in \Gamma$, $\psi \equiv \varphi$, or $\psi$ is an axiom, and the same reasoning as in the induction basis applies.) Then some previous steps in the derivation are $\chi \to \psi$ and $\chi$, for some formula $\chi$, i.e., $\Gamma \cup \{\varphi\} \vdash \chi \to \psi$ and $\Gamma \cup \{\varphi\} \vdash \chi$, and the respective derivations are shorter, so the inductive hypothesis applies to them. We thus have both:

$$\Gamma \vdash \varphi \to (\chi \to \psi);$$
$$\Gamma \vdash \varphi \to \chi.$$

But also

$$\Gamma \vdash (\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi)),$$

by eq. (6.8), and two applications of Proposition 6.19 give $\Gamma \vdash \varphi \to \psi$, as required. □

Notice how eq. (6.7) and eq. (6.8) were chosen precisely so that the Deduction Theorem would hold.

The following are some useful facts about derivability, which we leave as exercises.

pl:axd:ded: prop:derivfacts **Proposition 6.21.**

pl:axd:ded: derivfacts:a 1. *$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;*

pl:axd:ded: derivfacts:b 2. *If $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$ then $\Gamma \cup \{\psi\} \vdash \varphi$ (Contraposition);*

pl:axd:ded: derivfacts:c 3. *$\{\varphi, \neg\varphi\} \vdash \psi$ (Ex Falso Quodlibet, Explosion);*

4. *$\{\neg\neg\varphi\} \vdash \varphi$ (Double Negation Elimination);* *pl:axd:ded: derivfacts:d*

5. *If $\Gamma \vdash \neg\neg\varphi$ then $\Gamma \vdash \varphi$;* *pl:axd:ded: derivfacts:e*

**Problem 6.3.** Prove Proposition 6.21

## 6.6 Derivability and Consistency

pl:axd:prv: sec

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

**Proposition 6.22.** *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.* *pl:axd:prv: prop:provability-contr*

*Proof.* If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \bot$. By Proposition 6.14, $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{\varphi\}$. By Proposition 6.16, $\Gamma \vdash \bot$, i.e., $\Gamma$ is inconsistent. □

**Proposition 6.23.** *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.* *pl:axd:prv: prop:prov-incons*

*Proof.* First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by Proposition 6.15. $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$ by Proposition 6.14. We also have $\vdash \neg\varphi \to (\varphi \to \bot)$ by eq. (6.10). So by two applications of Proposition 6.19, we have $\Gamma \cup \{\neg\varphi\} \vdash \bot$.

Now assume $\Gamma \cup \{\neg\varphi\}$ is inconsistent, i.e., $\Gamma \cup \{\neg\varphi\} \vdash \bot$. By the deduction theorem, $\Gamma \vdash \neg\varphi \to \bot$. $\Gamma \vdash (\neg\varphi \to \bot) \to \neg\neg\varphi$ by eq. (6.13), so $\Gamma \vdash \neg\neg\varphi$ by Proposition 6.19. Since $\Gamma \vdash \neg\neg\varphi \to \varphi$ (eq. (6.14)), we have $\Gamma \vdash \varphi$ by Proposition 6.19 again. □

**Problem 6.4.** Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

**Proposition 6.24.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then $\Gamma$ is inconsistent.* *pl:axd:prv: prop:explicit-inc*

*Proof.* $\Gamma \vdash \neg\varphi \to (\varphi \to \bot)$ by eq. (6.10). $\Gamma \vdash \bot$ by two applications of Proposition 6.19. □

**Proposition 6.25.** *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.* *pl:axd:prv: prop:provability-exhaustive*

*Proof.* Exercise. □

**Problem 6.5.** Prove Proposition 6.25

## 6.7 Derivability and the Propositional Connectives

pl:axd:ppr:sec

pl:axd:ppr:prop:provability-land

**Proposition 6.26.**

pl:axd:ppr:prop:provability-land-left

1. *Both* $\varphi \wedge \psi \vdash \varphi$ *and* $\varphi \wedge \psi \vdash \psi$

pl:axd:ppr:prop:provability-land-right

2. $\varphi, \psi \vdash \varphi \wedge \psi$.

*Proof.* 1. From eq. (6.1) and eq. (6.1) by modus ponens.

2. From eq. (6.3) by two applications of modus ponens.

□

pl:axd:ppr:prop:provability-lor

**Proposition 6.27.**

1. $\varphi \vee \psi, \neg\varphi, \neg\psi$ *is inconsistent.*

2. *Both* $\varphi \vdash \varphi \vee \psi$ *and* $\psi \vdash \varphi \vee \psi$.

*Proof.* 1. From eq. (6.9) we get $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \bot)$ and $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \bot)$. So by the deduction theorem, we have $\{\neg\varphi\} \vdash \varphi \rightarrow \bot$ and $\{\neg\psi\} \vdash \psi \rightarrow \bot$. From eq. (6.6) we get $\{\neg\varphi, \neg\psi\} \vdash (\varphi \vee \psi) \rightarrow \bot$. By the deduction theorem, $\{\varphi \vee \psi, \neg\varphi, \neg\psi\} \vdash \bot$.

2. From eq. (6.4) and eq. (6.5) by modus ponsens.

□

pl:axd:ppr:prop:provability-lif

**Proposition 6.28.**

pl:axd:ppr:prop:provability-lif-left

1. $\varphi, \varphi \rightarrow \psi \vdash \psi$.

pl:axd:ppr:prop:provability-lif-right

2. *Both* $\neg\varphi \vdash \varphi \rightarrow \psi$ *and* $\psi \vdash \varphi \rightarrow \psi$.

*Proof.* 1. We can derive:

| | | |
|---|---|---|
| 1. | $\varphi$ | Hyp |
| 2. | $\varphi \rightarrow \psi$ | Hyp |
| 3. | $\psi$ | 1, 2, MP |

2. By eq. (6.10) and eq. (6.7) and the deduction theorem, respectively.

□

## 6.8 Soundness

explanation A derivation system, such as axiomatic deduction, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that pl:axd:sou:sec

1. every derivable $\varphi$ is valid;
2. if $\varphi$ is derivable from some others $\Gamma$, it is also a consequence of them;
3. if a set of formulas $\Gamma$ is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

**Proposition 6.29.** *If $\varphi$ is an axiom, then $\mathfrak{v} \vDash \varphi$ for each valuation $\mathfrak{v}$.*

*Proof.* Do truth tables for each axiom to verify that they are tautologies. □

**Theorem 6.30** (Soundness)**.** *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.* *pl:axd:sou:thm:soundness*

*Proof.* By induction on the length of the derivation of $\varphi$ from $\Gamma$. If there are no steps justified by inferences, then all formulas in the derivation are either instances of axioms or are in $\Gamma$. By the previous proposition, all the axioms are tautologies, and hence if $\varphi$ is an axiom then $\Gamma \vDash \varphi$. If $\varphi \in \Gamma$, then trivially $\Gamma \vDash \varphi$.

If the last step of the derivation of $\varphi$ is justified by modus ponens, then there are formulas $\psi$ and $\psi \to \varphi$ in the derivation, and the induction hypothesis applies to the part of the derivation ending in those formulas (since they contain at least one fewer steps justified by an inference). So, by induction hypothesis, $\Gamma \vDash \psi$ and $\Gamma \vDash \psi \to \varphi$. Then $\Gamma \vDash \varphi$ by Theorem 1.16. □

**Corollary 6.31.** *If $\vdash \varphi$, then $\varphi$ is a tautology.* *pl:axd:sou:cor:weak-soundness*

**Corollary 6.32.** *If $\Gamma$ is satisfiable, then it is consistent.* *pl:axd:sou:cor:consistency-soundness*

*Proof.* We prove the contrapositive. Suppose that $\Gamma$ is not consistent. Then $\Gamma \vdash \bot$, i.e., there is a derivation of $\bot$ from $\Gamma$. By Theorem 6.30, any valuation $\mathfrak{v}$ that satisfies $\Gamma$ must satisfy $\bot$. Since $\mathfrak{v} \nvDash \bot$ for every valuation $\mathfrak{v}$, no $\mathfrak{v}$ can satisfy $\Gamma$, i.e., $\Gamma$ is not satisfiable. □

# Chapter 7

# The Completeness Theorem

## 7.1 Introduction

pl:com:int:sec

The completeness theorem is one of the most fundamental results about logic. It comes in two formulations, the equivalence of which we'll prove. In its first formulation it says something fundamental about the relationship between semantic consequence and our proof system: if a sentence $\varphi$ follows from some sentences $\Gamma$, then there is also a derivation that establishes $\Gamma \vdash \varphi$. Thus, the proof system is as strong as it can possibly be without proving things that don't actually follow.

In its second formulation, it can be stated as a model existence result: every consistent set of sentences is satisfiable. Consistency is a proof-theoretic notion: it says that our proof system is unable to produce certain derivations. But who's to say that just because there are no derivations of a certain sort from $\Gamma$, it's guaranteed that there is valuation $\mathfrak{v}$ with $\mathfrak{v} \vDash \Gamma$? Before the completeness theorem was first proved—in fact before we had the proof systems we now do—the great German mathematician David Hilbert held the view that consistency of mathematical theories guarantees the existence of the objects they are about. He put it as follows in a letter to Gottlob Frege:

> If the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.

Frege vehemently disagreed. The second formulation of the completeness theorem shows that Hilbert was right in at least the sense that if the axioms are consistent, then *some* valuation exists that makes them all true.

These aren't the only reasons the completeness theorem—or rather, its proof—is important. It has a number of important consequences, some of which we'll discuss separately. For instance, since any derivation that shows $\Gamma \vdash \varphi$ is finite and so can only use finitely many of the sentences in $\Gamma$, it follows by the completeness theorem that if $\varphi$ is a consequence of $\Gamma$, it is already a consequence of a finite subset of $\Gamma$. This is called *compactness*. Equivalently, if every finite subset of $\Gamma$ is consistent, then $\Gamma$ itself must be consistent.

Although the compactness theorem follows from the completeness theorem via the detour through derivations, it is also possible to use the *the proof of* the completeness theorem to establish it directly. For what the proof does is take a set of sentences with a certain property—consistency—and constructs a structure out of this set that has certain properties (in this case, that it satisfies the set). Almost the very same construction can be used to directly establish compactness, by starting from “finitely satisfiable” sets of sentences instead of consistent ones.

## 7.2 Outline of the Proof

The proof of the completeness theorem is a bit complex, and upon first reading it, it is easy to get lost. So let us outline the proof. The first step is a shift of perspective, that allows us to see a route to a proof. When completeness is thought of as “whenever $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$,” it may be hard to even come up with an idea: for to show that $\Gamma \vdash \varphi$ we have to find a derivation, and it does not look like the hypothesis that $\Gamma \vDash \varphi$ helps us for this in any way. For some proof systems it is possible to directly construct a derivation, but we will take a slightly different tack. The shift in perspective required is this: completeness can also be formulated as: “if $\Gamma$ is consistent, it has a model.” Perhaps we can use the information in $\Gamma$ together with the hypothesis that it is consistent to construct a model. After all, we know what kind of model we are looking for: one that is as $\Gamma$ describes it! pl:com:out:sec

If $\Gamma$ contains only propositional variables, it is easy to construct a model for it. All we have to do is come up with a valuation $\mathfrak{v}$ such that $\mathfrak{v} \vDash p$a for all $p \in \Gamma$. Well, let $\mathfrak{v}(p) = \mathbb{T}$ iff $p \in \Gamma$.

Now suppose $\Gamma$ contains some formula $\neg\psi$, with $\psi$ atomic. We might worry that the construction of $\mathfrak{v}$ interferes with the possibility of making $\neg\psi$ true. But here's where the consistency of $\Gamma$ comes in: if $\neg\psi \in \Gamma$, then $\psi \notin \Gamma$, or else $\Gamma$ would be inconsistent. And if $\psi \notin \Gamma$, then according to our construction of $\mathfrak{v}$, $\mathfrak{v} \nvDash \psi$, so $\mathfrak{v} \vDash \neg\psi$. So far so good.

What if $\Gamma$ contains complex, non-atomic formulas? Say it contains $\varphi \land \psi$. To make that true, we should proceed as if both $\varphi$ and $\psi$ were in $\Gamma$. And if $\varphi \lor \psi \in \Gamma$, then we will have to make at least one of them true, i.e., proceed as if one of them was in $\Gamma$.

This suggests the following idea: we add additional formulas to $\Gamma$ so as to (a) keep the resulting set consistent and (b) make sure that for every possible atomic sentence $\varphi$, either $\varphi$ is in the resulting set, or $\neg\varphi$ is, and (c) such that, whenever $\varphi \land \psi$ is in the set, so are both $\varphi$ and $\psi$, if $\varphi \lor \psi$ is in the set, at least one of $\varphi$ or $\psi$ is also, etc. We keep doing this (potentially forever). Call the set of all formulas so added $\Gamma^*$. Then our construction above would provide us with a structure $\mathfrak{v}$ for which we could prove, by induction, that all sentences in $\Gamma^*$ are true in it, and hence also all sentence in $\Gamma$ since $\Gamma \subseteq \Gamma^*$. It turns out that guaranteeing (a) and (b) is enough. A set of sentences for which (b) holds is called *complete*. So our task will be to extend the consistent set $\Gamma$ to a consistent and complete set $\Gamma^*$.

So here's what we'll do. First we investigate the properties of complete consistent sets, in particular we prove that a complete consistent set contains $\varphi \wedge \psi$ iff it contains both $\varphi$ and $\psi$, $\varphi \vee \psi$ iff it contains at least one of them, etc. (Proposition 7.2). We'll then take the consistent set $\Gamma$ and show that it can be extended to a consistent and complete set $\Gamma^*$ (Lemma 7.3). This set $\Gamma^*$ is what we'll use to define our valuation $\mathfrak{v}(\Gamma^*)$. The valuation is determined by the propositional variables in $\Gamma^*$ (Definition 7.4). We'll use the properties of complete consistent sets to show that indeed $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff $\varphi \in \Gamma^*$ (Lemma 7.5), and thus in particular, $\mathfrak{v}(\Gamma^*) \vDash \Gamma$.

## 7.3 Complete Consistent Sets of Sentences

pl:com:ccs:sec
pl:com:ccs:def:complete-set **Definition 7.1** (Complete set)**.** A set $\Gamma$ of sentences is *complete* iff for any sentence $\varphi$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Complete sets of sentences leave no questions unanswered. For any sentence $A$, $\Gamma$ "says" if $\varphi$ is true or false. The importance of complete sets extends beyond the proof of the completeness theorem. A theory which is complete and axiomatizable, for instance, is always decidable. explanation

Complete consistent sets are important in the completeness proof since we can guarantee that every consistent set of sentences $\Gamma$ is contained in a complete consistent set $\Gamma^*$. A complete consistent set contains, for each sentence $\varphi$, either $\varphi$ or its negation $\neg\varphi$, but not both. This is true in particular for atomic sentences, so from a complete consistent set in a language suitably expanded by constant symbols, we can construct a structure where the interpretation of predicate symbols is defined according to which atomic sentences are in $\Gamma^*$. This structure can then be shown to make all sentences in $\Gamma^*$ (and hence also all those in $\Gamma$) true. The proof of this latter fact requires that $\neg\varphi \in \Gamma^*$ iff $\varphi \notin \Gamma^*$, $(\varphi \vee \psi) \in \Gamma^*$ iff $\varphi \in \Gamma^*$ or $\psi \in \Gamma^*$, etc. explanation

In what follows, we will often tacitly use the properties of reflexivity, monotonicity, and transitivity of $\vdash$ (see sections 3.6, 4.5, 5.5 and 6.4).

*pl:com:ccs:prop:ccs* **Proposition 7.2.** *Suppose $\Gamma$ is complete and consistent. Then:*

*pl:com:ccs:prop:ccs-prov-in* 1. *If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.*

*pl:com:ccs:prop:ccs-and* 2. *$\varphi \wedge \psi \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$.*

*pl:com:ccs:prop:ccs-or* 3. *$\varphi \vee \psi \in \Gamma$ iff either $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

*pl:com:ccs:prop:ccs-if* 4. *$\varphi \rightarrow \psi \in \Gamma$ iff either $\varphi \notin \Gamma$ or $\psi \in \Gamma$.*

*Proof.* Let us suppose for all of the following that $\Gamma$ is complete and consistent.

1. If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.

   Suppose that $\Gamma \vdash \varphi$. Suppose to the contrary that $\varphi \notin \Gamma$. Since $\Gamma$ is complete, $\neg\varphi \in \Gamma$. By Propositions 4.17, 5.17, 3.19 and 6.24, $\Gamma$ is inconsistent. This contradicts the assumption that $\Gamma$ is consistent. Hence, it cannot be the case that $\varphi \notin \Gamma$, so $\varphi \in \Gamma$.

2. $\varphi \land \psi \in \Gamma$ iff both $\varphi \in \Gamma$ and $\psi \in \Gamma$:

   For the forward direction, suppose $\varphi \land \psi \in \Gamma$. Then by Propositions 4.19, 5.19, 3.21 and 6.26, item (1), $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$. By (1), $\varphi \in \Gamma$ and $\psi \in \Gamma$, as required.

   For the reverse direction, let $\varphi \in \Gamma$ and $\psi \in \Gamma$. By Propositions 4.19, 5.19, 3.21 and 6.26, item (2), $\Gamma \vdash \varphi \land \psi$. By (1), $\varphi \land \psi \in \Gamma$.

3. First we show that if $\varphi \lor \psi \in \Gamma$, then either $\varphi \in \Gamma$ or $\psi \in \Gamma$. Suppose $\varphi \lor \psi \in \Gamma$ but $\varphi \notin \Gamma$ and $\psi \notin \Gamma$. Since $\Gamma$ is complete, $\neg\varphi \in \Gamma$ and $\neg\psi \in \Gamma$. By Propositions 4.20, 5.20, 3.22 and 6.27, item (1), $\Gamma$ is inconsistent, a contradiction. Hence, either $\varphi \in \Gamma$ or $\psi \in \Gamma$.

   For the reverse direction, suppose that $\varphi \in \Gamma$ or $\psi \in \Gamma$. By Propositions 4.20, 5.20, 3.22 and 6.27, item (2), $\Gamma \vdash \varphi \lor \psi$. By (1), $\varphi \lor \psi \in \Gamma$, as required.

4. For the forward direction, suppose $\varphi \rightarrow \psi \in \Gamma$, and suppose to the contrary that $\varphi \in \Gamma$ and $\psi \notin \Gamma$. On these assumptions, $\varphi \rightarrow \psi \in \Gamma$ and $\varphi \in \Gamma$. By Propositions 4.21, 5.21, 3.23 and 6.28, item (1), $\Gamma \vdash \psi$. But then by (1), $\psi \in \Gamma$, contradicting the assumption that $\psi \notin \Gamma$.

   For the reverse direction, first consider the case where $\varphi \notin \Gamma$. Since $\Gamma$ is complete, $\neg\varphi \in \Gamma$. By Propositions 4.21, 5.21, 3.23 and 6.28, item (2), $\Gamma \vdash \varphi \rightarrow \psi$. Again by (1), we get that $\varphi \rightarrow \psi \in \Gamma$, as required.

   Now consider the case where $\psi \in \Gamma$. By Propositions 4.21, 5.21, 3.23 and 6.28, item (2) again, $\Gamma \vdash \varphi \rightarrow \psi$. By (1), $\varphi \rightarrow \psi \in \Gamma$.

□

**Problem 7.1.** Complete the proof of Proposition 7.2.

## 7.4 Lindenbaum's Lemma

explanation We now prove a lemma that shows that any consistent set of sentences is contained in some set of sentences which is not just consistent, but also complete. pl:com:lin:sec The proof works by adding one sentence at a time, guaranteeing at each step that the set remains consistent. We do this so that for every $\varphi$, either $\varphi$ or $\neg\varphi$ gets added at some stage. The union of all stages in that construction then contains either $\varphi$ or its negation $\neg\varphi$ and is thus complete. It is also consistent, since we made sure at each stage not to introduce an inconsistency.

pl:com:lin: lem:lindenbaum **Lemma 7.3** (Lindenbaum's Lemma). *Every consistent set $\Gamma$ in a language $\mathcal{L}$ can be extended to a complete and consistent set $\Gamma^*$.*

*Proof.* Let $\Gamma$ be consistent. Let $\varphi_0$, $\varphi_1$, ... be an enumeration of all the sentences of $\mathcal{L}$. Define $\Gamma_0 = \Gamma$, and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_n\} & \text{otherwise.} \end{cases}$$

Let $\Gamma^* = \bigcup_{n \geq 0} \Gamma_n$.

Each $\Gamma_n$ is consistent: $\Gamma_0$ is consistent by definition. If $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$, this is because the latter is consistent. If it isn't, $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}$. We have to verify that $\Gamma_n \cup \{\neg\varphi_n\}$ is consistent. Suppose it's not. Then *both* $\Gamma_n \cup \{\varphi_n\}$ and $\Gamma_n \cup \{\neg\varphi_n\}$ are inconsistent. This means that $\Gamma_n$ would be inconsistent by Propositions 4.17, 5.17, 3.19 and 6.24, contrary to the induction hypothesis.

For every $n$ and every $i < n$, $\Gamma_i \subseteq \Gamma_n$. This follows by a simple induction on $n$. For $n = 0$, there are no $i < 0$, so the claim holds automatically. For the inductive step, suppose it is true for $n$. We have $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ or $= \Gamma_n \cup \{\neg\varphi_n\}$ by construction. So $\Gamma_n \subseteq \Gamma_{n+1}$. If $i < n$, then $\Gamma_i \subseteq \Gamma_n$ by inductive hypothesis, and so $\subseteq \Gamma_{n+1}$ by transitivity of $\subseteq$.

From this it follows that every finite subset of $\Gamma^*$ is a subset of $\Gamma_n$ for some $n$, since each $\psi \in \Gamma^*$ not already in $\Gamma_0$ is added at some stage $i$. If $n$ is the last one of these, then all $\psi$ in the finite subset are in $\Gamma_n$. So, every finite subset of $\Gamma^*$ is consistent. By Propositions 4.14, 5.14, 3.16 and 6.18, $\Gamma^*$ is consistent.

Every sentence of $\mathrm{Frm}(\mathcal{L})$ appears on the list used to define $\Gamma^*$. If $\varphi_n \notin \Gamma^*$, then that is because $\Gamma_n \cup \{\varphi_n\}$ was inconsistent. But then $\neg\varphi_n \in \Gamma^*$, so $\Gamma^*$ is complete. □

## 7.5 Construction of a Model

pl:com:mod: sec We are now ready to define a valuation that makes all $\varphi \in \Gamma$ true. To do this, we first apply Lindenbaum's Lemma: we get a complete consistent $\Gamma^* \supseteq \Gamma$. We let the propositional variables in $\Gamma^*$ determine $\mathfrak{v}(\Gamma^*)$. explanation

pl:com:mod: defn:termmodel **Definition 7.4.** Suppose $\Gamma^*$ is a complete consistent set of formulas. Then we let

$$\mathfrak{v}(\Gamma^*)(p) = \begin{cases} \mathbb{T} & \text{if } p \in \Gamma^* \\ \mathbb{F} & \text{if } p \notin \Gamma^* \end{cases}$$

pl:com:mod: lem:truth **Lemma 7.5** (Truth Lemma). $\mathfrak{v}(\Gamma^*) \vDash \varphi$ *iff* $\varphi \in \Gamma^*$.

*Proof.* We prove both directions simultaneously, and by induction on $\varphi$.

1. $\varphi \equiv \bot$: $\mathfrak{v}(\Gamma^*) \nvDash \bot$ by definition of satisfaction. On the other hand, $\bot \notin \Gamma^*$ since $\Gamma^*$ is consistent.

2. $\varphi \equiv \top$: $\mathfrak{v}(\Gamma^*) \vDash \top$ by definition of satisfaction. On the other hand, $\top \in \Gamma^*$ since $\Gamma^*$ is consistent and complete, and $\Gamma^* \vdash \top$.

3. $\varphi \equiv p$: $\mathfrak{v}(\Gamma^*) \vDash p$ iff $\mathfrak{v}(\Gamma^*)(p) = \mathbb{T}$ (by the definition of satisfaction) iff $p \in \Gamma^*$ (by the construction of $\mathfrak{v}(\Gamma^*)$).

4. $\varphi \equiv \neg\psi$: $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff $\mathfrak{M}(\Gamma^*) \nvDash \psi$ (by definition of satisfaction). By induction hypothesis, $\mathfrak{M}(\Gamma^*) \nvDash \psi$ iff $\psi \notin \Gamma^*$. Since $\Gamma^*$ is consistent and complete, $\psi \notin \Gamma^*$ iff $\neg\psi \in \Gamma^*$.

5. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff we have both $\mathfrak{v}(\Gamma^*) \vDash \psi$ and $\mathfrak{v}(\Gamma^*) \vDash \chi$ (by definition of satisfaction) iff both $\psi \in \Gamma^*$ and $\chi \in \Gamma^*$ (by the induction hypothesis). By Proposition 7.2(2), this is the case iff $(\psi \wedge \chi) \in \Gamma^*$.

6. $\varphi \equiv \psi \vee \chi$: $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff at $\mathfrak{v}(\Gamma^*) \vDash \psi$ or $\mathfrak{v}(\Gamma^*) \vDash \chi$ (by definition of satisfaction) iff $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \vee \chi) \in \Gamma^*$ (by Proposition 7.2(3)).

7. $\varphi \equiv \psi \rightarrow \chi$: $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff $\mathfrak{M}(\Gamma^*) \nvDash \psi$ or $\mathfrak{M}(\Gamma^*) \vDash \chi$ (by definition of satisfaction) iff $\psi \notin \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \rightarrow \chi) \in \Gamma^*$ (by Proposition 7.2(4)).

□

## 7.6 The Completeness Theorem

explanation Let's combine our results: we arrive at the completeness theorem. pl:com:cth:sec

**Theorem 7.6** (Completeness Theorem). *Let $\Gamma$ be a set of sentences. If $\Gamma$ is consistent, it is satisfiable.* pl:com:cth:thm:completeness

*Proof.* Suppose $\Gamma$ is consistent. By Lemma 7.3, there is a $\Gamma^* \supseteq \Gamma$ which is consistent and complete. By Lemma 7.5, $\mathfrak{v}(\Gamma^*) \vDash \varphi$ iff $\varphi \in \Gamma^*$. From this it follows in particular that for all $\varphi \in \Gamma$, $\mathfrak{v}(\Gamma^*) \vDash \varphi$, so $\Gamma$ is satisfiable. □

**Corollary 7.7** (Completeness Theorem, Second Version). *For all $\Gamma$ and $\varphi$ sentences: if $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.* pl:com:cth:cor:completeness

*Proof.* Note that the $\Gamma$'s in Corollary 7.7 and Theorem 7.6 are universally quantified. To make sure we do not confuse ourselves, let us restate Theorem 7.6 using a different variable: for any set of sentences $\Delta$, if $\Delta$ is consistent, it is satisfiable. By contraposition, if $\Delta$ is not satisfiable, then $\Delta$ is inconsistent. We will use this to prove the corollary.

Suppose that $\Gamma \vDash \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable by Proposition 1.15. Taking $\Gamma \cup \{\neg\varphi\}$ as our $\Delta$, the previous version of Theorem 7.6 gives us that $\Gamma \cup \{\neg\varphi\}$ is inconsistent. By Propositions 4.16, 5.16, 3.18 and 6.23, $\Gamma \vdash \varphi$. □

**Problem 7.2.** Use Corollary 7.7 to prove Theorem 7.6, thus showing that the two formulations of the completeness theorem are equivalent.

**Problem 7.3.** In order for a derivation system to be complete, its rules must be strong enough to prove every unsatisfiable set inconsistent. Which of the rules of derivation were necessary to prove completeness? Are any of these rules not used anywhere in the proof? In order to answer these questions, make a list or diagram that shows which of the rules of derivation were used in which results that lead up to the proof of Theorem 7.6. Be sure to note any tacit uses of rules in these proofs.

## 7.7 The Compactness Theorem

pl:com:com:sec One important consequence of the completeness theorem is the compactness theorem. The compactness theorem states that if each *finite* subset of a set of sentences is satisfiable, the entire set is satisfiable—even if the set itself is infinite. This is far from obvious. There is nothing that seems to rule out, at first glance at least, the possibility of there being infinite sets of sentences which are contradictory, but the contradiction only arises, so to speak, from the infinite number. The compactness theorem says that such a scenario can be ruled out: there are no unsatisfiable infinite sets of sentences each finite subset of which is satisfiable. Like the completeness theorem, it has a version related to entailment: if an infinite set of sentences entails something, already a finite subset does.

**Definition 7.8.** A set $\Gamma$ of formulas is *finitely satisfiable* if and only if every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable.

*pl:com:com:thm:compactness* **Theorem 7.9** (Compactness Theorem)**.** *The following hold for any sentences $\Gamma$ and $\varphi$:*

1. *$\Gamma \vDash \varphi$ iff there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vDash \varphi$.*
2. *$\Gamma$ is satisfiable if and only if it is finitely satisfiable.*

*Proof.* We prove (2). If $\Gamma$ is satisfiable, then there is a valuation $\mathfrak{v}$ such that $\mathfrak{v} \vDash \varphi$ for all $\varphi \in \Gamma$. Of course, this $\mathfrak{v}$ also satisfies every finite subset of $\Gamma$, so $\Gamma$ is finitely satisfiable.

Now suppose that $\Gamma$ is finitely satisfiable. Then every finite subset $\Gamma_0 \subseteq \Gamma$ is satisfiable. By soundness (Corollaries 4.24, 5.26, 3.28 and 6.32), every finite subset is consistent. Then $\Gamma$ itself must be consistent by Propositions 4.14, 5.14, 3.16 and 6.18. By completeness (Theorem 7.6), since $\Gamma$ is consistent, it is satisfiable. □

**Problem 7.4.** Prove (1) of Theorem 7.9.

## 7.8 A Direct Proof of the Compactness Theorem

pl:com:cpd:sec

We can prove the Compactness Theorem directly, without appealing to the Completeness Theorem, using the same ideas as in the proof of the completeness theorem. In the proof of the Completeness Theorem we started with a consistent set $\Gamma$ of sentences, expanded it to a consistent and complete set $\Gamma^*$ of sentences, and then showed that in the valuation $\mathfrak{v}(\Gamma^*)$ constructed from $\Gamma^*$, all sentences of $\Gamma$ are true, so $\Gamma$ is satisfiable.

We can use the same method to show that a finitely satisfiable set of sentences is satisfiable. We just have to prove the corresponding versions of the results leading to the truth lemma where we replace "consistent" with "finitely satisfiable."

**Proposition 7.10.** *Suppose $\Gamma$ is complete and finitely satisfiable. Then:* pl:com:cpd:prop:fsat-ccs

1. $(\varphi \land \psi) \in \Gamma$ *iff both* $\varphi \in \Gamma$ *and* $\psi \in \Gamma$.
2. $(\varphi \lor \psi) \in \Gamma$ *iff either* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$.
3. $(\varphi \to \psi) \in \Gamma$ *iff either* $\varphi \notin \Gamma$ *or* $\psi \in \Gamma$.

**Problem 7.5.** Prove Proposition 7.10. Avoid the use of $\vdash$.

**Lemma 7.11.** *Every finitely satisfiable set $\Gamma$ can be extended to a complete and finitely satisfiable set $\Gamma^*$.* pl:com:cpd:lem:fsat-lindenbaum

**Problem 7.6.** Prove Lemma 7.11. (Hint: the crucial step is to show that if $\Gamma_n$ is finitely satisfiable, then either $\Gamma_n \cup \{\varphi_n\}$ or $\Gamma_n \cup \{\neg\varphi_n\}$ is finitely satisfiable.)

**Theorem 7.12** (Compactness)**.** *$\Gamma$ is satisfiable if and only if it is finitely satisfiable.* pl:com:cpd:thm:compactness-direct

*Proof.* If $\Gamma$ is satisfiable, then there is a valuation $\mathfrak{v}$ such that $pSatv\varphi$ for all $\varphi \in \Gamma$. Of course, this $\mathfrak{v}$ also satisfies every finite subset of $\Gamma$, so $\Gamma$ is finitely satisfiable.

Now suppose that $\Gamma$ is finitely satisfiable. By Lemma 7.11, $\Gamma$ can be extended to a complete and finitely satisfiable set $\Gamma^*$. Construct the valuation $\mathfrak{v}(\Gamma^*)$ as in Definition 7.4. The proof of the Truth Lemma (Lemma 7.5) goes through if we replace references to Proposition 7.2. □

**Problem 7.7.** Write out the complete proof of the Truth Lemma (Lemma 7.5) in the version required for the proof of Theorem 7.12.

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