Chapter udf

Modal Tableaux

Draft chapter on prefixed tableaux for modal logic. Needs more examples, completeness proofs, and discussion of how one can find countermodels from unsuccessful searches for closed tableaux.

tab.1 Introduction

nml:tab:int:sec

Tableaux are certain (downward-branching) trees of signed formulas, i.e., pairs consisting of a truth value sign ($\mathbb{T}$ or $\mathbb{F}$) and a sentence

$$\mathbb{T}\,\varphi \text{ or } \mathbb{F}\,\varphi.$$

A tableau begins with a number of *assumptions*. Each further signed formula is generated by applying one of the inference rules. Some inference rules add one or more signed formulas to a tip of the tree; others add two new tips, resulting in two branches. Rules result in signed formulas where the formula is less complex than that of the signed formula to which it was applied. When a branch contains both $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$, we say the branch is *closed*. If every branch in a tableau is closed, the entire tableau is closed. A closed tableau constitutes a derivation that shows that the set of signed formulas which were used to begin the tableau are unsatisfiable. This can be used to define a $\vdash$ relation: $\Gamma \vdash \varphi$ iff there is some finite set $\Gamma_0 = \{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ such that there is a closed tableau for the assumptions

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

For modal logics, we have to both extend the notion of signed formula and add rules that cover $\Box$ and $\Diamond$ In addition to a sign($\mathbb{T}$ or $\mathbb{F}$), formulas in modal tableaux also have *prefixes* σ. The prefixes are non-empty sequences of positive integers, i.e., $\sigma \in (\mathbb{Z}^+)^* \setminus \{\Lambda\}$. When we write such prefixes without the surrounding $\langle\ \rangle$, and separate the individual elements by .'s instead of ,'s.

$\dfrac{\sigma\,\mathbb{T}\,\neg\varphi}{\sigma\,\mathbb{F}\,\varphi}\,\neg\mathbb{T}$	$\dfrac{\sigma\,\mathbb{F}\,\neg\varphi}{\sigma\,\mathbb{T}\,\varphi}\,\neg\mathbb{F}$
$\dfrac{\sigma\,\mathbb{T}\,\varphi\land\psi}{\begin{array}{c}\sigma\,\mathbb{T}\,\varphi\\ \sigma\,\mathbb{T}\,\psi\end{array}}\,\land\mathbb{T}$	$\dfrac{\sigma\,\mathbb{F}\,\varphi\land\psi}{\sigma\,\mathbb{F}\,\varphi \quad \mid \quad \sigma\,\mathbb{F}\,\psi}\,\land\mathbb{F}$
$\dfrac{\sigma\,\mathbb{T}\,\varphi\lor\psi}{\sigma\,\mathbb{T}\,\varphi \quad \mid \quad \sigma\,\mathbb{T}\,\psi}\,\lor\mathbb{T}$	$\dfrac{\sigma\,\mathbb{F}\,\varphi\lor\psi}{\begin{array}{c}\sigma\,\mathbb{F}\,\varphi\\ \sigma\,\mathbb{F}\,\psi\end{array}}\,\lor\mathbb{F}$
$\dfrac{\sigma\,\mathbb{T}\,\varphi\to\psi}{\sigma\,\mathbb{F}\,\varphi \quad \mid \quad \sigma\,\mathbb{T}\,\psi}\,\to\mathbb{T}$	$\dfrac{\sigma\,\mathbb{F}\,\varphi\to\psi}{\begin{array}{c}\sigma\,\mathbb{T}\,\varphi\\ \sigma\,\mathbb{F}\,\psi\end{array}}\,\to\mathbb{F}$

Table tab.1: Prefixed tableau rules for the propositional connectives

nml:tab:rul: tab:prop-rules

If σ is a prefix, then $\sigma.n$ is $\sigma \frown \langle n\rangle$; e.g., if $\sigma = 1.2.1$, then $\sigma.3$ is 1.2.1.3. So for instance,

$$1.2\,\mathbb{T}\,\Box\varphi\to\varphi$$

is a *prefixed signed formula* (or just a *prefixed formula* for short).

Intuitively, the prefix names a world in a model that might satisfy the formulas on a branch of a tableau, and if σ names some world, then $\sigma.n$ names a world accessible from (the world named by) σ.

tab.2 Rules for K

The rules for the regular propositional connectives are the same as for regular propositional signed tableaux, just with prefixes added. In each case, the rule applied to a signed formula $\sigma\,S\,\varphi$ produces new formulas that are also prefixed by σ. This should be intuitively clear: e.g., if $\varphi\land\psi$ is true at (a world named by) σ, then φ and ψ are true at σ (and not at any other world). We collect the propositional rules in Table tab.1. nml:tab:rul: sec

The closure condition is the same as for ordinary tableaux, although we require that not just the formulas but also the prefixes must match. So a branch is closed if it contains both

$$\sigma\,\mathbb{T}\,\varphi \quad\text{and}\quad \sigma\,\mathbb{F}\,\varphi$$

for some prefix σ and formula φ.

The rules for setting up assumptions is also as for ordinary tableaux, except that for assumptions we always use the prefix 1. (It does not matter which

$\dfrac{\sigma\,\mathbb{T}\,\Box\varphi}{\sigma.n\,\mathbb{T}\,\varphi}\,\Box\mathbb{T}$ $\sigma.n$ is used	$\dfrac{\sigma\,\mathbb{F}\,\Box\varphi}{\sigma.n\,\mathbb{F}\,\varphi}\,\Box\mathbb{F}$ $\sigma.n$ is new
$\dfrac{\sigma\,\mathbb{T}\,\Diamond\varphi}{\sigma.n\,\mathbb{T}\,\varphi}\,\Diamond\mathbb{T}$ $\sigma.n$ is new	$\dfrac{\sigma\,\mathbb{F}\,\Diamond\varphi}{\sigma.n\,\mathbb{F}\,\varphi}\,\Diamond\mathbb{F}$ $\sigma.n$ is used

Table tab.2: The modal rules for K.

nml:tab:rul:
tab:rules-K

prefix we use, as long as it's the same for all assumptions.) So, e.g., we say that

$$\psi_1, \dots, \psi_n \vdash \varphi$$

iff there is a closed tableau for the assumptions

$$1\,\mathbb{T}\,\psi_1, \dots, 1\,\mathbb{T}\,\psi_n, 1\,\mathbb{F}\,\varphi.$$

For the modal operators $\Box$ and $\Diamond$, the prefix of the conclusion of the rule applied to a formula with prefix σ is $\sigma.n$. However, which n is allowed depends on whether the sign is $\mathbb{T}$ or $\mathbb{F}$.

The $\mathbb{T}\Box$ rule extends a branch containing $\sigma\,\mathbb{T}\,\Box\varphi$ by $\sigma.n\,\mathbb{T}\,\varphi$. Similarly, the $\mathbb{F}\Diamond$ rule extends a branch containing $\sigma\,\mathbb{F}\,\Diamond\varphi$ by $\sigma.n\,\mathbb{F}\,\varphi$. They can only be applied for a prefix $\sigma.n$ which *already* occurs on the branch in which it is applied. Let's call such a prefix "used" (on the branch).

The $\mathbb{F}\Box$ rule extends a branch containing $\sigma\,\mathbb{F}\,\Box\varphi$ by $\sigma.n\,\mathbb{F}\,\varphi$. Similarly, the $\mathbb{T}\Diamond$ rule extends a branch containing $\sigma\,\mathbb{T}\,\Diamond\varphi$ by $\sigma.n\,\mathbb{T}\,\varphi$. These rules, however, can only be applied for a prefix $\sigma.n$ which *does not* already occur on the branch in which it is applied. We call such prefixes "new" (to the branch).

The rules are given in Table tab.2.

The requirement that the restriction that the prefix for $\Box\mathbb{T}$ must be used is necessary as otherwise we would count the following as a closed tableau:

1.	$1\,\mathbb{T}$	$\Box\varphi$	Assumption
2.	$1\,\mathbb{F}$	$\Diamond\varphi$	Assumption
3.	$1.1\,\mathbb{T}$	φ	$\Box\mathbb{T}\,1$
4.	$1.1\,\mathbb{F}$	φ	$\Diamond\mathbb{F}\,2$
	$\otimes$		

But $\Box\varphi \nvDash \Diamond\varphi$, so our proof system would be unsound. Likewise, $\Diamond\varphi \nvDash \Box\varphi$, but without the restriction that the prefix for $\Box\mathbb{F}$ must be new, this would be a closed tableau:

1.	$1\,\mathbb{T}\ \ \Diamond\varphi$	Assumption
2.	$1\,\mathbb{F}\ \ \Box\varphi$	Assumption
3.	$1.1\,\mathbb{T}\ \ \varphi$	$\Diamond\mathbb{T}\,1$
4.	$1.1\,\mathbb{F}\ \ \varphi$	$\Box\mathbb{F}\,2$
	$\otimes$	

tab.3 Tableaux for K

Example tab.1. We give a closed tableau that shows $\vdash (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$. nml:tab:prk:sec

1.	$1\,\mathbb{F}\ \ (\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$		Assumption
2.	$1\,\mathbb{T}\ \ \Box\varphi \wedge \Box\psi$		$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \ \Box(\varphi \wedge \psi)$		$\to\mathbb{F}\,1$
4.	$1\,\mathbb{T}\ \ \Box\varphi$		$\wedge\mathbb{T}\,2$
5.	$1\,\mathbb{T}\ \ \Box\psi$		$\wedge\mathbb{T}\,2$
6.	$1.1\,\mathbb{F}\ \ \varphi \wedge \psi$		$\Box\mathbb{F}\,3$
7.	$1.1\,\mathbb{F}\ \ \varphi$	$1.1\,\mathbb{F}\ \ \psi$	$\wedge\mathbb{F}\,6$
8.	$1.1\,\mathbb{T}\ \ \varphi$	$1.1\,\mathbb{T}\ \ \psi$	$\Box\mathbb{T}\,4$; $\Box\mathbb{T}\,5$
	$\otimes$	$\otimes$	

Example tab.2. We give a closed tableau that shows $\vdash \Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$:

1.	$1\,\mathbb{F}\ \ \Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$		Assumption
2.	$1\,\mathbb{T}\ \ \Diamond(\varphi \vee \psi)$		$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \ \Diamond\varphi \vee \Diamond\psi$		$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\ \ \Diamond\varphi$		$\vee\mathbb{F}\,3$
5.	$1\,\mathbb{F}\ \ \Diamond\psi$		$\vee\mathbb{F}\,3$
6.	$1.1\,\mathbb{T}\ \ \varphi \vee \psi$		$\Diamond\mathbb{T}\,2$
7.	$1.1\,\mathbb{T}\ \ \varphi$	$1.1\,\mathbb{T}\ \ \psi$	$\vee\mathbb{T}\,6$
8.	$1.1\,\mathbb{F}\ \ \varphi$	$1.1\,\mathbb{F}\ \ \psi$	$\Diamond\mathbb{F}\,4$; $\Diamond\mathbb{F}\,5$
	$\otimes$	$\otimes$	

Problem tab.1. Find closed tableaux in **K** for the following formulas:

1. $\Box\neg p \to \Box(p \to q)$
2. $(\Box p \vee \Box q) \to \Box(p \vee q)$

3. $\Diamond p \to \Diamond(p \lor q)$

4. $\Box(p \land q) \to \Box p$

tab.4 Soundness for K

nml:tab:sou:sec

> This soundness proof reuses the soundness proof for classical propositional logic, i.e., it proves everything from scratch. That's ok if you want a self-contained soundness proof. If you already have seen soundness for ordinary tableau this will be repetitive. It's planned to make it possible to switch between self-contained version and a version building on the non-modal case.

In order to show that prefixed tableaux are sound, we have to show that if explanation

$$1\,\mathbb{T}\,\psi_1, \dots, 1\,\mathbb{T}\,\psi_n, 1\,\mathbb{F}\,\varphi$$

has a closed tableau then $\psi_1, \dots, \psi_n \vDash \varphi$. It is easier to prove the contrapositive: if for some $\mathfrak{M}$ and world w, $\mathfrak{M}, w \Vdash \psi_i$ for all $i = 1, \dots, n$ but $\mathfrak{M}, w \nVdash \varphi$, then no tableau can close. Such a countermodel shows that the initial assumptions of the tableau are satisfiable. The strategy of the proof is to show that whenever all the prefixed formulas on a tableau branch are satisfiable, any application of a rule results in at least one extended branch that is also satisfiable. Since closed branches are unsatisfiable, any tableau for a satisfiable set of prefixed formulas must have at least one open branch.

In order to apply this strategy in the modal case, we have to extend our definition of "satisfiable" to modal modals and prefixes. With that in hand, however, the proof is straightforward.

Definition tab.3. Let P be some set of prefixes, i.e., $P \subseteq (\mathbb{Z}^+)^* \setminus \{\Lambda\}$ and let $\mathfrak{M}$ be a model. A function $f \colon P \to W$ is an *interpretation of* P in $\mathfrak{M}$ if, whenever σ and $\sigma.n$ are both in P, then $Rf(\sigma)f(\sigma.n)$.

Relative to an interpretation of prefixes P we can define:

1. $\mathfrak{M}$ satisfies $\sigma\,\mathbb{T}\,\varphi$ iff $\mathfrak{M}, f(\sigma) \Vdash \varphi$.

2. $\mathfrak{M}$ satisfies $\sigma\,\mathbb{F}\,\varphi$ iff $\mathfrak{M}, f(\sigma) \nVdash \varphi$.

Definition tab.4. Let Γ be a set of prefixed formulas, and let $P(\Gamma)$ be the set of prefixes that occur in it. If f is an interpretation of $P(\Gamma)$ in $\mathfrak{M}$, we say that $\mathfrak{M}$ satisfies Γ with respect to f, $\mathfrak{M}, f \Vdash \Gamma$, if $\mathfrak{M}$ satisfies every prefixed formula in Γ with respect to f. Γ is *satisfiable* iff there is a model $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that $\mathfrak{M}, f \Vdash \Gamma$.

Proposition tab.5. *If Γ contains both $\sigma\,\mathbb{T}\,\varphi$ and $\sigma\,\mathbb{F}\,\varphi$, for some formula φ and prefix σ, then Γ is unsatisfiable.*

Proof. There cannot be a model $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that both $\mathfrak{M}, f(\sigma) \Vdash \varphi$ and $\mathfrak{M}, f(\sigma) \nVdash \varphi$. □

Theorem tab.6 (Soundness). *If Γ has a closed tableau, Γ is unsatisfiable.* nml:tab:sou: thm:tableau-soundness

Proof. We call a branch of a tableau satisfiable iff the set of signed formulas on it is satisfiable, and let's call a tableau satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable tableau by one of the rules of inference always results in a satisfiable tableau. This will prove the theorem: any closed tableau results by applying rules of inference to the tableau consisting only of assumptions from Γ. So if Γ were satisfiable, any tableau for it would be satisfiable. A closed tableau, however, is clearly not satisfiable, since all its branches are closed and closed branches are unsatisfiable.

Suppose we have a satisfiable tableau, i.e., a tableau with at least one satisfiable branch. Applying a rule of inference either adds signed formulas to a branch, or splits a branch in two. If the tableau has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended tableau, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let Γ be the set of signed formulas on that branch, and let $\sigma\,S\,\varphi \in \Gamma$ be the signed formula to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, i.e., Γ together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable. First, we consider the possible inferences with only one premise.

1. The branch is expanded by applying $\neg\mathbb{T}$ to $\sigma\,\mathbb{T}\,\neg\psi \in \Gamma$. Then the extended branch contains the signed formulas $\Gamma \cup \{\sigma\,\mathbb{F}\,\psi\}$. Suppose $\mathfrak{M}, f \Vdash \Gamma$. In particular, $\mathfrak{M}, f(\sigma) \Vdash \neg\psi$. Thus, $\mathfrak{M}, f(\sigma) \nVdash \psi$, i.e., $\mathfrak{M}$ satisfies $\sigma\,\mathbb{F}\,\psi$ with respect to f.

2. The branch is expanded by applying $\neg\mathbb{F}$ to $\sigma\,\mathbb{F}\,\neg\psi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\land\mathbb{T}$ to $\sigma\,\mathbb{T}\,\psi \land \chi \in \Gamma$, which results in two new signed formulas on the branch: $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \Vdash \psi \land \chi$. Then $\mathfrak{M}, f(\sigma) \Vdash \psi$ and $\mathfrak{M}, f(\sigma) \Vdash \chi$. This means that $\mathfrak{M}$ satisfies both $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$ with respect to f.

4. The branch is expanded by applying $\lor\mathbb{F}$ to $\mathbb{F}\,\psi \lor \chi \in \Gamma$: Exercise.

5. The branch is expanded by applying $\to\mathbb{F}$ to $\sigma\,\mathbb{F}\,\psi \to \chi \in \Gamma$: This results in two new signed formulas on the branch: $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{F}\,\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \nVdash \psi \to \chi$. Then $\mathfrak{M}, f(\sigma) \Vdash \psi$ and $\mathfrak{M}, f(\sigma) \nVdash \chi$. This means that $\mathfrak{M}, f$ satisfies both $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{F}\,\chi$.

6. The branch is expanded by applying $\Box\mathbb{T}$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma.n\,\mathbb{T}\,\psi$ on the branch, for some $\sigma.n \in P(\Gamma)$ (since $\sigma.n$ must be used). Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since f is an interpretation of prefixes and both σ, $\sigma.n \in P(\Gamma)$, we know that $Rf(\sigma)f(\sigma.n)$. Hence, $\mathfrak{M}, f(\sigma.n) \Vdash \psi$, i.e., $\mathfrak{M}, f$ satisfies $\sigma.n\,\mathbb{T}\,\psi$.

7. The branch is expanded by applying $\Box\mathbb{F}$ to $\sigma\,\mathbb{F}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma.n\,\mathbb{F}\,\varphi$, where $\sigma.n$ is a new prefix on the branch, i.e., $\sigma.n \notin P(\Gamma)$. Since Γ is satisfiable, there is a $\mathfrak{M}$ and interpretation f of $P(\Gamma)$ such that $\mathfrak{M}, f \vDash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \nVdash \Box\psi$. We have to show that $\Gamma \cup \{\sigma.n\,\mathbb{F}\,\psi\}$ is satisfiable. To do this, we define an interpretation of $P(\Gamma) \cup \{\sigma.n\}$ as follows:

   Since $\mathfrak{M}, f(\sigma) \nVdash \Box\psi$, there is a $w \in W$ such that $Rf(\sigma)w$ and $\mathfrak{M}, w \nVdash \psi$. Let f' be like f, except that $f'(\sigma.n) = w$. Since $f'(\sigma) = f(\sigma)$ and $Rf(\sigma)w$, we have $Rf'(\sigma)f'(\sigma.n)$, so f' is an interpretation of $P(\Gamma)\cup\{\sigma.n\}$. Obviously $\mathfrak{M}, f'(\sigma.n) \nVdash \psi$. Since $f(\sigma') = f'(\sigma')$ for all prefixes $\sigma' \in P(\Gamma)$, $\mathfrak{M}, f' \Vdash \Gamma$. So, $\mathfrak{M}, f'$ satisfies $\Gamma \cup \{\sigma.n\,\mathbb{F}\,\psi\}$.

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying $\wedge\mathbb{F}$ to $\sigma\,\mathbb{F}\,\psi \wedge \chi \in \Gamma$, which results in two branches, a left one continuing through $\sigma\,\mathbb{F}\,\psi$ and a right one through $\sigma\,\mathbb{F}\,\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \nVdash \psi \wedge \chi$. Then $\mathfrak{M}, f(\sigma) \nVdash \psi$ or $\mathfrak{M}, f(\sigma) \nVdash \chi$. In the former case, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\psi$, i.e., the left branch is satisfiable. In the latter, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\chi$, i.e., the right branch is satisfiable.

2. The branch is expanded by applying $\vee\mathbb{T}$ to $\sigma\,\mathbb{T}\,\psi \vee \chi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\to\mathbb{T}$ to $\sigma\,\mathbb{T}\,\psi \to \chi \in \Gamma$: Exercise. □

Problem tab.2. Complete the proof of Theorem tab.6.

nml:tab:sou: cor:entailment-soundness

Corollary tab.7. *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.*

Proof. If $\Gamma \vdash \varphi$ then for some $\psi_1, \dots, \psi_n \in \Gamma$, $\Delta = \{1\,\mathbb{F}\,\varphi, 1\,\mathbb{T}\,\psi_1, \dots, 1\,\mathbb{T}\,\psi_n\}$ has a closed tableau. We want to show that $\Gamma \vDash \varphi$. Suppose not, so for some $\mathfrak{M}$ and w, $\mathfrak{M}, w \Vdash \psi_i$ for $i = 1, \dots, n$, but $\mathfrak{M}, w \nVdash \varphi$. Let $f(1) = w$; then f is an interpretation of $P(\Delta)$ into $\mathfrak{M}$, and $\mathfrak{M}$ satisfies Δ with respect to f. But by Theorem tab.6, Δ is unsatisfiable since it has a closed tableau, a contradiction. So we must have $\Gamma \vdash \varphi$ after all. □

nml:tab:sou: cor:weak-soundness

Corollary tab.8. *If $\vdash \varphi$ then φ is true in all models.*

$\dfrac{\sigma\,\mathbb{T}\,\Box\varphi}{\sigma\,\mathbb{T}\,\varphi}\,\mathrm{T}\Box$	$\dfrac{\sigma\,\mathbb{F}\,\Diamond\varphi}{\sigma\,\mathbb{F}\,\varphi}\,\mathrm{T}\Diamond$
$\dfrac{\sigma\,\mathbb{T}\,\Box\varphi}{\sigma\,\mathbb{T}\,\Diamond\varphi}\,\mathrm{D}\Box$	$\dfrac{\sigma\,\mathbb{F}\,\Diamond\varphi}{\sigma\,\mathbb{F}\,\Box\varphi}\,\mathrm{D}\Diamond$
$\dfrac{\sigma.n\,\mathbb{T}\,\Box\varphi}{\sigma\,\mathbb{T}\,\varphi}\,\mathrm{B}\Box$	$\dfrac{\sigma.n\,\mathbb{F}\,\Diamond\varphi}{\sigma\,\mathbb{F}\,\varphi}\,\mathrm{B}\Diamond$
$\dfrac{\sigma\,\mathbb{T}\,\Box\varphi}{\sigma.n\,\mathbb{T}\,\Box\varphi}\,4\Box$ $\sigma.n$ is used	$\dfrac{\sigma\,\mathbb{F}\,\Diamond\varphi}{\sigma.n\,\mathbb{F}\,\Diamond\varphi}\,4\Diamond$ $\sigma.n$ is used
$\dfrac{\sigma.n\,\mathbb{T}\,\Box\varphi}{\sigma\,\mathbb{T}\,\Box\varphi}\,4\mathrm{r}\Box$	$\dfrac{\sigma.n\,\mathbb{F}\,\Diamond\varphi}{\sigma\,\mathbb{F}\,\Diamond\varphi}\,4\mathrm{r}\Diamond$

Table tab.3: More modal rules.

nml:tab:mru: tab:more-rules

Logic	R is ...	Rules
$\mathbf{T} = \mathbf{KT}$	reflexive	T$\Box$, T$\Diamond$
$\mathbf{D} = \mathbf{KD}$	serial	D$\Box$, D$\Diamond$
$\mathbf{K4}$	transitive	4$\Box$, 4$\Diamond$
$\mathbf{B} = \mathbf{KTB}$	reflexive, symmetric	T$\Box$, T$\Diamond$ B$\Box$, B$\Diamond$
$\mathbf{S4} = \mathbf{KT4}$	reflexive, transitive	T$\Box$, T$\Diamond$, 4$\Box$, 4$\Diamond$
$\mathbf{S5} = \mathbf{KT4B}$	reflexive, transitive, euclidean	T$\Box$, T$\Diamond$, 4$\Box$, 4$\Diamond$, 4r$\Box$, 4r$\Diamond$

Table tab.4: Tableau rules for various modal logics.

nml:tab:mru: tab:logics-rules

tab.5 Rules for Other Accessibility Relations

In order to deal with logics determined by special accessibility relations, we consider the additional rules in Table tab.3. nml:tab:mru: sec

Adding these rules results in systems that are sound and complete for the logics given in Table tab.4.

Example tab.9. We give a closed tableau that shows $\mathbf{S5} \vdash 5$, i.e., $\Box\varphi \to \Box\Diamond\varphi$.

1.	$1\,\mathbb{F}\ \ \Box\varphi \to \Box\Diamond\varphi$	Assumption
2.	$1\,\mathbb{T}\ \ \Box\varphi$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \ \Box\Diamond\varphi$	$\to\mathbb{F}\,1$
4.	$1.1\,\mathbb{F}\ \ \Diamond\varphi$	$\Box\mathbb{F}\,3$
5.	$1\,\mathbb{F}\ \ \Diamond\varphi$	$4\mathrm{r}\Diamond\ 4$
6.	$1.1\,\mathbb{F}\ \ \varphi$	$\Diamond\mathbb{F}\,5$
7.	$1.1\,\mathbb{T}\ \ \varphi$	$\Box\mathbb{T}\,2$
	$\otimes$	

Problem tab.3. Give closed tableaux that show the following:

1. $\mathbf{KT5} \vdash \mathrm{B}$;
2. $\mathbf{KT5} \vdash 4$;
3. $\mathbf{KDB4} \vdash \mathrm{T}$;
4. $\mathbf{KB4} \vdash 5$;
5. $\mathbf{KB5} \vdash 4$;
6. $\mathbf{KT} \vdash \mathrm{D}$.

tab.6 Soundness for Additional Rules

nml:tab:msn: sec

We say a rule is sound for a class of models if, whenever a branch in a tableau is satisfiable in a model from that class, the branch resulting from applying the rule is also satisfiable in a model from that class.

nml:tab:msn: prop:soundness-T

Proposition tab.10. *T$\Box$ and T$\Diamond$ are sound for reflexive models.*

Proof.

1. The branch is expanded by applying T$\Box$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \psi$, i.e., $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\psi$.

2. The branch is expanded by applying T$\Diamond$ to $\sigma\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{F}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$. Since R is reflexive, we know that $Rf(\sigma)f(\sigma)$. Hence, $\mathfrak{M}, f(\sigma) \nVdash \psi$, i.e., $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\psi$. $\Box$

nml:tab:msn: prop:soundness-D

Proposition tab.11. *D$\Box$ and D$\Diamond$ are sound for serial models.*

Proof.

1. The branch is expanded by applying D$\Box$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \Vdash \psi$, and hence $\mathfrak{M}, f(\sigma) \Vdash \Diamond\psi$. So, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\Diamond\psi$.

2. The branch is expanded by applying D$\Diamond$ to $\sigma\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{F}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$. Since R is serial, there is a $w \in W$ such that $Rf(\sigma)w$. Then $\mathfrak{M}, w \nVdash \psi$, and hence $\mathfrak{M}, f(\sigma) \nVdash \Box\psi$. So, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\Box\psi$. $\Box$

Proposition tab.12. B$\Box$ *and* B$\Diamond$ *are sound for symmetric models.* *nml:tab:msn:prop:soundness-B*

Proof.

1. The branch is expanded by applying B$\Box$ to $\sigma.n\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, $\mathfrak{M}, f(\sigma) \Vdash \psi$. Hence, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\psi$.

2. The branch is expanded by applying B$\Diamond$ to $\sigma.n\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{F}\,\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Since R is symmetric, $Rf(\sigma.n)f(\sigma)$. Since $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$, $\mathfrak{M}, f(\sigma) \nVdash \psi$. Hence, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\psi$. $\Box$

Proposition tab.13. 4$\Box$ *and* 4$\Diamond$ *are sound for transitive models.* *nml:tab:msn:prop:soundness-4*

Proof.

1. The branch is expanded by applying 4$\Box$ to $\sigma\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma.n\,\mathbb{T}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$ and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$, and $\mathfrak{M}, f$ satisfies $\sigma.n\,\mathbb{T}\,\Box\psi$.

2. The branch is expanded by applying 4$\Diamond$ to $\sigma\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma.n\,\mathbb{F}\,\Diamond\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$ and $\sigma.n$ must be used, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma.n)w$. Since R is transitive, $Rf(\sigma)w$. Since $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$, $\mathfrak{M}, w \nVdash \psi$. Hence, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$, and $\mathfrak{M}, f$ satisfies $\sigma.n\,\mathbb{F}\,\Diamond\psi$. $\Box$

nml:tab:msn: prop:soundness-4r **Proposition tab.14.** *$4r\Box$ and $4r\Diamond$ are sound for euclidean models.*

Proof. 1. The branch is expanded by applying $4r\Box$ to $\sigma.n\,\mathbb{T}\,\Box\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \Vdash \Box\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma).n \Vdash \Box\psi$, $\mathfrak{M}, w \Vdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$, and $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{T}\,\Box\psi$.

2. The branch is expanded by applying $4r\Diamond$ to $\sigma.n\,\mathbb{F}\,\Diamond\psi \in \Gamma$: This results in a new signed formula $\sigma\,\mathbb{T}\,\Box\psi$ on the branch. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular, $\mathfrak{M}, f(\sigma.n) \nVdash \Diamond\psi$. Since f is an interpretation of prefixes on the branch into $\mathfrak{M}$, we know that $Rf(\sigma)f(\sigma.n)$. Now let w be any world such that $Rf(\sigma)w$. Since R is euclidean, $Rf(\sigma.n)w$. Since $\mathfrak{M}, f(\sigma).n \nVdash \Diamond\psi$, $\mathfrak{M}, w \nVdash \psi$. Hence, $\mathfrak{M}, f(\sigma) \nVdash \Diamond\psi$, and $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\Diamond\psi$. $\Box$

nml:tab:msn: cor:soundness-logics **Corollary tab.15.** *The tableau systems given in Table tab.4 are sound for the respective classes of models.*

tab.7 Simple Tableaux for S5

nml:tab:s5: sec **S5** is sound and complete with respect to the class of universal models, i.e., models where every world is accessible from every world. In universal models the accessibility relation doesn't matter: "there is a world w where $\mathfrak{M}, w \Vdash \varphi$" is true if and only if there is such a w that's accessible from u. So in **S5**, we can define models as simply a set of worlds and a valuation V. This suggests that we should be able to simplify the tableau rules as well. In the general case, we take as prefixes sequences of positive integers, so that we can keep track of which such prefixes name worlds which are accessible from others: $\sigma.n$ names a world accessible from σ. But in **S5** any world is accessible from any world, so there is no need to so keep track. Instead, we can use positive integers as prefixes. The simplified rules are given in Table tab.5.

Example tab.16. We give a simplified closed tableau that shows **S5** $\vdash 5$, i.e., $\Diamond\varphi \to \Box\Diamond\varphi$.

1.	$1\,\mathbb{F}\;\;\Diamond\varphi \to \Box\Diamond\varphi$	Assumption
2.	$1\,\mathbb{T}\;\;\Diamond\varphi$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\;\;\Box\Diamond\varphi$	$\to\mathbb{F}\,1$
4.	$2\,\mathbb{F}\;\;\Diamond\varphi$	$\Box\mathbb{F}\,3$
5.	$3\,\mathbb{T}\;\;\varphi$	$\Diamond\mathbb{T}\,2$
6.	$3\,\mathbb{F}\;\;\varphi$	$\Diamond\mathbb{F}\,4$
	$\otimes$	

$\dfrac{n\,\mathbb{T}\,\Box\varphi}{m\,\mathbb{T}\,\varphi}\,\Box\mathbb{T}$ m is used	$\dfrac{n\,\mathbb{F}\,\Box\varphi}{m\,\mathbb{F}\,\varphi}\,\Box\mathbb{F}$ m is new
$\dfrac{n\,\mathbb{T}\,\Diamond\varphi}{m\,\mathbb{T}\,\varphi}\,\Diamond\mathbb{T}$ m is new	$\dfrac{n\,\mathbb{F}\,\Diamond\varphi}{m\,\mathbb{F}\,\varphi}\,\Diamond\mathbb{F}$ m is used

Table tab.5: Simplified rules for **S5**.

nml:tab:s5: tab:rules-S5

tab.8 Completeness for K

explanation

nml:tab:cpl: sec

To show that the method of tableaux is complete, we have to show that whenever there is no closed tableau to show $\Gamma \vdash \varphi$, then $\Gamma \nvDash \varphi$, i.e., there is a countermodel. But "there is no closed tableau" means that every way we could try to construct one has to fail to close. The trick is to see that if every such way fails to close, then a specific, *systematic and exhaustive* way also fails to close. And this systematic and exhaustive way would close if a closed tableau exists. The single tableau will contain, among its open branches, all the information required to define a countermodel. The countermodel given by an open branch in this tableau will contain the all the prefixes used on that branch as the worlds, and a propositional variable p is true at σ iff $\sigma\,\mathbb{T}\,p$ occurs on the branch.

Definition tab.17. A branch in a tableau is called complete if, whenever it contains a prefixed formula $\sigma\,S\,\varphi$ to which a rule can be applied, it also contains

1. the prefixed formulas that are the corresponding conclusions of the rule, in the case of propositional stacking rules;
2. one of the corresponding conclusion formulas in the case of propositional branching rules;
3. at least one possible conclusion in the case of modal rules that require a new prefix;
4. the corresponding conclusion for every prefix occurring on the branch in the case of modal rules that require a used prefix.

explanation

For instance, a complete branch contains $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$ whenever it contains $\mathbb{T}\,\psi \wedge \chi$. If it contains $\sigma\,\mathbb{T}\,\psi \vee \chi$ it contains at least one of $\sigma\,\mathbb{F}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$.

If it contains $\sigma\,\mathbb{F}\,\Box$ it also contains $\sigma.n\,\mathbb{F}\,\Box$ for at least one n. And whenever it contains $\sigma\,\mathbb{T}\,\Box$ it also contains $\sigma.n\,\mathbb{T}\,\Box$ for every n such that $\sigma.n$ is used on the branch.

nml:tab:cpl: prop:complete-tableau **Proposition tab.18.** *Every finite Γ has a tableau in which every branch is complete.*

Proof. Consider an open branch in a tableau for Γ. There are finitely many prefixed formulas in the branch to which a rule could be applied. In some fixed order (say, top to bottom), for each of these prefixed formulas for which the conditions (1)–(4) do not already hold, apply the rules that can be applied to it to extend the branch. In some cases this will result in branching; apply the rule at the tip of each resulting branch for all remaining prefixed formulas. Since the number of prefixed formulas is finite, and the number of used prefixes on the branch is finite, this procedure eventually results in (possibly many) branches extending the original branch. Apply the procedure to each, and repeat. But by construction, every branch is closed. $\Box$

nml:tab:cpl: thm:tableau-completeness **Theorem tab.19 (Completeness).** *If Γ has no closed tableau, Γ is satisfiable.*

Proof. By the proposition, Γ has a tableau in which every branch is complete. Since it has no closed tableau, it thas has a tableau in which at least one branch is open and complete. Let Δ be the set of prefixed formulas on the branch, and $P(\Delta)$ the set of prefixes occurring in it.

We define a model $\mathfrak{M}(\Delta) = \langle P(\Delta), R, V\rangle$ where the worlds are the prefixes occurring in Δ, the accessibility relation is given by:

$$R\sigma\sigma' \quad \text{iff} \quad \sigma' = \sigma.n \quad \text{for some } n$$

and

$$V(p) = \{\sigma : \sigma\,\mathbb{T}\,p \in \Delta\}.$$

We show by induction on φ that if $\sigma\,\mathbb{T}\,\varphi \in \Delta$ then $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$, and if $\sigma\,\mathbb{F}\,\varphi \in \Delta$ then $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

1. $\varphi \equiv p$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$ then $\sigma \in V(p)$ (by definition of V) and so $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$ then $\sigma\,\mathbb{T}\,\varphi \notin \Delta$, since the branch would otherwise be closed. So $\sigma \notin V(p)$ and thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

2. $\varphi \equiv \neg\psi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then $\sigma\,\mathbb{F}\,\psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ and thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then $\sigma\,\mathbb{T}\,\psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

3. $\varphi \equiv \psi \wedge \chi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then both $\sigma\,\mathbb{T}\,\psi \in \Delta$ and $\sigma\,\mathbb{T}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then either $\sigma\,\mathbb{F}\,\psi \in \Delta$ or $\sigma\,\mathbb{F}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

4. $\varphi \equiv \psi \vee \chi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then either $\sigma\,\mathbb{T}\,\psi \in \Delta$ or $\sigma\,\mathbb{T}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then both $\sigma\,\mathbb{F}\,\psi \in \Delta$ and $\sigma\,\mathbb{F}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, both $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ and $\mathfrak{M}(\Delta), \sigma \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

5. $\varphi \equiv \psi \rightarrow \chi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then either $\sigma\,\mathbb{F}\,\psi \in \Delta$ or $\sigma\,\mathbb{T}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, either $\mathfrak{M}(\Delta), \sigma \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then both $\sigma\,\mathbb{T}\,\psi \in \Delta$ and $\sigma\,\mathbb{F}\,\chi \in \Delta$ since the branch is complete. By induction hypothesis, both $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

6. $\varphi \equiv \Box\psi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then, since the branch is complete, $\sigma.n\,\mathbb{T}\,\psi \in \Delta$ for every $\sigma.n$ used on the branch, i.e., for every $\sigma' \in P(\Delta)$ such that $R\sigma\sigma'$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma' \Vdash \psi$ for every σ' such that $R\sigma\sigma'$. Therefore, $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then for some $\sigma.n$, $\sigma.n\,\mathbb{F}\,\psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma.n \nVdash \psi$. Since $R\sigma(\sigma.n)$, there is a σ' such that $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

7. $\varphi \equiv \Diamond\psi$: If $\sigma\,\mathbb{T}\,\varphi \in \Delta$, then for some $\sigma.n$, $\sigma.n\,\mathbb{T}\,\psi \in \Delta$ since the branch is complete. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma.n \Vdash \psi$. Since $R\sigma(\sigma.n)$, there is a σ' such that $\mathfrak{M}(\Delta), \sigma' \Vdash \psi$. Thus $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   If $\sigma\,\mathbb{F}\,\varphi \in \Delta$, then, since the branch is complete, $\sigma.n\,\mathbb{F}\,\psi \in \Delta$ for every $\sigma.n$ used on the branch, i.e., for every $\sigma' \in P(\Delta)$ such that $R\sigma\sigma'$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$ for every σ' such that $R\sigma\sigma'$. Therefore, $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

Since $\Gamma \subseteq \Delta$, $\mathfrak{M}(\Delta) \Vdash \Gamma$. □

Problem tab.4. Complete the proof of Theorem tab.19.

Corollary tab.20. *If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.* *nml:tab:cpl: cor:entailment-completeness*

Corollary tab.21. *If φ is true in all models, then $\vdash \varphi$.* *nml:tab:cpl: cor:weak-completeness*

tab.9 Countermodels from Tableaux

nml:tab:cou:sec The proof of the completeness theorem doesn't just show that if $\vDash \varphi$ then $\vdash \varphi$, it also gives us a method for constructing countermodels to φ if $\nvDash A$. In the case of **K**, this method constitutes a *decision procedure*. For suppose $\nvDash \varphi$. Then the proof of Proposition tab.18 gives a method for constructing a complete tableau. The method in fact always terminates. The propositional rules for **K** only add prefixed formulas of lower complexity, i.e., each propositional rule need only be applied once on a branch for any signed formula $\sigma\, S\, \varphi$. New prefixes are only generated by the $\Box\mathbb{F}$ and $\Diamond\mathbb{T}$ rules, and also only have to be applied once (and produce a single new prefix). $\Box\mathbb{T}$ and $\Diamond\mathbb{F}$ have to be applied potentially multiple times, but only once per prefix, and only finitely many new prefixes are generated. So the construction either results in a closed branch or a complete branch after finitely many stages. explanation

Once a tableau with an open complete branch is constructed, the proof of Theorem tab.19 gives us an explict model that satisfies the original set of prefixed formulas. So not only is it the case that if $\Gamma \vDash \varphi$, then a closed tableau exists and $\Gamma \vdash \varphi$, if we look for the closed tableau in the right way and end up with a "complete" tableau, we'll not only know that $\Gamma \nvDash \varphi$ but actually be able to construct a countermodel.

Example tab.22. We know that $\nvDash \Box(p \lor q) \to (\Box p \lor \Box q)$. The construction of a tableau begins with:

1.	$1\,\mathbb{F}\;\; \Box(p \lor q) \to (\Box p \lor \Box q)$ ✓	Assumption
2.	$1\,\mathbb{T}\;\; \Box(p \lor q)$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\;\; \Box p \lor \Box q$ ✓	$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\;\; \Box p$ ✓	$\lor\mathbb{F}\,3$
5.	$1\,\mathbb{F}\;\; \Box q$ ✓	$\lor\mathbb{F}\,3$
6.	$1.1\,\mathbb{F}\;\; p$ ✓	$\Box\mathbb{F}\,4$
7.	$1.2\,\mathbb{F}\;\; q$ ✓	$\Box\mathbb{F}\,5$

The tableau is of course not finished yet. In the next step, we consider the only line without a checkmark: the prefixed formula $1\,\mathbb{T}\,\Box(p \lor q)$ on line 2. The construction of the closed tableau says to apply the $\Box\mathbb{T}$ rule for every prefix used on the branch, i.e., for both 1.1 and 1.2:

1.	$1\,\mathbb{F}\;\; \Box(p \lor q) \to (\Box p \lor \Box q)$ ✓	Assumption
2.	$1\,\mathbb{T}\;\; \Box(p \lor q)$	$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\;\; \Box p \lor \Box q$ ✓	$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\;\; \Box p$ ✓	$\lor\mathbb{F}\,3$
5.	$1\,\mathbb{F}\;\; \Box q$ ✓	$\lor\mathbb{F}\,3$
6.	$1.1\,\mathbb{F}\;\; p$ ✓	$\Box\mathbb{F}\,4$
7.	$1.2\,\mathbb{F}\;\; q$ ✓	$\Box\mathbb{F}\,5$
8.	$1.1\,\mathbb{T}\;\; p \lor q$	$\Box\mathbb{T}\,2$
9.	$1.2\,\mathbb{T}\;\; p \lor q$	$\Box\mathbb{T}\,2$

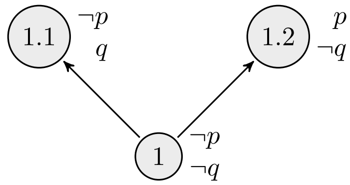

Figure tab.1: A countermodel to $\Box(p \lor q) \to (\Box p \lor \Box q)$.

nml:tab:cou: fig:counter-Box

Now lines 2, 8, and 9, don't have checkmarks. But no new prefix has been added, so we apply $\lor\mathbb{T}$ to lines 8 and 9, on all resulting branches (as long as they don't close):

1.	$1\,\mathbb{F}\ \Box(p \lor q) \to (\Box p \lor \Box q)$ ✓		Assumption
2.	$1\,\mathbb{T}\ \Box(p \lor q)$		$\to\mathbb{F}\,1$
3.	$1\,\mathbb{F}\ \Box p \lor \Box q$ ✓		$\to\mathbb{F}\,1$
4.	$1\,\mathbb{F}\ \Box p$ ✓		$\lor\mathbb{F}\,3$
5.	$1\,\mathbb{F}\ \Box q$ ✓		$\lor\mathbb{F}\,3$
6.	$1.1\,\mathbb{F}\ p$ ✓		$\Box\mathbb{F}\,4$
7.	$1.2\,\mathbb{F}\ q$ ✓		$\Box\mathbb{F}\,5$
8.	$1.1\,\mathbb{T}\ p \lor q$ ✓		$\Box\mathbb{T}\,2$
9.	$1.2\,\mathbb{T}\ p \lor q$ ✓		$\Box\mathbb{T}\,2$
10.	$1.1\,\mathbb{T}\ p$ ✓ $\otimes$	$1.1\,\mathbb{T}\ q$ ✓	$\lor\mathbb{T}\,8$
11.		$1.2\,\mathbb{T}\ p$ ✓ | $1.2\,\mathbb{T}\ q$ ✓ $\otimes$	$\lor\mathbb{T}\,9$

There is one remaining open branch, and it is complete. From it we define the model with worlds $W = \{1, 1.1, 1.2\}$ (the only prefixes appearing on the open branch), the accessibility relation $R = \{\langle 1, 1.1\rangle, \langle 1, 1.2\rangle\}$, and the assignment $V(p) = \{1.2\}$ (because line 11 contains $1.2\,\mathbb{T}\,p$) and $V(q) = \{1.1\}$ (because line 10 contains $1.1\,\mathbb{T}\,q$). The model is pictured in Figure tab.1, and you can verify that it is a countermodel to $\Box(p \lor q) \to (\Box p \lor \Box q)$.

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Bibliography