

## tab.1 Soundness for K

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sec

This soundness proof reuses the soundness proof for classical propositional logic, i.e., it proves everything from scratch. That's ok if you want a self-contained soundness proof. If you already have seen soundness for ordinary tableau this will be repetitive. It's planned to make it possible to switch between self-contained version and a version building on the non-modal case.

In order to show that prefixed **tableaux** are sound, we have to show that if

[explanation](#)

$$1 \mathbb{T} \psi_1, \dots, 1 \mathbb{T} \psi_n, 1 \mathbb{F} \varphi$$

has a closed **tableau** then  $\psi_1, \dots, \psi_n \models \varphi$ . It is easier to prove the contrapositive: if for some  $\mathfrak{M}$  and world  $w$ ,  $\mathfrak{M}, w \Vdash \psi_i$  for all  $i = 1, \dots, n$  but  $\mathfrak{M}, w \not\Vdash \varphi$ , then no **tableau** can close. Such a countermodel shows that the initial assumptions of the **tableau** are satisfiable. The strategy of the proof is to show that whenever all the prefixed **formulas** on a **tableau** branch are satisfiable, any application of a rule results in at least one extended branch that is also satisfiable. Since closed branches are unsatisfiable, any **tableau** for a satisfiable set of prefixed **formulas** must have at least one open branch.

In order to apply this strategy in the modal case, we have to extend our definition of “satisfiable” to modal modals and prefixes. With that in hand, however, the proof is straightforward.

**Definition tab.1.** Let  $P$  be some set of prefixes, i.e.,  $P \subseteq (\mathbb{Z}^+)^* \setminus \{A\}$  and let  $\mathfrak{M}$  be a model. A function  $f: P \rightarrow W$  is an *interpretation of  $P$*  in  $\mathfrak{M}$  if, whenever  $\sigma$  and  $\sigma.n$  are both in  $P$ , then  $Rf(\sigma)f(\sigma.n)$ .

Relative to an interpretation of prefixes  $P$  we can define:

1.  $\mathfrak{M}$  satisfies  $\sigma \mathbb{T} \varphi$  iff  $\mathfrak{M}, f(\sigma) \Vdash \varphi$ .
2.  $\mathfrak{M}$  satisfies  $\sigma \mathbb{F} \varphi$  iff  $\mathfrak{M}, f(\sigma) \not\Vdash \varphi$ .

**Definition tab.2.** Let  $\Gamma$  be a set of prefixed **formulas**, and let  $P(\Gamma)$  be the set of prefixes that occur in it. If  $f$  is an interpretation of  $P(\Gamma)$  in  $\mathfrak{M}$ , we say that  $\mathfrak{M}$  satisfies  $\Gamma$  with respect to  $f$ ,  $\mathfrak{M}, f \Vdash \Gamma$ , if  $\mathfrak{M}$  satisfies every prefixed **formula** in  $\Gamma$  with respect to  $f$ .  $\Gamma$  is *satisfiable* iff there is a model  $\mathfrak{M}$  and interpretation  $f$  of  $P(\Gamma)$  such that  $\mathfrak{M}, f \Vdash \Gamma$ .

**Proposition tab.3.** *If  $\Gamma$  contains both  $\sigma \mathbb{T} \varphi$  and  $\sigma \mathbb{F} \varphi$ , for some **formula**  $\varphi$  and prefix  $\sigma$ , then  $\Gamma$  is unsatisfiable.*

*Proof.* There cannot be a model  $\mathfrak{M}$  and interpretation  $f$  of  $P(\Gamma)$  such that both  $\mathfrak{M}, f(\sigma) \Vdash \varphi$  and  $\mathfrak{M}, f(\sigma) \not\Vdash \varphi$ .  $\square$

**Theorem tab.4 (Soundness).** *If  $\Gamma$  has a closed **tableau**,  $\Gamma$  is unsatisfiable.*

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thm:tableau-soundness*

*Proof.* We call a branch of a **tableau** satisfiable iff the set of **signed formulas** on it is satisfiable, and let's call a **tableau** satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable **tableau** by one of the rules of inference always results in a satisfiable **tableau**. This will prove the theorem: any closed **tableau** results by applying rules of inference to the **tableau** consisting only of assumptions from  $\Gamma$ . So if  $\Gamma$  were satisfiable, any **tableau** for it would be satisfiable. A closed **tableau**, however, is clearly not satisfiable, since all its branches are closed and closed branches are unsatisfiable.

Suppose we have a satisfiable **tableau**, i.e., a **tableau** with at least one satisfiable branch. Applying a rule of inference either adds **signed formulas** to a branch, or splits a branch in two. If the **tableau** has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended **tableau**, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let  $\Gamma$  be the set of **signed formulas** on that branch, and let  $\sigma S\varphi \in \Gamma$  be the **signed formula** to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, i.e.,  $\Gamma$  together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable. First, we consider the possible inferences with only one premise.

1. The branch is expanded by applying  $\neg\mathbb{T}$  to  $\sigma\mathbb{T}\neg\psi \in \Gamma$ . Then the extended branch contains the **signed formulas**  $\Gamma \cup \{\sigma\mathbb{F}\psi\}$ . Suppose  $\mathfrak{M}, f \Vdash \Gamma$ . In particular,  $\mathfrak{M}, f(\sigma) \Vdash \neg\psi$ . Thus,  $\mathfrak{M}, f(\sigma) \not\models \psi$ , i.e.,  $\mathfrak{M}$  satisfies  $\sigma\mathbb{F}\psi$  with respect to  $f$ .
2. The branch is expanded by applying  $\neg\mathbb{F}$  to  $\sigma\mathbb{F}\neg\psi \in \Gamma$ : Exercise.
3. The branch is expanded by applying  $\wedge\mathbb{T}$  to  $\sigma\mathbb{T}\psi \wedge \chi \in \Gamma$ , which results in two new **signed formulas** on the branch:  $\sigma\mathbb{T}\psi$  and  $\sigma\mathbb{T}\chi$ . Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular  $\mathfrak{M}, f(\sigma) \Vdash \psi \wedge \chi$ . Then  $\mathfrak{M}, f(\sigma) \Vdash \psi$  and  $\mathfrak{M}, f(\sigma) \Vdash \chi$ . This means that  $\mathfrak{M}$  satisfies both  $\sigma\mathbb{T}\psi$  and  $\sigma\mathbb{T}\chi$  with respect to  $f$ .
4. The branch is expanded by applying  $\vee\mathbb{F}$  to  $\mathbb{F}\psi \vee \chi \in \Gamma$ : Exercise.
5. The branch is expanded by applying  $\rightarrow\mathbb{F}$  to  $\sigma\mathbb{F}\psi \rightarrow \chi \in \Gamma$ : This results in two new **signed formulas** on the branch:  $\sigma\mathbb{T}\psi$  and  $\sigma\mathbb{F}\chi$ . Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular  $\mathfrak{M}, f(\sigma) \not\models \psi \rightarrow \chi$ . Then  $\mathfrak{M}, f(\sigma) \Vdash \psi$  and  $\mathfrak{M}, f(\sigma) \not\models \chi$ . This means that  $\mathfrak{M}, f$  satisfies both  $\sigma\mathbb{T}\psi$  and  $\sigma\mathbb{F}\chi$ .
6. The branch is expanded by applying  $\Box\mathbb{T}$  to  $\sigma\mathbb{T}\Box\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma.n\mathbb{T}\psi$  on the branch, for some  $\sigma.n \in P(\Gamma)$  (since  $\sigma.n$  must be used). Suppose  $\mathfrak{M}, f \Vdash \Gamma$ , in particular,  $\mathfrak{M}, f(\sigma) \Vdash \Box\psi$ . Since  $f$  is an interpretation of prefixes and both  $\sigma, \sigma.n \in P(\Gamma)$ , we know that  $Rf(\sigma)f(\sigma.n)$ . Hence,  $\mathfrak{M}, f(\sigma.n) \Vdash \psi$ , i.e.,  $\mathfrak{M}, f$  satisfies  $\sigma.n\mathbb{T}\psi$ .

7. The branch is expanded by applying  $\Box\mathbb{F}$  to  $\sigma\mathbb{F}\Box\psi \in \Gamma$ : This results in a new **signed formula**  $\sigma.n\mathbb{F}\psi$ , where  $\sigma.n$  is a new prefix on the branch, i.e.,  $\sigma.n \notin P(\Gamma)$ . Since  $\Gamma$  is satisfiable, there is a  $\mathfrak{M}$  and interpretation  $f$  of  $P(\Gamma)$  such that  $\mathfrak{M}, f \models \Gamma$ , in particular  $\mathfrak{M}, f(\sigma) \not\models \Box\psi$ . We have to show that  $\Gamma \cup \{\sigma.n\mathbb{F}\psi\}$  is satisfiable. To do this, we define an interpretation of  $P(\Gamma) \cup \{\sigma.n\}$  as follows:

Since  $\mathfrak{M}, f(\sigma) \not\models \Box\psi$ , there is a  $w \in W$  such that  $Rf(\sigma)w$  and  $\mathfrak{M}, w \not\models \psi$ . Let  $f'$  be like  $f$ , except that  $f'(\sigma.n) = w$ . Since  $f'(\sigma) = f(\sigma)$  and  $Rf(\sigma)w$ , we have  $Rf'(\sigma)f'(\sigma.n)$ , so  $f'$  is an interpretation of  $P(\Gamma) \cup \{\sigma.n\}$ . Obviously  $\mathfrak{M}, f'(\sigma.n) \not\models \psi$ . Since  $f(\sigma') = f'(\sigma')$  for all prefixes  $\sigma' \in P(\Gamma)$ ,  $\mathfrak{M}, f' \models \Gamma$ . So,  $\mathfrak{M}, f'$  satisfies  $\Gamma \cup \{\sigma.n\mathbb{F}\psi\}$ .

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying  $\wedge\mathbb{F}$  to  $\sigma\mathbb{F}\psi \wedge \chi \in \Gamma$ , which results in two branches, a left one continuing through  $\sigma\mathbb{F}\psi$  and a right one through  $\sigma\mathbb{F}\chi$ . Suppose  $\mathfrak{M}, f \models \Gamma$ , in particular  $\mathfrak{M}, f(\sigma) \not\models \psi \wedge \chi$ . Then  $\mathfrak{M}, f(\sigma) \not\models \psi$  or  $\mathfrak{M}, f(\sigma) \not\models \chi$ . In the former case,  $\mathfrak{M}, f$  satisfies  $\sigma\mathbb{F}\psi$ , i.e., the left branch is satisfiable. In the latter,  $\mathfrak{M}, f$  satisfies  $\sigma\mathbb{F}\chi$ , i.e., the right branch is satisfiable.
2. The branch is expanded by applying  $\vee\mathbb{T}$  to  $\sigma\mathbb{T}\psi \vee \chi \in \Gamma$ : Exercise.
3. The branch is expanded by applying  $\rightarrow\mathbb{T}$  to  $\sigma\mathbb{T}\psi \rightarrow \chi \in \Gamma$ : Exercise.  $\square$

**Problem tab.1.** Complete the proof of **Theorem tab.4**.

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cor:entailment-soundness*

**Corollary tab.5.** *If  $\Gamma \vdash \varphi$  then  $\Gamma \models \varphi$ .*

*Proof.* If  $\Gamma \vdash \varphi$  then for some  $\psi_1, \dots, \psi_n \in \Gamma$ ,  $\Delta = \{1\mathbb{F}\varphi, 1\mathbb{T}\psi_1, \dots, 1\mathbb{T}\psi_n\}$  has a closed **tableau**. We want to show that  $\Gamma \models \varphi$ . Suppose not, so for some  $\mathfrak{M}$  and  $w$ ,  $\mathfrak{M}, w \models \psi_i$  for  $i = 1, \dots, n$ , but  $\mathfrak{M}, w \not\models \varphi$ . Let  $f(1) = w$ ; then  $f$  is an interpretation of  $P(\Delta)$  into  $\mathfrak{M}$ , and  $\mathfrak{M}$  satisfies  $\Delta$  with respect to  $f$ . But by **Theorem tab.4**,  $\Delta$  is unsatisfiable since it has a closed **tableau**, a contradiction. So we must have  $\Gamma \vdash \varphi$  after all.  $\square$

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**Corollary tab.6.** *If  $\vdash \varphi$  then  $\varphi$  is true in all models.*

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